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CREMONA TRANSFORMATIONS

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CREMONA TRANSFORMATIONS
IN PLANE AND SPACE

BY

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PREFACE

CREMONA transformations are powerful tools in many lines of research; the aim of this book is to bring together all that has so far been published on their construction and use, as regards points and loci in two and three dimensions. The most important application is to the resolution of singularities of curves and surfaces, which is treated fully in Chapters VII and XVI. The useful and interesting part of a transformation is the way in which it changes the neighbourhood of its fundamental elements, and the fundamental systems are first studied, both in general and in the examples which occur most often in the literature. Special attention is drawn to elements of contact.

A historical account of the subject is given in Chapter XVII; Cayley and Cremona are still leaders, though we British have fallen behind the rest of the world in their track. Pure and analytical methods are here used together; a curve or surface, a function and an equation are treated as the same thing under different aspects, and certain liberties of language are taken in this connection, which make the sentences shorter but not less clear.

The bibliography shows how varied is the mass of material; yet the space theory is far from complete, and it is hoped that the list of outstanding problems on p. 394 may attract more workers. The most promising line of advance is probably from space of higher dimensions: this I must leave to other writers.

By the untimely death of Miss Grace Sadd, a mathematician of promise, and my friend and fellow-worker, the book has suffered loss and delay. Chapter III is mainly her work, also the collection of material for Chapters V and VI, and much helpful criticism of all the MS. which then existed.

I offer grateful thanks to Mr Arthur Berry, who first introduced me to the subject; to Mr T. L. Wren, for his skilled and tireless work on the proofs; to many correspondents who have helped with the bibliography; and to the Cambridge University Press for their care in the printing.

HILDA P. HUDSON

London, 1927

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Part I

CREMONA PLANE TRANSFORMATIONS

CHAPTER I

OUTLINE OF THE GENERAL PLANE THEORY

1. Cremona and Riemann transformations (61, 71, 80, 385*).

A *Cremona plane transformation* is a 1, 1 relation between the points P, P' of two planes S, S'. Let P be given by its homogeneous coordinates x, y, z; then the coordinates x', y', z' of P' are determined uniquely in terms of those of P, and there exists a set of relations

$$x' : y' : z' = \phi_1 : \phi_2 : \phi_3, \quad \dots\dots\dots\dots(1)$$

where the ϕ are homogeneous polynomials in x, y, z, of the same degree, n say, assumed to have no common factor. These are the equations of the *direct* transformation T, and n is its *degree*; they carry us from the given point P of S to the corresponding point or *homologue* P' of S'. We denote this by $P \sim P'$, or $P \overset{T}{\sim} P'$.

But a set of equations of this form does not necessarily determine a Cremona transformation, because so far we have only made T unique one way. If P', and therefore x', y', z', are given, equations (1) give

$$x'\phi_3 - z'\phi_1 = 0, \quad y'\phi_3 - z'\phi_2 = 0.$$

These are the equations of two curves in S, of degree n, intersecting in n^2 points, which, unless ϕ_1, ϕ_2, ϕ_3 are specialized, are all *free*, that is, variable with the parameters x', y', z'. Then P may be any one of these n^2 intersections, all of which correspond to the one given point P'. The relation between P, P' is n^2, 1, and is not 1, 1 except in the trivial case $n = 1$, when T is a linear transformation or homography, equivalent to a change of the frame of reference.

If, however, we do not consider all the points of S, but only those of a certain curve f, then by T, this curve $\sim$ a certain curve f' in S', on which P' lies. In general, only one of the n^2 corresponding positions of P lies on f, and we do obtain a 1, 1 relation between the points of the two curves. This is a *Riemann transformation* between the curves, not a Cremona transformation between the planes. A Cremona transformation between S, S' always gives rise to a 1, 1 relation between the points of any curve in S and its homologue in S', but the converse is not true.

Conditions have been found that a given 1, 1 relation between two curves may be contained in a Cremona plane transformation (46, 225).

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* References to bibliography, pp. 396 ff.

2. The reverse transformation.

Analytically, the equations (1) are not in general equivalent to a similar set with the planes interchanged; but if we add the equation of a curve $f=0$, we can solve rationally in the form

$$x:y:z=\phi_1':\phi_2':\phi_3', \quad \text{.........................(2)}$$

where the ϕ' are homogeneous polynomials in x', y', z', free from common factors, of degree n' say; and we can also find the eliminant $f'(x', y', z')=0$, which is the equation of the homologue of f. Equations (2) give the transformation T^{-1} which is the *reverse* of T, and carries us from P' to P.

But now, for all positions of P', let the two curves in S which determine P have certain fixed points in common, either simple or multiple, equivalent to E simple intersections; these lie on each of ϕ_1, ϕ_2, ϕ_3, and on any linear combination of them, that is, on any member of the net (ϕ) which they determine. Then the n^2 intersections fall into two groups, E fixed and n^2-E free; it is only the free intersections that $\sim P'$ in T, whose order is reduced from n^2, 1 to n^2-E, 1. The elimination of z from (1) gives an equation in x/y of degree n^2, with E fixed roots, and a constant factor of degree E is dropped, leaving an equation of degree n^2-E for the values of x/y at P.

If T is a Cremona transformation, the order is 1, 1 and $E=n^2-1$. Then (1) are rationally reversible in the form (2), and the coordinates of P are determined uniquely in terms of those of P', without the aid of $f=0$; we have a 1, 1 transformation between the points of S, S', and not only between the points of f, f'.

Now if we substitute from (2) in (1), we obtain an identity

$$x':y':z'\equiv\phi_1(\phi_1', \phi_2', \phi_3'):\phi_2(\phi_1', \phi_2', \phi_3'):\phi_3(\phi_1', \phi_2', \phi_3'), \quad \text{...(3)}$$

and there is a factor of proportionality

$$\frac{1}{x'}\phi_1(\phi_1', \phi_2', \phi_3')\equiv\frac{1}{y'}\phi_2(\phi_1', \phi_2', \phi_3')\equiv\frac{1}{z'}\phi_3(\phi_1', \phi_2', \phi_3')\equiv G'(x', y', z') \text{ say,}$$

where G' is a polynomial in x', y', z' of degree $nn'-1$ (p. 15).

Let P' be a general point of S', lying on two general lines

$$a_1x'+a_2y'+a_3z', \quad b_1x'+b_2y'+b_3z';$$

these $\sim$ two general n-ics of the net (ϕ),

$$a_1\phi_1+a_2\phi_2+a_3\phi_3, \quad b_1\phi_1+b_2\phi_2+b_3\phi_3.$$

Let P be one of their n^2 intersections; then $P\sim$ a point (x', y', z'), where, by (2) and (3),

$$(a_1x'+a_2y'+a_3z')\,G'=0 \text{ and } (b_1x'+b_2y'+b_3z')\,G'=0,$$

that is, either $P\sim P'$ or $P\sim$ a point Q' of G'. But Q' also lies on $(c_1x'+c_2y'+c_3z')\,G'$, and if $P\sim Q'$, then P lies on $c_1\phi_1+c_2\phi_2+c_3\phi_3$, the general member of (ϕ). Hence all but one of the intersections of any two members of (ϕ) are base points common to the whole net.

3. Homaloidal nets.

The homologue in S of the net of lines of S' is the *first homaloidal net* of T,

$$(\phi) \equiv c_1\phi_1 + c_2\phi_2 + c_3\phi_3,$$

and similarly the lines of $S \sim$ the *second homaloidal net* (ϕ') in S'. Each is ∞^2, being linear in the three homogeneous parameters c_1, c_2, c_3, and any two of its members have one and only one point of intersection which varies with c_1, c_2, c_3, and $\sim$ the single point of intersection of the two corresponding lines in the other plane.

The curves ϕ, ϕ' are *homaloids*. By definition, each is representable, point by point, upon a line of the other plane, and is therefore rational (unicursal, of genus or deficiency 0). The particular homaloids ϕ_1, ϕ_2, ϕ_3, which $\sim$ the sides x', y', z' of the triangle of reference in S', will be called the *base homaloids* of S. When the net (ϕ) is given in S, the selection of the base homaloids is equivalent to the choice of the frame of reference in S'.

Some writers use *homaloid* as equivalent to *rational curve*; but there are too many words for this meaning, and it is convenient to reserve one term for the narrower use. A rational curve is not a homaloid, as here defined, unless it can form part of a homaloidal net (216). The family of all plane n-ics is of freedom $\frac{1}{2}n(n+3)$, and the general member is of genus $\frac{1}{2}(n-1)(n-2)$. A set of multiple base points, which reduce the genus to 0, may so reduce the freedom as not to leave a net, but only a pencil or a single curve; for example, ten given d.p.'s may define a single sextic curve, but they cannot define a sextic family.

We shall now prove that the degrees of the two homaloidal nets, that is, the degrees of the direct and reverse transformations, are the same; this is not true in three dimensions, and is the first great divergence of the plane and space theories. For, if l, ϕ are a general line and homaloid, and ϕ', l' their homologues, we have a 1, 1 relation between the n intersections of l, ϕ and the n' intersections of ϕ', l'; and all the points of each set are general and variable with the parameters of l, ϕ. Therefore

$$n = n',$$

either being the number of pairs of corresponding points that lie one on each of two given lines l, l' in S, S' respectively. We can therefore denote the degree of a plane transformation by a single suffix, T_n.

But though the two homaloidal nets are of the same degree, they are not always of the same nature; if different, they are called *conjugate*. The simplest transformation in which two different types appear is the T_6 for which $(\phi) \equiv 1^4 4^2 3^1$, this denoting that (ϕ) has one four-fold, four double and three simple base points; and $(\phi') \equiv 3^3 1^2 4^1$. We notice that there are eight points in each base, falling into three groups of 1, 3 and 4 points of equal multiplicity; and it is only the multiplicities of the groups that differ in S, S' (81 and pp. 23, 122).

4. The fundamental system.

The base points of the homaloidal net are called *fundamental points* or *F-points*, and form the *first F-system* of the transformation; we denote it by H. The multiple F-points reduce the genus to 0, and together with the simple F-points reduce the family to a net, and absorb all but one of the free intersections of two general members. The F-points ΣO, which compose H and the second F-system H' in S', may be either simple or multiple, and (ϕ) may be required merely to pass through O with one or more branches, or to have contact of a given order with a given curve at O, or to satisfy different contact conditions for the different branches. F-points of contact, in the plane but not in space, can for most purposes be considered as the limit of distinct adjacent points.

Since an n-ic with an n-fold point breaks up, the multiplicity i of O can vary from 1 to $n-1$ inclusive.

It has been shown that a general member of a linear plane family can have no multiple points except the base points; all the general members have exactly the same singularities, in number, position and kind. An exceptional member may have an additional multiple point (28 and pp. 12, 15).

5. Homologues.

After carrying out the algebraical substitutions of T_n on any entity P, whether a point, set of points, curve or family of curves, we often have to separate a fixed part, before arriving at the proper homologue P'. Thus if P is a point, of the n^2 solutions of (2), n^2-1 are fixed and must be set aside, leaving the one variable point P'. In any 1, 1 transformation, the homologue of a variable entity is variable with the same freedom. The natures of P, P' are the same if these are general, but may be different for special elements; we shall see that F-points $\sim$ special curves, not points (p. 14).

By (2), a general μ-ic $f_\mu(x, y, z) \sim$ the curve $f_\mu(\phi_1', \phi_2', \phi_3')$, or say $f'_{\mu'}(x', y', z')$, of degree $\mu' = \mu n$. It contains the whole of H' taken μ times; that is, an F-point which is i-fold on (ϕ') is μi-fold on f'.

If f contains any F-points, the result of substitution breaks up into fixed curves to be set aside, corresponding to the F-points, and a residual f' of degree $< \mu n$, which is the proper homologue of f, and $\sim$ all its general points. For example, if f is ϕ, then f' reduces to a line.

If we are given a curve f_μ, and can find a T_n with so many F-points on f, and of such multiplicities for T_n and for f, that after substitution curves drop away of total degree $> \mu n - \mu$, then T_n transforms f into a curve f' of lower degree; or the degree may not be lower, but the nature of f' or of its singularities may be simpler. This is a valuable method of studying the properties of the original curve f (342).

6. The equations of condition.

A homaloidal net has three characteristic properties:

(i) It is linear and ∞^2.

(ii) Two general members have one and only one free intersection.

(iii) All its members are rational.

These three properties are invariant under Cremona transformation: parameters entering linearly remain such, and therefore a net $\sim$ another net; the free intersection $\sim$ a free intersection, and no other can arise; and genus is also invariant. Hence any Cremona transformation changes one homaloidal net into another.

The family (ϕ) acquires these properties by virtue of containing the F-system H; we are therefore interested in three numbers belonging to each element of H:

(i) its *postulation* P: the number of independent conditions that it presents to (ϕ);

(ii) its *equivalence* E: the number of free intersections of two general members that it absorbs;

(iii) the *reduction of genus* R of (ϕ) due to it.

For a simple F-point, $P=1$, $E=1$, $R=0$; for an i-fold F-point, $P=\frac{1}{2}i(i+1)$, $E=i^2$, $R=\frac{1}{2}i(i-1)$. Before discussing these, we write down the equations giving the total values of P, E and R for the whole of H.

The unrestricted family of plane n-ics has $\frac{1}{2}(n+1)(n+2)$ linearly independent terms; two members meet in n^2 free points, and the genus is $\frac{1}{2}(n-1)(n-2)$. Let this be reduced to (ϕ), with 3 linearly independent members, 1 free intersection and genus 0, by σ_i F-points of multiplicity i, where $1 \leqslant i \leqslant n-1$. Then

$$\begin{aligned}\Sigma P &= \Sigma \tfrac{1}{2}i(i+1)\,\sigma_i = \tfrac{1}{2}(n+1)(n+2)-3,\\ \Sigma E &= \Sigma\, i^2\sigma_i \qquad\qquad = n^2-1,\\ \Sigma R &= \Sigma \tfrac{1}{2}i(i-1)\,\sigma_i = \tfrac{1}{2}(n-1)(n-2),\end{aligned}$$

the sums extending to all values of i that occur in H.

These are equivalent to the two *equations of condition* satisfied by the constants i, σ_i of the F-system

$$\Sigma i\sigma_i = 3n-3, \quad \Sigma i^2\sigma_i = n^2-1.$$

In forming these equations we have assumed that the numbers for the whole system are the sums of those for the separate points. We need only assume it for two of P, E and R; it follows for the third, since the equations are not independent. For some linear families it is not true; for example, the postulation of the nine base points of a cubic pencil is only 8.

But if the homaloidal condition of total equivalence holds, it implies both the others. In the first place, if the linear family (ϕ) is such that two general members have one and only one free intersection, (ϕ) is at least a net; for all the intersections of a pencil are base points. There are therefore at least three linearly independent members ϕ_1, ϕ_2, ϕ_3, and the net which they determine is contained in (ϕ).

From these, by equations (1), we define a Cremona transformation T_n, by which the net $(\phi_1, \phi_2, \phi_3) \sim$ the net of lines in S'. Now any fourth member ϕ_4 of (ϕ) has only one free intersection with the net (ϕ_1, ϕ_2, ϕ_3); by T_n therefore, it $\sim$ a curve in S' having only one free intersection with the net of lines of S', and therefore itself a line of the net. Hence ϕ_4 is a member of the net (ϕ_1, ϕ_2, ϕ_3) which contains every curve of (ϕ); and since it is represented point by point on a line of S', it is rational. Hence (ϕ) is a net of rational curves, and satisfies the conditions of postulation and genus.

The same argument can be put into analytical form. Let P_1, P_2 be any two general points and let $[\phi_1]_1$ denote the value of ϕ_1 at P_1. Then the four determinants of the array

$$\left\| \begin{matrix} \phi_1, & \phi_2, & \phi_3, & \phi_4 \\ [\phi_1]_1, & [\phi_2]_1, & [\phi_3]_1, & [\phi_4]_1 \\ [\phi_1]_2, & [\phi_2]_2, & [\phi_3]_2, & [\phi_4]_2 \end{matrix} \right\|,$$

which are connected by two linear identities, represent four curves of (ϕ) passing through both P_1 and P_2. By the condition of equivalence, no two of these are linearly independent; hence there is a third linear identity connecting them. Since ϕ_1, ϕ_2, ϕ_3 are linearly independent, this identity effectively involves ϕ_4, and expresses it as a linear function of ϕ_1, ϕ_2, ϕ_3.

Thus all the conditions presented to (ϕ) by H are independent. This property holds for certain linear families other than homaloidal (32, 61, 145, 230).

7. Characteristics (239·2, 239·4).

A set of numbers n, i, σ_i, satisfying the equations of condition, is called a *characteristic*; we shall write it in either of the forms

$$n;\ \sigma_1^{i_1} \ldots \sigma_a^{i_a} \ldots \sigma_\beta^{i_\beta} \quad \text{or} \quad n;\ i_1, i_2 \ldots i_\sigma.$$

In the first form, the F-points are divided into their *groups*, that is, sets of equal multiplicities, and $i_1 > \ldots > i_a > \ldots > i_\beta$, being the different multiplicities that occur in H, in descending order. In the second, $i_1 \geqq i_2 \geqq \ldots \geqq i_\sigma$, being the multiplicities of all the different F-points, and a consecutive set of σ_a of them are equal to i_a. The total number of F-points is $\sigma \equiv \sum\limits_a \sigma_a$.

Every T_n and every (ϕ) has a characteristic, but the converse is not true. Unless certain inequalities also hold, a solution of the equations of condition defines a degenerate family of curves, and does not lead to a proper geometrical transformation. For example, if $i_1 + i_2 > n$, a line joining two F-points is part of every curve of the family. A characteristic is *geometrical* or *arithmetical* according as the net which it determines is proper or degenerate. In the first case, but not in the second, it belongs to a Cremona transformation.

If T_n is given, the characteristic is completely determined, but if a geometrical characteristic is given, the transformation, though determined in its main features, can be varied by the F-points having special positions, either adjacent to each other, or lying together on curves of lower degree than for the most general transformation having the same characteristic. It can also be metrically specialized. For most purposes, but with important exceptions, such varieties can be regarded as limiting cases.

The general transformation with a given characteristic depends on $2\sigma+8$ parameters. Since the F-points are in general position, the first homaloidal net is determined by their 2σ coordinates. Then T_n is completely determined by selecting the base homaloids, involving the ratios of nine arbitrary coefficients; this is equivalent to an arbitrary choice of the corners of the triangle of reference and the unit point in S'. The total number of parameters is $2\sigma+8$. The coordinates of an F-point in S', or of a pair of corresponding points, can replace two of these parameters; for example, T_4; $3^2 3^1$ is completely determined by any ten F-points in the two planes (232, 344).

By the same argument with the planes reversed, T_n^{-1} depends on $2\sigma'+8$ parameters. But T_n determines its reverse; hence

$$\sigma = \sigma',$$

that is, the total number of F-points is the same for the two planes.

Since $n=n'$, the constants i', $\sigma'_{i'}$ of the second F-system satisfy the same equations as i, σ_i, and give the *conjugate* characteristic. For many transformations the two sets are alike, and the characteristic is *self-conjugate.* In any case, they are related, the numbers $\sigma'_{i'}$ being the same as the σ_i, in the same or different order (p. 23).

We shall also show that, for n general, there are always at least two self-conjugate characteristics (pp. 28, 61).

8. Postulation and equivalence.

The simplest F-element is an ordinary simple point O common to all the homaloids. This has no effect upon the genus; it presents one condition to the parameters, found by substituting the coordinates of O in $\phi=0$; and it fixes at O one of the n^2 intersections of two general homaloids ϕ, ψ.

If O is a d.p. it reduces the genus by 1. Take O as $(0, 0, 1)$ and let

$$\phi \equiv \sum_{\alpha=0}^{n} z^{n-\alpha} u_\alpha,$$

where u_α is a homogeneous function of x, y only, of degree α. The conditions for a d.p. at O are $u_0=0$, $u_1 \equiv 0$; the first expresses that O lies on ϕ and the vanishing of the two coefficients of u_1 expresses that there is no definite tangent to ϕ at O. Hence $P=3$.

In the same way, if O is i-fold, the reduction of genus is $\frac{1}{2}i(i-1)$, and the conditions are that $u_0, u_1 \dots u_{i-1} \equiv 0$, in number

$$P = 1 + 2 + \dots + i = \tfrac{1}{2}i(i+1).$$

To find the equivalence when O is i-fold, let

$$\phi \equiv z^{n-i}u_i + z^{n-i-1}u_{i+1} + \dots + u_n,$$

and let another general homaloid of the family be

$$\psi \equiv z^{n-i}v_i + z^{n-i-1}v_{i+1} + \dots + v_n,$$

where the v are also homogeneous functions of x, y only.

The equation of the set of lines joining O to the common points of ϕ, ψ, other than O, is found by eliminating z between $\phi = 0$, $\psi = 0$. The resultant is a determinant of $2(n-i)$ rows and columns

$$\begin{vmatrix} u_i, & u_{i+1}, & \dots\dots\dots & u_n, & 0, & \dots & 0 \\ v_i, & v_{i+1}, & \dots\dots\dots & v_n, & 0, & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & u_i, & u_{i+1}, & \dots\dots\dots & u_n \\ 0, & 0, & \dots & v_i, & v_{i+1}, & \dots\dots\dots & v_n \end{vmatrix},$$

giving an equation for x/y of degree

$$i + 2(i + 1 + \dots + n - 1) + n = n^2 - i^2.$$

This is the number of free intersections of ϕ, ψ, and the number absorbed at O is

$$E = i^2.$$

The result is intuitive, since each of the i branches of ϕ meets each of the i branches of ψ; but the method can be extended to complicated cases where intuition may fail.

If a transformation has only simple F-points, both P and E are equal to the number of F-points, and the equations of condition are

$$\sigma = 3n - 3 = n^2 - 1,$$

whence $$n = 1 \text{ or } 2, \quad \sigma = 0 \text{ or } 3.$$

Hence, besides the linear transformation T_1, with no F-points at all, the only Cremona transformation with no multiple F-points is quadratic, with three simple F-points. This T_2 is treated fully in Chapter III.

The result also follows from the condition of genus, since lines and conics are the only rational curves with no multiple points.

If H consists of σ_1 simple and σ_2 double F-points only, the condition of genus gives $\sigma_2 = \frac{1}{2}(n-1)(n-2)$, and from any of the other conditions, $\sigma_1 = (n-1)(5-n)$. Since $\sigma_2 > 0$, $\sigma_1 \geqq 0$, we have $3 \leqq n \leqq 5$, and such characteristics are possible with $n = 3$, 4 and 5, and are given in Table I. See also 337 and p. 71.

9. The de Jonquières and symmetric transformations.

If H contains an F-point of maximum multiplicity $n-1$, the curves ϕ, of degree n with an $(n-1)$-fold point, are *monoids*. They are rational, and can have no other multiple base points, and the rest of H consists of σ_1 simple points, where both the equations of condition require $\sigma_1 = 2n-2$. Conversely, if all the F-points except one are simple, that one is $(n-1)$-fold, since (ϕ) is rational.

The characteristic $n;\ 1^{n-1}(2n-2)^1$ is geometrical, for if the F-points are in general positions the net of monoids is proper and defines a transformation called the *de Jonquières transformation*, which we denote by T_J and discuss in Chapter VI. The multiple F-point O is called the *vertex* of (ϕ) and of T_J.

The chief property of T_J is that it transforms the pencil of lines through O into another pencil of lines. For each line l through O meets each ϕ and each line of S in one free point only; hence $(l) \sim$ a pencil of curves meeting each line of S' and each ϕ' in one free point only, that is, a pencil of lines with $n-1$ intersections with (ϕ') absorbed at the vertex O' of the pencil, which is therefore an $(n-1)$-fold F-point of H', and (ϕ') are monoids, T_J^{-1} is also a T_J and the characteristic is self-conjugate. Conversely, if two pencils of lines correspond, the transformation is T_J, for the vertices are $(n-1)$-fold F-points (177).

T_J has the maximum range of multiplicity of F-points, from $n-1$ to 1. A transformation with the minimum range, that is, with all its F-points of equal multiplicity, is called *symmetric* and denoted by T_{sym}. There are only a finite number of them (p. 70).

There is no transformation with just two multiple F-points. If $i_3 = \ldots = i_\sigma = 1$, since $i_1 + i_2 \leqq n$ (p. 6), the condition of genus gives

$$i_1(i_1-1) + i_2(i_2-1) = (n-1)(n-2) \geqq (i_1+i_2-1)(i_1+i_2-2),$$

whence $(i_1-1)(i_2-1) \leqq 0$, and $i_2 = 1$. There is only one multiple F-point and the transformation is T_J (pp. 72, 76).

10. Noether's inequality (61, 285, 286, 335).

From the equations of condition there follows the important result:

The sum of the multiplicities of the three highest F-points exceeds the degree n, provided $n > 1$.

There are at least three F-points; for if there were not more than two, of multiplicities $i_1, i_2 \geqq 0$, since the line joining them meets ϕ in not more than n points,

$$i_1 + i_2 \leqq n;$$

and by the first equation of condition,

$$3n-3 = i_1 + i_2 \leqq n,$$

whence $n = 1$.

Thus if $n \geqq 2$, the number of F-points is $\sigma \geqq 3$.

Now with the second notation of p. 6, the equations of condition are

$$\sum_{\alpha=1}^{\sigma} i_\alpha = 3n - 3, \qquad \sum_{\alpha=1}^{\sigma} i_\alpha^2 = n^2 - 1.$$

Multiply the first by i_3 and subtract from the second:

$$i_1(i_1 - i_3) + i_2(i_2 - i_3) - \sum_{\alpha=3}^{\sigma} i_\alpha(i_3 - i_\alpha) = n^2 - 1 - 3i_3(n-1).$$

This can be written

$$(n-1)\{i_1 + i_2 + i_3 - (n+1)\}$$
$$= (i_1 - i_3)(n - 1 - i_1) + (i_2 - i_3)(n - 1 - i_2) + \sum_{\alpha=3}^{\sigma} i_\alpha(i_3 - i_\alpha).$$

Now $\quad n > 1, \quad i_1 \geqq i_2 \geqq i_3, \quad n - 1 \geqq i_1, \quad \geqq i_2,$

and every term in $\Sigma \geqq 0$; hence

$$i_1 + i_2 + i_3 \geqq n + 1,$$

which is Noether's inequality.

Another way of arranging the equation is

$$(n - 1 + i_1 + i_2 - 2i_3)\{i_1 + i_2 + i_3 - (n+1)\}$$
$$= 2\{(i_1 - 1)(i_2 - 1) - (i_3 - 1)^2\} + \sum_{\alpha=3}^{\sigma} i_\alpha(i_3 - i_\alpha),$$

from which the same conclusion follows; for alternative proofs, see **26**, **67**, **239·1**, **286**, **335**.

In Noether's inequality we can have the sign = only if each term on the R.H.S. of either form of the equation vanishes separately. This requires

$$i_3 = i_\alpha \quad (\alpha = 3 \ldots \sigma),$$

and either $\quad i_1 = i_2 = i_3 = i_\alpha$, or $i_1 = n - 1, \quad i_2 = i_3 = i_\alpha = 1,$

that is, T_n is either T_{sym} or T_J. For these two extreme types,

$$i_1 + i_2 + i_3 = n + 1,$$

and for every other type

$$i_1 + i_2 + i_3 \geqq n + 2.$$

The inequality holds for many linear families other than homaloidal. Let $(f_n) \equiv \Sigma O_\alpha^{i_\alpha}$ be of genus p, and let q be its apparent freedom, that is, calculated on the assumption that the base points all present independent conditions to (f):

$$\Sigma \tfrac{1}{2} i_\alpha(i_\alpha - 1) = \tfrac{1}{2}(n-1)(n-2) - p,$$
$$\Sigma \tfrac{1}{2} i_\alpha(i_\alpha + 1) = \tfrac{1}{2} n(n+3) - q,$$

whence $\quad \Sigma i_\alpha = 3n - 1 + p - q,$

$$\Sigma i_\alpha^2 = n^2 + 1 - p - q.$$

We find

$$(n + i_1 + i_2 - 2i_3)(i_1 + i_2 + i_3 - n)$$
$$= 2\{i_1(i_2 - i_3) + (i_1 - i_3)(i \ - 1)\} + \sum_{\alpha=3}^{\sigma} i_\alpha(i_3 - i_\alpha)$$
$$+ i_3(-1 - p + q) + 2i_1 - p - q + 1;$$

it follows that
$$i_1 + i_2 + i_3 \geqq n + 1$$
for all families such that
$$i_3(-1-p+q) + 2i_1 - p - q \geqq 0 \quad (26).$$

A similar method gives an upper limit for σ the total number of F-points. Multiply the first equation of condition by n and subtract the second:
$$\sum_{\alpha=1}^{\sigma} i_\alpha (n - i_\alpha) = (n-1)(2n-1).$$
Now Σ has σ terms, and each $\geqq 1\,(n-1)$:
$$\sigma(n-1) \leqq (n-1)(2n-1).$$
Hence if $n > 1$, $\sigma \leqq 2n-1$.

This limit is actually reached for T_J.

11. Contact as a limit.

A multiple point is an F-element of an essentially different nature from any set of simple points. If $i=1$, whatever the number of simple points, $R=0$ and $P=E$; but if $i>1$, then $R>0$ and $E-P=\frac{1}{2}i(i-1)>0$.

On the other hand, in the plane but not in space, as far as regards P, E and R, an F-point of contact can be regarded as the limit of a set of distinct F-points. Let O (0, 0, 1) be a simple point of simple contact; let
$$\phi \equiv z^n u_0 + z^{n-1}u_1 + \ldots + u_n,$$
$$\psi \equiv z^n v_0 + z^{n-1}v_1 + \ldots + v_n.$$

The condition of passing through O is $u_0 = v_0 = 0$; then the tangents to ϕ, ψ at O are u_1, v_1, and the condition of contact is that these are the same line. The two coefficients of v_1, instead of being independent parameters of the family, are in a constant ratio. Thus, contact involves one condition beyond the condition of incidence, and altogether for a point of contact, $P=2$. The proper form for ψ is
$$\psi \equiv az^{n-1}u_1 + z^{n-2}v_2 + \ldots + v_n.$$

To study E, in this and more complicated cases, we deduce an auxiliary curve Λ, meeting ϕ in as many points at O as ψ does, but not in contact with ϕ. We eliminate the common tangent u_1, and form
$$\Lambda \equiv \psi - a\phi \equiv z^{n-2}(v_2 - au_2) + \ldots + v_n - au_n,$$
which has an ordinary d.p. at O, with a general pair of tangents $v_2 - au_2$, neither branch touching u_1. Hence O is simple on ϕ and double on Λ, and absorbs two free intersections of ϕ, Λ, and therefore of ϕ, ψ; and $E=2$.

Since O is not singular on either ϕ or ψ, we have $R=0$.

Thus a simple point of simple contact is equivalent to two distinct simple points, as regards P, E and R. The same method applies both to higher contact at a simple point and to contact of any kind at a multiple point; or we can prove the same result by resolving the singularity by auxiliary transformations (Chapter VII).

All general members of the pencil $(\phi) \equiv c_1\phi + c_2\psi$ touch ϕ at O; but on the special member Λ of this family, O is not a simple point with u_1 as tangent, but a d.p. with no special relation to u_1. It would be inconvenient always to except Λ from general statements about (ϕ), and we make the convention that it may still be said to touch u_1; and in general, that if a special member Λ of a family (ϕ) satisfies all the same conditions at a given point O, as the general ϕ, and one or more in addition, so as to have a higher singularity, then Λ will be said to have the same multiplicity and order of contact as (ϕ). This convention will shorten our work; where there is any risk of confusion, more explicit statements will be made.

Now ϕ meets u_1 in two points at O and a general line through O in one point only; Λ also meets u_1 in two points at O, and in that sense touches it; but in the same sense it touches every line through O.

12. Multiple point as a limit.

Though contact is always the limit of intersection in distinct points, it is not true, conversely, that such a limit always leads to contact: it leads instead to a common multiple point, if more than two points coalesce in such a way that in the limit no linear branch of an algebraic curve passes through them all.

Let two F-points O_1, $O_2 \rightarrow O\ (0, 0, 1)$, both moving on a fixed line y through O. Let their coordinates be $(\xi_1\epsilon, 0, 1)$, $(\xi_2\epsilon, 0, 1)$; in the limit, $\epsilon = 0$, and ξ_1, ξ_2 remain finite.

Since ϕ contains O, O_1, O_2, it has the form

$$\phi \equiv y\,(x, y, z)^{n-1} + x\,(x - \xi_1\epsilon z)\,(x - \xi_2\epsilon z)\,(x, y, z)^{n-3}.$$

In the limit $$\phi \equiv y\,(x, y, z)^{n-1} + x^3\,(x, y, z)^{n-3},$$

and has a simple point at O, where it osculates the fixed line y.

Next, let $O_1\ (\xi\epsilon, 0, 1)$ and $O_2\ (0, \eta\epsilon, 1) \rightarrow O$ along different fixed lines y, x; in the limit, $\epsilon = 0$ and ξ, η are finite. As long as the three points are distinct, the radius of the circle OO_1O_2 is of the same order of magnitude as $\epsilon\sqrt{\xi^2 + \eta^2}$. As O_1, $O_2 \rightarrow O$, this circle would $\rightarrow$ the circle of curvature of ϕ at O if the branch were linear, and its radius would $\rightarrow 0$. Hence any curve through O, O_1, O_2 is singular at O in the limit.

As long as $\epsilon > 0$,

$$\phi \equiv (x\,(x - \xi\epsilon z),\quad y\,(y - \eta\epsilon z),\quad xy \between x, y, z)^{n-2},$$

and in the limit, $$\phi \equiv (x^2, y^2, xy \between x, y, z)^{n-2},$$

which is the general curve with a d.p. at O. The directions of approach of O_1, O_2 to O have no effect upon the final form of ϕ.

The conditions of passing through O, O_1 require in the limit that ϕ shall touch y, the path of approach of O_1 to O; it must also touch x, the path of approach of O_2. These are incompatible conditions for the tangent line to ϕ, which does not exist: ϕ acquires a d.p. at O, and in that sense touches both lines, and also every other line through O, and its final form is independent of the directions of approach of O_1, O_2 provided these are different from one another.

We might have been misled to expect that in the limit ϕ would have a d.p. with two fixed tangents x, y; but then ϕ would satisfy the conditions of osculating y and also x, meeting each in three points at O, two of one branch and one of the other. This case is the limit of intersection in one double and two simple distinct points, or five simple points, namely O, O_1, O_3 of y and O, O_2, O_4 of x. If first $O_1, O_2 \to O$, then ϕ acquires a d.p. with general tangents; if then $O_3, O_4 \to O$, the tangents become fixed, and (ϕ) has a d.p. of contact, for which $P=5$, $E=6$, $R=1$.

For the set of three distinct points O, O_1, O_2, we have $P=E=3$; for the d.p. O with two variable tangents, $P=3$ still, but $E=4$. Any two curves ϕ, ψ through O, O_1, O_2 have a fourth free intersection, variable with the parameters of ψ, which $\to O$ in the limit, and becomes fixed instead of free, without imposing any fresh conditions. Thus E is raised, without P changing.

Assume that ϕ, ψ have a common point $P(x, y, z)$, in the neighbourhood of O, that is, such that x, y are small when ϵ is small, and z is finite. Then if we neglect third order quantities, we can replace ϕ by an approximating conic through O, O_1, O_2, say,

$$\phi_1 \equiv c_1 x(x-\xi\epsilon z)+c_2 y(y-\eta\epsilon z)+c_3 xy,$$

and ψ by another conic of the same form

$$\psi_1 \equiv d_1 x(x-\xi\epsilon z)+d_2 y(y-\eta\epsilon z)+d_3 xy,$$

where $c_1 \ldots d_3$ are finite.

The intersections of ϕ_1, ψ_1 are the same as those of the two independent linear functions of them, formed by eliminating the second and first terms respectively, say,

$$ax(x-\xi\epsilon z)+xy, \quad by(y-\eta\epsilon z)+xy,$$

where a, b are finite functions of the parameters of ϕ, ψ. These two line-pairs meet in O, O_1, O_2 and a fourth point $P_1(x_1, y_1, z_1)$, where

$$x_1 : y_1 : z_1 = \epsilon b(\eta - a\xi) : \epsilon a(\xi - b\eta) : 1-ab.$$

We verify that x_1, y_1 are small of the same order as ϵ, and z_1 is finite, so that $P_1 \to O$ as $\epsilon \to 0$; and P_1 is an approximation to an intersection P of ϕ, ψ, which also $\to O$.

As long as $\epsilon > 0$, the point P is variable, its coordinates depending effectively on the parameters of ϕ, ψ which determine a, b. But in the limit, $\epsilon=0$, $x_1=y_1=0$, and P_1 and P are fixed at O.

A d.p. can also arise in the limit when $O_1, O_2 \to O$ in the same direction, if they move along a cuspidal branch. Let them be $(\epsilon^2, \pm\epsilon^3, 1)$; then they move on the two branches of the semi-cubical parabola $y^2=x^3$, and both $\to O$

in the direction of the cuspidal tangent y. The general curve through O, O_1, O_2 is

$$\phi \equiv (x(x-\epsilon^2 z), \quad y(x-\epsilon^2 z), \quad (y^2-\epsilon^2 x^2) \between x, y, z)^{n-2},$$

and in the limit, $$\phi \equiv (x^2, xy, y^2 \between x, y, z)^{n-2},$$

and again O is a d.p. with general tangent lines.

13. The principal system.

A Cremona transformation breaks down at the F-points, where x, y, z are such that ϕ_1, ϕ_2, ϕ_3 all vanish at once. Then equations (1) cease to be significant, and (2) cease to be independent, but reduce to one equation instead of two, representing a curve instead of a point. If O is an F-point, lying on every ϕ, its homologue is an element in S' incident with every line, and therefore a curve j', not a point or set of discrete points. This is called a *principal curve* or *P-curve* in S', and the set of P-curves corresponding to the whole of H is the *second P-system* J' of T_n. Similarly there is the first P-system J in S, corresponding to H'.

Cremona and most other writers define *fundamental* and *principal* (*Grund-*, *Haupt-*) as both having both the meanings which we have assigned to them respectively; elsewhere we find *cardinal* (223, 343), *singular* (130), and for space, *skeleton* (13, 374). This is sheer waste of language; our convention follows 76; see also p. 342 of 385.

If O is an i-fold F-point, as a point P describes ϕ once, it passes i times through O; the homologue P' describes a line l' once, and is incident i times with j'; hence j' is of degree i. The total number of P-curves in J' is the same as that of F-points in H, and they fall into groups of equal degrees in the same way as the points O fall into groups of equal multiplicities. The total degree of J' is $\Sigma i = 3n-3$, and that of J is the same.

The breakdown of the 1, 1 transformation at the F-points is related to the convention, that of the n^2 intersections of two homaloids, the base points are to be set aside. Algebraically, the reverse equations (2) are not in all respects equivalent to (1), but only after the rejection of the possibility $G'=0$ (p. 2). This factor G' represents the P-curves, with certain multiplicities; and at points where $G'=0$, the reversal is no longer unique.

If any curve f_μ contains an i-fold F-point O, the result of substitution contains the whole of j_i', and breaks up into j_i', corresponding to the single point O, and a residual f' of degree $\mu n - i$, corresponding to all the general points of f, which is the proper homologue. If O is simple on f, then f meets (ϕ) in $\mu n - i$ variable points instead of μn; this shows that j_i is only dropped once; if O is γ-fold on f, then j'_i is dropped and the degree of f' is lowered by γi.

Thus the degree of the homologue of a general curve f of degree μ, with γ_a branches through O_a, is

$$\mu' = \mu n - \Sigma \gamma_a i_a.$$

For example, if the homologue of a non-singular cubic f_3 is also cubic,

$$\mu = \mu' = 3, \quad \gamma = 0 \text{ or } 1,$$

and

$$\Sigma \gamma_a i_a = 3n - 3$$

$$= \Sigma i_a \text{ from the first equation of condition;}$$

hence every $\gamma = 1$, and f_3 passes simply through all the F-points (46).

If f is a homaloid, $\mu = n$, $\mu' = 1$, $\gamma = i$; hence the total dropped factor is

$$G' = \Pi j'^i,$$

the product extending to all the P-curves. We verify the degree

$$1 = n^2 - \Sigma i^2,$$

giving another geometrical interpretation of the second equation of condition.

The P-system J' can be found, without investigating the correspondence of its different curves with the points of H, by taking the different factors, each once, of

$$G' \equiv \sum_{a=1}^{3} c_a \phi_a (\phi_1', \phi_2', \phi_3')/(c_1 x' + c_2 y' + c_3 z'),$$

which is an integral function of x', y', z', independent of c_1, c_2, c_3; similarly J can be found from

$$G \equiv \sum_{a=1}^{3} c_a \phi_a' (\phi_1, \phi_2, \phi_3)/(c_1 x + c_2 y + c_3 z),$$

but this requires a knowledge of the functions ϕ' as well as ϕ, so that equations (1) must first be solved. Also every j or j' which $\sim$ a multiple F-point appears repeated.

14. The Jacobian.

We can also find the P-curves, each once, without solving the equations: together they form the *Jacobian* J of (ϕ), which is the locus of additional d.p.'s of members of the net.

If $Q(x, y, z)$ is a d.p. on $\phi \equiv c_1\phi_1 + c_2\phi_2 + c_3\phi_3$, the tangent to ϕ at Q is indeterminate; the conditions are

$$\frac{\partial \phi}{\partial x} \equiv c_1 \frac{\partial \phi_1}{\partial x} + c_2 \frac{\partial \phi_2}{\partial x} + c_3 \frac{\partial \phi_3}{\partial x} = 0, \quad \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial z} = 0.$$

Eliminating c_1, c_2, c_3, we obtain the Jacobian

$$J \equiv \begin{vmatrix} \frac{\partial \phi_1}{\partial x}, & \frac{\partial \phi_1}{\partial y}, & \frac{\partial \phi_1}{\partial z} \\ \frac{\partial \phi_2}{\partial x}, & \frac{\partial \phi_2}{\partial y}, & \frac{\partial \phi_2}{\partial z} \\ \frac{\partial \phi_3}{\partial x}, & \frac{\partial \phi_3}{\partial y}, & \frac{\partial \phi_3}{\partial z} \end{vmatrix}.$$

We have to show that every P-point lies on J, and that every point of J is principal.

If ϕ contains a point Q of a P-curve j, then l' the homologue of ϕ contains the F-point O' that $\sim j$; hence ϕ contains the whole of j. All the ordinary points of j present the same single condition to (ϕ), which is the same as the condition that O' presents to l'. The pencil of ϕ's through Q all contain the adjacent point of j; we can make one member contain a point adjacent to Q in a different direction. This ϕ has a d.p. at Q, and therefore Q is a point of J.

Conversely, if Q is any point of J, since the general ϕ is rational, the particular ϕ which has an additional d.p. at Q is of negative genus and breaks up, and Q is a common point of the components. By continuity, only one of the components $\sim$ the general points of the line l' which $\sim \phi$, the other component $j \sim$ some special point O' on l', which, having a curve as homologue, is an F-point, and j is a P-curve on which Q lies. Hence every point of J is principal.

There is a 1, 1 relation between the pencil (l') of lines through O' and the ∞^1 points Q of j, d.p.'s of the ϕ's of the pencil that $\sim (l')$; hence every P-curve is rational. If a point $P' \rightarrow O'$ along any path f', the homologue $P \rightarrow$ a definite point Q, common to f, j, and associated with l', the tangent to f' at O'. We can say that $Q \sim$ the direction of l' at O', or that $Q \sim$ the point adjacent to O' on l'. The effect of T_n on O' is to spread out the ∞^1 points of its first neighbourhood into the curve j (p. 131).

If any curve f has γ branches through O with distinct tangents, the homologue f' meets j' in γ distinct points Q', other than F-points, associated with the tangents to f at O; these are simple points on f'. If f has a higher singularity at O, some of the tangents coinciding, then some of the points Q' coincide, and f' either touches j' at a Q', or is singular there; but its multiplicity cannot exceed that of f at O, and the nature of the singularity of Q' is always simpler than that of O.

Let O' (0, 0, 1) be an i-fold F-point. Then ϕ_1', ϕ_2', ϕ_3' vanish here, together with all their derivates up to order $i-1$ inclusive, but at least one derivate of order i does not vanish.

Let P' $(\xi\epsilon, \eta\epsilon, 1) \rightarrow O'$ as $\epsilon \rightarrow 0$ along a fixed line l', which depends on ξ/η. The homologue P has coordinates proportional to

$$\phi_\alpha'(\xi\epsilon, \eta\epsilon, 1) \equiv \phi_\alpha'(0, 0, 1) + \epsilon\left(\xi\frac{\partial}{\partial x'} + \eta\frac{\partial}{\partial y'}\right)\phi_\alpha'(0, 0, 1) + \dots$$

$$+ \epsilon^n\left(\xi\frac{\partial}{\partial x'} + \eta\frac{\partial}{\partial y'}\right)^n \phi_\alpha'(0, 0, 1) \quad (\alpha = 1, 2, 3).$$

The first i terms of each series vanish, and as long as $\epsilon > 0$ we can cancel ϵ^i, but no higher power of ϵ; as $P' \rightarrow O'$, so $P \rightarrow$ the definite point Q whose coordinates are

$$\left(\xi\frac{\partial}{\partial x'} + \eta\frac{\partial}{\partial y'}\right)^i \phi_\alpha'(0, 0, 1),$$

which do not all vanish.

This gives the coordinates of Q explicitly as rational functions of degree i of the parameter ξ/η, and for variation of this, that is, for a rotation of l' about O', the locus of Q is a rational curve j of degree i.

The total degree of Σj is $\Sigma i\sigma_i$; the degree of the determinant J is $3(n-1)$. These are equal, giving a geometrical meaning to the first equation of condition.

Thus J gives the P-curves more simply than G. If $J \equiv G$, then from their degrees, $3(n-1) = n^2 - 1$, only satisfied by $n=1$ or 2, when there are no multiple F-points, and no repeated factors in G.

Every i-fold F-point O is $(3i-1)$-fold on J. If O is $(0, 0, 1)$, the lowest terms in x, y in J are apparently of degree $3i-2$, but these vanish identically, and the lowest effective terms are of degree $3i-1$. These multiplicities exactly determine J; and since it breaks up into σ rational curves, its genus is $1-\sigma$. These two properties are expressed by the identities

$$\Sigma \tfrac{1}{2}(3i-1) \,.\, 3i = \tfrac{1}{2}(3n-3) \,.\, 3n,$$

$$\Sigma \tfrac{1}{2}(3i-1)(3i-2) = \tfrac{1}{2}(3n-4)(3n-5) + \sigma - 1.$$

An auxiliary Cremona transformation V applied to (ϕ) and J transforms them into another homaloidal family (ϕ_1) and its Jacobian, save for a possible set of P-curves of V in either plane. For since any j has no free intersections with (ϕ), its homologue by V has no free intersections with (ϕ_1), and is either a P-curve of (ϕ_1) or an F-point of V. This also follows from the algebraic property of transformation of Jacobians:

$$\frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(x_1, y_1, z_1)} \equiv \frac{\partial(\phi_1, \phi_2, \phi_3)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(x_1, y_1, z_1)}.$$

Other properties of J, including its behaviour at an F-point of contact, are given in 115.

CHAPTER II

CLEBSCH'S THEOREM

1. Incidences of P-curves and F-points (70, 80, 107, 334).

Let there be σ F-points O_α in S, of multiplicities i_α $(\alpha = 1, 2, \dots \sigma)$; and σ' F-points O_β' in S', of multiplicities i_β' $(\beta = 1, 2, \dots \sigma')$. Now $O_\alpha \sim$ a P-curve j_α' in S', of degree i_α; let it have $i'_{\beta\alpha}$ branches through O_β'. Similarly, $O_\beta' \sim$ a P-curve j_β of degree i_β' with $i_{\alpha\beta}$ branches through O_α.

The number of incidences of the two elements O_α, j_β is the same as that of their homologues j_α', O_β':

$$i_{\alpha\beta} = i'_{\beta\alpha}.$$

Certain geometrical facts are expressed by the following relations between the incidence numbers i_α, i_β', $i_{\alpha\beta}$:

$$\begin{array}{llll}
\sum\limits_{\alpha=1}^{\sigma} i_\alpha = 3(n-1) & \dots\dots(1), & \sum\limits_{\beta=1}^{\sigma'} i_\beta' = 3(n-1) & \dots\dots(1'), \\
\sum\limits_{\alpha} i_\alpha^2 = n^2 - 1 & \dots\dots(2), & \sum\limits_{\beta} i_\beta'^2 = n^2 - 1 & \dots\dots(2'), \\
\sum\limits_{\alpha} i_{\alpha\beta} = 3i_\beta' - 1 & \dots\dots(3), & \sum\limits_{\beta} i_{\alpha\beta} = 3i_\alpha - 1 & \dots\dots(3'), \\
\sum\limits_{\alpha} i_\alpha i_{\alpha\beta} = i_\beta' n & \dots\dots(4), & \sum\limits_{\beta} i_\beta' i_{\alpha\beta} = i_\alpha n & \dots\dots(4'), \\
\sum\limits_{\alpha} i_{\alpha\beta}^2 = i_\beta'^2 + 1 & \dots\dots(5), & \sum\limits_{\beta} i_{\alpha\beta}^2 = i_\alpha^2 + 1 & \dots\dots(5'), \\
\sum\limits_{\alpha} i_{\alpha\beta} i_{\alpha\gamma} = i_\beta' i_\gamma' \quad (\beta \neq \gamma) & \dots(6), & \sum\limits_{\beta} i_{\alpha\beta} i_{\gamma\beta} = i_\alpha i_\gamma \quad (\alpha \neq \gamma) & \dots(6').
\end{array}$$

We assume that the F-points are in general positions, and we deal with characteristics rather than with transformations that may be specialized (p. 7). The F-points must present independent conditions to a curve of degree n (p. 5), but they may be specially related to a curve of lower degree; for example, three may lie on a line if $n \geqq 3$. We can construct examples by compounding two transformations, whose F points in the common plane, each set by itself in general position, are specially related to each other, as when two of one system are collinear with one of the other. In such cases, the relations may cease to hold (11, 38).

In the case of adjacent F-points also, the theorems must be rightly interpreted. For example, the total number of *distinct* F-points need not be the same in both planes (183 and p. 112).

2. Degenerate P-curves.

If the F-points in S have such special positions as to oblige the P-curve j, which $\sim O'$, to break up into γ distinct components $j_1, j_2 \dots j_\gamma$, then j is of degree > 1, and O' is a multiple F-point. The incidences of any line l with j

are rationally separable into γ distinct sets, according to the component of j on which they lie; hence the branches of ϕ' through O' fall into γ distinct sets, each set satisfying different conditions. Since they all satisfy the condition of incidence at O', the different conditions are those of contact, either of different orders, or with different fixed curves. At most one set satisfies no contact condition; hence O' is a multiple F-point adjacent to which are at least $\gamma - 1$ F-points of lower multiplicity.

Conversely, if the branches of ϕ' through O' fall into γ sets satisfying different conditions, the intersections of l with j are rationally separable into γ sets, and j breaks up into γ components.

For example, there is a cubic transformation (p. 112) with a double F-point O_1 and four simple F-points O_2, O_3, O_4, O_5 in each plane. In S', let O_2' be adjacent to O_1'; then in S we find that $O_3O_4O_5$ are collinear. Now $O_1' \sim$ the P-conic $O_1O_2O_3O_4O_5$; this breaks up into two lines, $O_3O_4O_5$ and O_1O_2. Of the two branches of ϕ_3' through O_1', one touches the fixed line $O_1'O_2'$, and satisfies a contact condition, and the other is in a general direction, and only satisfies a condition of incidence; these two incidences $\sim$ those of the corresponding line with O_1O_2 and with $O_3O_4O_5$ respectively.

Now let $O_3' \rightarrow O_1'$ along the fixed line $O_3'O_1'$; we find that $O_3O_4O_5 \rightarrow O_3O_1$, rotating about O_3, which remains fixed, and O_4, $O_5 \rightarrow O_1$ along fixed lines: H, like H', has two simple points adjacent to the d.p. Now the conic through $O_1O_2O_3O_4O_5$ breaks up into O_1O_2 and O_1O_3, and in the determining elements of j, the three simple points O_1, O_4, O_5 are replaced by O_1^2. The degenerate conic may be said to contain O_4, and to touch O_1O_4, because it is an exceptional member of a linear family of proper conics touching O_1O_4; but neither component of j by itself can be regarded as containing O_4, and similarly for O_5 (p. 12).

In the first case, the component O_1O_2 is exactly determined, and $O_3O_4O_5$ is over-determined, by the F-points that lie on it. The multiplicities of the F-points leave j rational, and the d.p. that reduces its genus to -1, and makes it break up, is not an F-point, but is the intersection of O_1O_2, $O_3O_4O_5$. This answers to the fact that one position of the tangent to the variable branch of ϕ' at O' coincides with the fixed tangent to the other branch, and there is a pencil of (ϕ') having a cusp at O'. So the two components of j have a common point which is not an F-point, and the general equations continue to hold, though they no longer express the genus of j.

In the second case, each component of j is exactly determined by the F-points on it, but the multiplicities of O_1, O_4, O_5 on j and J are not as assumed by the general equations, which cease to hold, unless $i_{\alpha\beta}$ is interpreted as the multiplicity of O_α in the determination of j_β, which may not be the same as its multiplicity on j_β when determined.

3. Interpretation of the equations.

These special cases being excluded, we assume that every j is proper. Equations (1), (2), (1′), (2′) are the equations of condition (p. 5); and (1) also expresses that J is of total degree $3n-3$. Now J has $3i_\alpha - 1$ branches through O_α (p. 17); hence (3′), and similarly (3) is found by considering J'.

A general line in S' does not pass through O'; therefore ϕ contains no general point of j, but meets it in F-points only: hence (4). Conversely, any curve meeting (ϕ) in F-points only is a P-curve; for it $\sim$ an element which is not incident with a general line, and is therefore a point. Thus j is exactly determined by its incidences at F-points:

$$\sum_\alpha \tfrac{1}{2} i_{\alpha\beta}(i_{\alpha\beta}+1) = \tfrac{1}{2}(i_\beta'+1)(i_\beta'+2) - 1.$$

Also j is rational by virtue of these multiple points:

$$\sum_\alpha \tfrac{1}{2} i_{\alpha\beta}(i_{\alpha\beta}-1) = \tfrac{1}{2}(i_\beta'-1)(i_\beta'-2).$$

These two relations are equivalent to (3), (5).

A common point of two P-curves j_β, j_γ is an F-point, for it $\sim$ an element incident with both O_β' and O_γ', and therefore a curve. Thus j_β, j_γ meet in F-points only. This is expressed by (6).

The remaining equations are proved by considering S'.

If j_β breaks up into γ components, it is still exactly determined, but its genus is $1-\gamma$; hence (3), (5) are replaced by

$$\sum_\alpha i_{\alpha\beta} \geqq 3i_\beta' - \gamma, \quad \sum_\alpha i_{\alpha\beta}^2 \leqq i_\beta'^2 + \gamma.$$

The relations can also be proved, without using J, by considering the transformation compounded of T_n and another, the two having one F-point common in the intermediate plane (182).

In S', let there be a homaloidal net (ϕ_μ') with O_β' as a simple F-point, the other F-points $\Sigma O_\gamma'^{\,i_\gamma'}$ being in general position. The equations of condition for (ϕ_μ') are

$$\Sigma i_\gamma' = 3\mu - 4, \quad \Sigma i_\gamma'^2 = \mu^2 - 2.$$

The homologue of this in S by T^{-1} is another homaloidal net (p. 5); the result of substitution is of degree μn and contains H^μ; from this there falls away the P-curve j_β, and there remains a family

$$(\phi_{\mu n - i_\beta'}) \equiv \sum_\alpha O_\alpha^{\mu i_\alpha - i_{\alpha\beta}} \sum_\gamma O_\gamma^{i_\gamma'},$$

for which the equations of condition are

$$\sum_\alpha (\mu i_\alpha - i_{\alpha\beta}) + 3\mu - 4 = 3(\mu n - i_\beta') - 3,$$

$$\sum_\alpha (\mu i_\alpha - i_{\alpha\beta})^2 + \mu^2 - 2 = (\mu n - i_\beta')^2 - 1.$$

These hold for all values of μ. Equate coefficients of powers of μ, and we have the relations (1)...(5).

For $n > 2$, we can show that, of the incidence numbers i_{11}, i_{22}, i_{33}, at least one > 0, where 1, 2, 3 are the suffixes of the three highest F-points in each plane, properly ordered if they are of equal multiplicities (**26**).

Thus for T_J with $n > 2$, we have $i_{11} = n - 2 > 0$; for T_2, $i_{11} = 0$ or > 0 according as the associate of O_1, or another simple F-point, is called O_1'.

The degree of a proper curve $\geqq$ the sum of the multiplicities of any two of its points, unless it is the line joining the points. Now if any of j_1, j_2, j_3 is a line, one of O_1', O_2', O_3' is a simple F-point; H' has at most two multiple F-points, and T is either T_J or T_2 (p. 9). If none of j_1, j_2, j_3 is a line,

$$i_{21} + i_{31} \leqq i_1',$$

and adding three such equations, we have

$$i_{23} + i_{32} + i_{31} + i_{13} + i_{12} + i_{21} \leqq i_1' + i_2' + i_3'.$$

The homologue of the line $O_2'O_3'$ is

$$k_{n-i_2'-i_3'} \equiv O_2^{i_2-i_{22}-i_{23}}\, O_3^{i_3-i_{32}-i_{33}},$$

whence $$i_2 + i_3 - i_{22} - i_{33} - i_{23} - i_{32} \leqq n - i_2' - i_3' + \epsilon_1,$$

where $\epsilon_1 = 0$ unless k is the line O_2O_3, when $\epsilon_1 = 1$. If $O_2'O_3'$ is a P-line, k does not exist, but the inequality holds with $\epsilon_1 = 0$. By addition,

$$2(i_1 + i_2 + i_3) - 2(i_{11} + i_{22} + i_{33}) - (i_1' + i_2' + i_3')$$
$$\leqq 3n - 2(i_1' + i_2' + i_3') + \epsilon_1 + \epsilon_2 + \epsilon_3.$$

But $$i_1 + i_2 + i_3 \geqq n + 1, \quad i_1' + i_2' + i_3' \geqq n + 1;$$

hence $$2(i_{11} + i_{22} + i_{33}) \geqq 3 - (\epsilon_1 + \epsilon_2 + \epsilon_3),$$

and $i_{11} + i_{22} + i_{33} > 0$ unless $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ and $i_1 + i_2 + i_3 = n + 1$.

In that case, the homologue of $O_2'O_3'$ is O_2O_3,

$$n - i_2' - i_3' = n - i_2 - i_3 = 1,$$

and two similar relations; whence

$$i_1 = i_2 = i_3 = i_1' = i_2' = i_3' = \tfrac{1}{2}(n - 1), \quad n = 5, \quad i_1 = 2 \text{ (p. 120)}.$$

Since O_1', O_2', O_3' are of equal multiplicities, their order is indifferent; we can permute the suffixes cyclically so that the homologue of O_2O_3 is called $O_3'O_1'$, and so on. Then $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$.

Hence in every case we can have

$$i_{11} + i_{22} + i_{33} > 0,$$

and at least one of these three numbers is positive.

4. Total number of F-points.

From these relations we can prove that the total number of F-points is the same in each plane, and is therefore the same as the number of P-curves. Sum (3) and (3'):

$$\sum_{\beta=1}^{\sigma'} \sum_{\alpha=1}^{\sigma} i_{\alpha\beta} = \sum_{\beta=1}^{\sigma'} (3i_\beta' - 1) = 9(n-1) - \sigma' \text{ by (1')},$$

$$\sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\sigma'} i_{\alpha\beta} = \sum_{\alpha=1}^{\sigma} (3i_\alpha - 1) = 9(n-1) - \sigma \text{ by (1)}.$$

The order of summation on the L.H.S. is indifferent; hence

$$\sigma' = \sigma.$$

This result can also be obtained by counting the number of double points on the homaloids of a given pencil (107).

5. Degenerate homaloids (32).

Since a general ϕ meets j_β in F-points only, but a pencil of ϕ's pass through any general point Q of j_β, these must break up into j_β and a pencil

$$(\phi_{n-i_\beta'}) \equiv \Sigma O_\alpha^{i_\alpha - i_{\alpha\beta}},$$

which $\sim$ the pencil of lines (l') through O_β', meeting (ϕ_n') in $n - i_\beta'$ variable points. From (1)...(6), we deduce the equations that express that $\phi_{n-i_\beta'}$ belongs to a pencil:

$$\sum_\alpha \tfrac{1}{2}(i_\alpha - i_{\alpha\beta})(i_\alpha - i_{\alpha\beta} + 1) = \tfrac{1}{2}(n - i_\beta' + 1)(n - i_\beta' + 2) - 2,$$

is rational:

$$\sum_\alpha \tfrac{1}{2}(i_\alpha - i_{\alpha\beta})(i_\alpha - i_{\alpha\beta} - 1) = \tfrac{1}{2}(n - i_\beta' - 1)(n - i_\beta' - 2),$$

has one free intersection with j_β:

$$\sum_\alpha i_{\alpha\beta}(i_\alpha - i_{\alpha\beta}) \qquad = i_\beta'(n - i_\beta') - 1,$$

has no free intersection with j_γ if $\gamma \neq \beta$:

$$\sum_\alpha i_{\alpha\gamma}(i_\alpha - i_{\alpha\beta}) \qquad = i_\gamma'(n - i_\beta').$$

Each l' through $O_\beta' \sim$ one $\phi_{n-i_\beta'}$ meeting j_β in one point Q; conversely, Q lies on one $\phi_{n-i_\beta'}$, and is associated in a 1, 1 relation with l' (p. 16). Also Q, being a point on the Jacobian, is a d.p. on the particular homaloid

$$\phi_n \equiv j_\beta \phi_{n-i_\beta'}.$$

6. Properties of two F-points (27, 32).

Let O_1, O_2 be any two F-points. Consider a curve f of degree i_β', subject to the same conditions of incidence as j_β, except that it has one branch more through O_1 and one branch fewer through O_2. Since j_β meets ϕ in $i_\beta' n$ points at the F-points, and is exactly determined, f meets ϕ in $i_\beta' n + i_1 - i_2$ points and has

$$\tfrac{1}{2} i_{1\beta}(i_{1\beta} + 1) + \tfrac{1}{2} i_{2\beta}(i_{2\beta} + 1) - \tfrac{1}{2}(i_{1\beta} + 1)(i_{1\beta} + 2) - \tfrac{1}{2}(i_{2\beta} - 1) i_{2\beta}$$
$$= i_{2\beta} - i_{1\beta} - 1 \text{ degrees of freedom.}$$

Now if $i_1 > i_2$, then f meets ϕ in more than $i_\beta' n$ points, and does not exist; hence

$$i_{1\beta} > i_{2\beta} - 1, \quad i_{1\beta} \geqq i_{2\beta}.$$

Thus if O_1 is of higher multiplicity than O_2 on ϕ, it is of higher or equal multiplicity on each P-curve. But if $i_1 = i_2$, then f meets ϕ in exactly $i_\beta' n$ points, and may exist; if it does, it is a P-curve and exactly determined:

$$i_{2\beta} = i_{1\beta} + 1, \quad i_{1\beta} = i_{2\beta} - 1.$$

Then j_β has multiplicities at O_1, O_2 differing by 1, and f has these interchanged.

By the same theorem for S', if j_1 is of higher degree than j_2, it is of higher or equal multiplicity at each F-point. This can also be proved as follows.

Let a curve $f_{i_2'}$ satisfy the same conditions as j_2 except that it has one branch fewer through one point O_α. Then f has $i_{\alpha 2}$ degrees of freedom, and meets j_1 in $i_{\alpha 1}$ points other than F-points. If $i_{\alpha 2} > i_{\alpha 1}$, we can make f contain $i_{\alpha 1} + 1$ other points of j_1, and therefore have j_1 as a component; hence $i_2' \geqq i_1'$. If $i_2' = i_1'$, then f coincides with j_1, and is exactly determined, $i_{\alpha 2} = i_{\alpha 1} + 1$; but if $i_2' < i_1'$, we must have $i_{\alpha 2} \leqq i_{\alpha 1}$.

These results can be made more precise. If $i_1' = i_2'$, from (5), (6),

$$\sum_\alpha (i_{\alpha 1} - i_{\alpha 2})^2 = 2.$$

The only way of expressing 2 as the sum of squares is $2 = 1 + 1$. Hence $i_{\alpha 1} = i_{\alpha 2}$ for all points O_α except two, which we take as O_1, O_2, and for these,

$$i_{11} = i_{12} + \epsilon_1, \quad i_{21} = i_{22} + \epsilon_2, \text{ where } \epsilon_1, \epsilon_2 = \pm 1.$$

Write down (3), (4), (5) for $\beta = 1$, 2 and subtract:

$$\begin{aligned}
&\epsilon_1 + \epsilon_2 && = 0, \text{ whence one of these, say } \epsilon_1 = +1, \text{ and } \epsilon_2 = -1,\\
&\epsilon_1 i_1 + \epsilon_2 i_2 && = 0, \text{ whence } i_1 = i_2,\\
&2\epsilon_1 i_{11} + 2\epsilon_2 i_{21} - 2 && = 0, \text{ whence } i_{21} = i_{12} = i_{11} - 1, \quad i_{22} = i_{11} = i_{21} + 1.
\end{aligned}$$

Thus if there are two P-curves j_1, j_2 of equal degree, they behave alike to each F-point except two, which are of equal multiplicities on (ϕ) and at which j_1 has multiplicities differing by 1, and j_2 has these multiplicities interchanged. Conversely, by an argument in S', if there are two F-points O_1, O_2 in S of equal multiplicities, they behave alike to each P-curve except two, which are of equal degree, and for which O_1 has multiplicities differing by 1, and O_2 has these multiplicities interchanged.

7. Clebsch's theorem (70, 80, 107).

The σ F-points of H fall into groups of equal multiplicity, and $\sigma = \sum_\alpha \sigma_\alpha$, where σ_α is the number of F-points of multiplicity i_α, and the sum extends to all the different multiplicities i_α that occur in H. Similarly for H' we have $\sigma' = \sum_\beta \sigma_\beta'$. The P-curves in each plane are grouped in the same way as the F-points in the other plane.

We shall now prove Clebsch's theorem, that the set of numbers σ_β' is the same as the set σ_α; either in a different order, the same number associated with different multiplicities i_α, i_β', or else, in the case of a self-conjugate characteristic, in the same order and associated with the same multiplicity.

Consider a group of $\sigma_\beta' \geqq 2$ P-curves $j_1, j_2 \ldots$ of equal degree i_β'. There are just two F-points O_1, O_2, of equal multiplicity i_α and belonging to the group σ_α, to which j_1, j_2 behave differently. If $\sigma_\alpha \geqq 3$, the common multiplicity

i_{31} of j_1, j_2 at any other point O_3 of the group is either equal to, or differs by ± 1 from, each of i_{11}, i_{21}, which themselves differ by ± 1; hence i_{31} has one or other of these values.

Now suppose there were a point O_3 for which $i_{31} = i_{11}$ and also a point O_4 for which $i_{41} = i_{21}$. Then a set of multiplicities the same as for j_1 except at these four points, which are replaced by $(O_1O_3)^{i_{21}}(O_2O_4)^{i_{11}}$, would exactly determine a curve of degree i_β' meeting (ϕ_n) in F-points only, and therefore a P-curve of the group σ_β'; but this would behave differently from j_1 at four F-points, which is impossible. Hence the multiplicities of j_1 at the points $O_3, O_4 \ldots$ of the group σ_α are either all i_{11} or all i_{21}, and we can choose the order of O_1, O_2 so that O_1 is the exceptional point, and $i_{21} = i_{31} = i_{41} = \ldots$, and similarly $i_{12} = i_{32} = i_{42} = \ldots$. Thus j_1 and j_2 first select from H the group σ_α, and then j_1 selects O_1 from the group, and j_2 selects O_2.

Any other j_3 of degree i_β' behaves differently at O_1 from either j_1 or j_2, and with one of them forms a pair with which the argument can be started; they select the same group σ_α, determined by O_1, and from it j_3 selects a different exceptional point O_3. The only incidence numbers that occur are i_{12} at all general points of the group and i_{11} at the exceptional point.

Now each of the σ_α points, taken to be exceptional, gives a set of multiplicities which defines a different j belonging to the group σ_β'; hence $\sigma_\alpha \leqq \sigma_\beta'$. But if we start with the two P-curves j_1', j_2' which $\sim O_1, O_2$, and carry through the argument with the planes interchanged, we find $\sigma_\beta' \leqq \sigma_\alpha$; hence

$$\sigma_\beta' = \sigma_\alpha.$$

Thus the two groups $\sigma_\alpha, \sigma_\beta'$ of F-points select each other, and if $\sigma_\beta' \geqq 3$, the curve j_1' selects O_1' just as j_1 selects O_1; the relation between the points of the two groups is symmetrical.

This sets up a 1, 1 relation between the two sets of groups, of two or more points each, in H, H', associated groups having the same number of points $\sigma_\alpha = \sigma_\beta'$; and if this number $\geqq 3$, there is also a 1, 1 relation between the points O_α, O_β' of two associated groups, each point being exceptional for the P-curve which $\sim$ the associated point, and such that $i_{\alpha\beta} = i_{\alpha\gamma} \pm 1$, where O_γ' is a general point of the group. The ambiguous sign is the same for the whole pair of groups, which are called *of positive* or *negative determination* according as it is + or − (271, which reverses the earlier convention of 268).

The association of points fails for groups of two points, for we can regard them as of either determination. We arbitrarily choose it to be positive.

There remain the groups of one point each. Since the total numbers of points in H, H' are the same, and also the numbers in all the groups of two or more points, the numbers of one-point groups are the same, and since the multiplicities in either plane are all different, we arbitrarily associate these points in order of decreasing multiplicities. This completes the 1, 1 relation between the two sets of points H, H' (74).

These results, all except $\epsilon = \pm 1$, can also be obtained as follows, without using the equations (27).

Let N_α be the number of different permutations of the σ_α multiplicities $i_{\alpha\beta}$ of j_β at the points O_α of one group. If these are all equal, $N_\alpha = 1$, and j_β behaves alike to all the group. If not, $N_\alpha \geqq \sigma_\alpha$, and j_β selects the group. Any set of permutations, one of each group, gives a set of multiplicities that exactly determines another j_γ of the same group σ_β' as j_β. Thus

$$\Pi \, \sigma_\alpha \leqq \Pi \, N_\alpha \leqq \sigma_\beta',$$

where Π extends to all the groups selected by j_β.

Each of these groups σ_α has two points O_1, O_2 at which j_β behaves differently; one of the permutations consists in interchanging these multiplicities; and there are two P-curves j_β, j_γ to which O_1 behaves differently. Hence σ_β' is one of the groups selected by j_1', and by the same argument in S',

$$\Pi' \sigma_\beta' \leqq \Pi' N_\beta \leqq \sigma_\alpha \leqq \Pi \, \sigma_\alpha \leqq \sigma_\beta',$$

where Π' extends to all the groups selected by j_1'.

Hence $\sigma_\alpha \leqq \sigma_\beta'$, and similarly $\sigma_\beta' \leqq \sigma_\alpha$, and

$$\sigma_\alpha = \sigma_\beta'.$$

Hence Π', and similarly Π, consists of a single factor, and for this, $N_\alpha = \sigma_\alpha$, and the permutations are of σ_α numbers all alike except one.

Thus if j_β selects any group, it selects only one, and if $\sigma_\alpha \geqq 3$, then j_β selects one point from the group. The rest follows as before.

In the determinant

$$\Delta = | \, i_{\alpha\beta} \, | = \begin{vmatrix} i_{11}, & i_{12}, & \ldots & i_{1\sigma'} \\ i_{21}, & i_{22}, & \ldots & i_{2\sigma'} \\ \cdots & \cdots & \cdots & \cdots \\ i_{\sigma 1}, & i_{\sigma 2}, & \ldots & i_{\sigma\sigma'} \end{vmatrix},$$

which is square since $\sigma = \sigma'$, each row belongs to a point O_α and each column to a point O_β'. Let them be so ordered that associated points have the same rank and that points of the same group are adjacent. Let Δ be divided into rectangles by lines separating the groups. Then Clebsch's theorem is that the vertical and horizontal divisions are the same, so that the rectangles about the leading diagonals are all squares. Also the numbers in each rectangle are all equal, except those on the leading diagonal, each of which differs by ± 1 from the rest of its square, the sign being constant for the square.

Now by (5′), (6′),

$$\Delta^2 = \begin{vmatrix} i_1^2 + 1, & i_1 i_2, & \ldots & i_1 i_\sigma \\ i_2 i_1, & i_2^2 + 1, & \ldots & i_2 i_\sigma \\ \cdots & \cdots & \cdots & \cdots \\ i_\sigma i_1, & i_\sigma i_2, & \ldots & i_\sigma^2 + 1 \end{vmatrix}$$

$$= i_1^2 + i_2^2 + \ldots + i_\sigma^2 + 1 = n^2 \text{ by (2)},$$

$$\Delta = \pm n.$$

The sign is indeterminate, because the order of the rows and columns in Δ is not completely assigned (70).

8. Properties of a general curve (114, 192, 268, 271).

If $f_\mu \equiv \Sigma O_\alpha^{\gamma_\alpha}$ is a general curve incident in any way with the F-points, then $\Pi j_\alpha'^{\gamma_\alpha}$ falls away (p. 14) from the homologue $f_{\mu'}' \equiv \Sigma O'_\beta{}^{\gamma_\beta'}$, where

$$\mu' = \mu n - \sum_\alpha \gamma_\alpha i_\alpha,$$

$$\gamma_\beta' = \mu i_\beta' - \sum_\alpha \gamma_\alpha i_{\alpha\beta}.$$

For example, if f_μ is a homaloid, $\mu = n$, $\gamma_\alpha = i_\alpha$, and from the equations of p. 18, we verify $\mu' = 1$, $\gamma_\beta' = 0$.

If f_μ is a P-curve, $\mu = i_\beta'$, $\gamma_\alpha = i_{\alpha\beta}$; for the homologue O_β' the equations give μ' and every $\gamma' = 0$, except $\gamma_\beta' = -1$; this suggests the convention that a point is a curve of degree 0, incident -1 times with itself (p. 320).

If f is a curve of degree i_β' with the same incidences as j_β, except that it passes $i_{\alpha\beta} - 1$ times through O_α, we find that f' is of the same nature as f, that is, it is of the same degree as j_α' and satisfies the same conditions except that it passes $i_{\alpha\beta} - 1$ times through O_β' (26).

By comparison with j_β, which is exactly determined and rational, we find that the freedom of the family (f) or (f') is

$$\tfrac{1}{2} i_{\alpha\beta}(i_{\alpha\beta} + 1) - \tfrac{1}{2}(i_{\alpha\beta} - 1) i_{\alpha\beta} \equiv i_{\alpha\beta},$$

its genus is $\quad \tfrac{1}{2} i_{\alpha\beta}(i_{\alpha\beta} - 1) - \tfrac{1}{2}(i_{\alpha\beta} - 1)(i_{\alpha\beta} - 2) \equiv i_{\alpha\beta} - 1,$

and the number of its free intersections is

$$i_{\alpha\beta}^2 - (i_{\alpha\beta} - 1)^2 - 1 \equiv 2 i_{\alpha\beta} - 2.$$

For any curve, we verify

$$3\mu - \Sigma\gamma_\alpha = 3\mu' - \Sigma\gamma_\beta',$$

$$\mu^2 - \Sigma\gamma_\alpha^2 = \mu'^2 - \Sigma\gamma_\beta'^2;$$

hence, these expressions are invariant for Cremona transformation.

A combination of these expresses the invariance of the genus p. Let R be the reduction of genus of f_μ due to all its multiple points other than H; these $\sim$ the multiple points of $f_{\mu'}'$ other than H', reducing the genus of $f_{\mu'}'$ by R also; then

$$\begin{aligned} p &= \tfrac{1}{2}(\mu - 1)(\mu - 2) - \Sigma \tfrac{1}{2}\gamma_\alpha(\gamma_\alpha - 1) - R \\ &= \tfrac{1}{2}(\mu' - 1)(\mu' - 2) - \Sigma \tfrac{1}{2}\gamma_\beta'(\gamma_\beta' - 1) - R. \end{aligned}$$

The incidences of f and f' at associated F-points are related. If O_1, O_2 are two points of the same group, and O_1', O_2' the associated F-points, we know

$$i_1' = i_2', \quad i_{11} - i_{21} = -(i_{12} - i_{22}) = \pm 1, \quad i_{\alpha 1} = i_{\alpha 2} \text{ if } \alpha > 2.$$

Hence $\quad \gamma_1' - \gamma_2' = \mp(\gamma_1 - \gamma_2),$

that is $\quad \gamma_1 \pm \gamma_1' = \gamma_2 \pm \gamma_2',$

with upper or lower signs according as the group is of positive or negative determination (268).

The degree of f_μ can be lowered by T_n if the multiplicities γ_a of the highest multiple points of f are high enough to make

$$\mu n - \Sigma \gamma_a i_a < \mu.$$

In particular, if f has no points of multiplicity > 2,

$$\mu' \geqq n\mu - 2\Sigma i_a, \text{ which } = n\mu - 6(n-1),$$
$$\mu' - \mu \geqq (\mu - 6)(n-1).$$

Thus no curve of degree $\geqq 6$, with d.p.'s only, can $\sim$ a curve of lower degree by a Cremona transformation (p. 41 of 385).

9. Direct determination of P-curves.

There is no simple rule connecting the multiplicity of any F-point with the degree of the associated P-curve, that is, connecting the multiplicities i_a, i_β' associated with the same group number $\sigma_a = \sigma_\beta'$ in H, H'; until the i_β' are known, H' is not completely determined.

We can always determine H', either by investigating the reverse transformation, or by finding J as a determinant; but we can often find all the P-curves more easily from some simple geometrical and arithmetical facts.

(i) Any j meets ϕ in F-points only.

(ii) The multiple points that determine j, in descending order of multiplicity for j, are the highest F-points of H, in descending order of multiplicity for ϕ (p. 22). Thus if there is a P-line, it joins O_1, O_2, or two other F-points of multiplicities i_1, i_2, and $i_1 + i_2 = n$. If there is a P-conic, it passes through five F-points of multiplicities $i_1, \dots i_5$, and the sum of these is $2n$.

(iii) The multiplicities of j at the points of any one group are either all equal, or all but one equal, and j can only partition one group. Hence the last of the F-points which determine j is the first, or the last but one, or the last, of some group.

Hence O_1 lies on every P-curve unless it belongs to a group of negative determination, and the associated P-curve passes simply through the rest of the top group, and not through O_1 nor through any other F-point. This is a rational curve determined by simple points only, and is therefore the line O_2O_3 or the conic $O_2O_3O_4O_5O_6$; the top group consists of three $\frac{1}{2}n$-fold or six $\frac{2}{5}n$-fold points. In the other plane, the P-curve of highest degree passes through all the F-points except possibly one of a group of three simple or six double points (27).

(iv) The total degree of J is $3n - 3$, and the multiplicity of O_a on it is $3i_a - 1$. This determines the last group of P-curves when we have found all the groups except one.

Considering P-lines, we have the following cases:

(1) There is no P-line if $i_2 < n - i_1$; then H' has no simple point.

(2) There is just one P-line j_1, and one simple point in H', if j_1 does not partition any group. Then O_1, O_2 may constitute the top group, of two points, with $i_1 = \frac{1}{2}n$;

(3) or O_1, O_2 may constitute the two top groups, of one point each, with

$$i_2 = n - i_1 < \tfrac{1}{2}n < i_1.$$

(4) The top group is partitioned by j_1, two of its points lie on j_1 and one does not; then $i_1 = \frac{1}{2}n$, the top group consists of three points and is of negative determination. There are three such P-lines, and three simple points in H'.

(5) The second group is partitioned by j_1 into one point on j_1 and $\sigma_2 - 1$ not on it; the top group consists of one point only, of multiplicity $i_1 > \frac{1}{2}n$; the second, of multiplicity $i_2 = n - i_1$, consists of any number σ_2 of points and is of positive determination. H' contains σ_2 simple points.

Incidentally, this shows that if $i_1 = \frac{1}{2}n$, then $\sigma_1 \leqq 3$ (p. 74); for if there are three or more $\frac{1}{2}n$-fold points, a P-line j_1 partitions them, two points lying on it, and there can be only one point not on j_1.

For T_J, n; $1^{n-1}(2n-2)^1$, there is a group of $2n-2$ P-lines, associated with the group of simple F-points. Then J can have only one other component, $j_{n-1} \equiv 1^{n-2}(2n-2)^1$; this is associated with the vertex, and the characteristic is self-conjugate. Conversely, if a group of P-lines is associated with a group of simple F-points, the transformation is T_J; for the P-line joins two F-points, one simple and therefore the other $(n-1)$-fold (30).

If j is a rational cubic, determined by $1^2 6^1$, it may partition the top group, which is then of positive determination, into one point double on j_3 and $\sigma_1 - 1 \leqq 6$ points simple on j_3; or it may partition a later group into a set of points simple on j_3 and a set not on it; or it may not partition any group. For example, with the characteristic $n = 4$; $1^3 6^1$, each j might be determined by 2, 6 or 7 points, and is either a line or a cubic; there is a group of six P-curves and a single one, and the total degree of J is 9. Hence J consists of one cubic and six lines, H' is $1^3 6^1$, and the characteristic is self-conjugate. For $n = 4$; $3^2 3^1$, we find three P-lines and three P-conics, and again the characteristic is self-conjugate.

Since these are the only two quartic characteristics (Table I), and they consist of different numbers σ_a, we see without any other proof that they cannot be conjugate to one another, and each must be self-conjugate. The same is true of the three quintic types. But two of the sextics

$$n = 6;\ 1^4 4^2 3^1 \text{ and } n = 6;\ 3^3 1^2 4^1$$

have the same set of σ_a. The first has four P-lines, associated with the second group, and H' contains 4^1, and is different from H, which contains 3^1. Hence this characteristic is not self-conjugate, and the pair must be conjugate to each other, as can be verified in detail. By the same argument, without going beyond P-lines, we find that there is one conjugate pair of T_7's, two of T_8's and three of T_9's. Among the T_{10}'s there are two (nos. 5, 10),

$$n = 10;\ 1^6 3^4 3^2 3^1 \text{ and } n = 10;\ 3^5 1^3 3^2 3^1,$$

which have the same numbers of simple and double points; and we must go as far as P-cubics, of which the first has one and the second none, to show that they are conjugate.

Characteristics are not necessarily conjugate if they have the same n and σ_a. For example (270), there are four T_{12}'s (nos. 6, 9, 16, 17):

$$n = 12;\ 1^8 4^4 3^2 3^1,\quad 1^7 3^5 4^2 3^1,\quad 3^6 1^4 4^2 3^1,\quad 3^6 3^3 1^2 4^1.$$

The first of these has four P-lines, and is therefore conjugate to the last, the only one with 4^1. The second and third both have $0^3 4^2 3^1$, and no P-cubics, four conics and three lines: we have to go to P-quartics, determined by either $1^3 8^1$ or $3^2 5^1$. Neither kind can be a P-curve for the second T_{12}, for it cannot have as many as 48 intersections with ϕ_{12} at F-points. This characteristic is therefore conjugate to the one with no four-fold point, that is, to itself. The third T_{12} must also be self-conjugate; it has one P-quartic of the type $3^2 5^1$, associated with its one four-fold point.

Many examples occur with $n = 16$. The following are the index numbers in Table I of the sets of three or more characteristics with the same (σ_a), conjugates being bracketed:

(15, 42), (32, 77), (41, 67), (62, 83).
(40, 86), (44, 70), (64, 76), (80, 84).
(31, 79), (35, 69), (57, 68).
(6, 49), 27, 46.
(18, 61), (29, 36).
(22, 71), (60, 63).
(23, 52), (48, 74).
(13, 34), 21.
43, (66, 85).

CHAPTER III

THE QUADRATIC PLANE TRANSFORMATION

I. PLANES DISTINCT

1. Description of the transformation (8, 15, 17, 43, 61, 80, 298, 372).

Consider a family of conics (ϕ) defined by the condition of passing through three fixed points O_1, O_2, O_3. Each point presents one linear condition to (ϕ), hence the total postulation is 3, and since there are five parameters in the equation of a conic, the family is ∞^2, that is, a net. Any two members of the family intersect in four points, of which three are fixed at O_1, O_2, O_3, and one is free. These two properties show that (ϕ) is a homaloidal net whose F-system H consists of the three simple points O_1, O_2, O_3.

Thus there exists a quadratic transformation T_2 which converts the net of conics (ϕ) in the plane S into the net of lines in the plane S'. Since a ϕ meets a line of S in two points, a line of S' meets a curve of the second homaloidal net (ϕ') in two points, and (ϕ') is a net of conics any two of which have one free intersection and three fixed, O_1', O_2', O_3' say: the reverse transformation T_2^{-1} is also quadratic.

Every general point of one plane $\sim$ one and only one of the other. The only points for which this does not hold are the F-points and their corresponding P-curves, which are lines since the F-points are simple. Since the P-lines in S meet (ϕ) in F-points only, they join pairs of O_1, O_2, O_3. Thus the F- and P-systems in S are the vertices and sides of the triangle $O_1O_2O_3$; and similarly in S'.

Let O_2O_3 be the P-line which $\sim O_1'$; this partitions the F-points of H into the two, O_2, O_3, that lie on it, and the one, O_1, that does not. Thus O_1, O_1' are associated F-points of H, H', and the single group of F-points in each plane is of negative determination.

This transformation occupies a unique position in the theory, since all other Cremona plane transformations can be resolved into series of T_2's. It is also of much use in deducing, from the properties of known curves, those of curves of higher degree, and in investigating higher singularities of plane curves.

2. Equations of $\mathbf{T}_2$.

The F-points O_1, O_2, O_3 cannot be collinear, since they lie on proper conics (ϕ). They may be adjacent; if so, the equations of this paragraph must be modified (pp. 35, 37). For the present, assume them distinct; they can then be taken as the vertices A_1, A_2, A_3 of the triangle of reference, and (ϕ) is circumscribed to this:

$$(\phi) \equiv c_1 . ayz + c_2 . bzx + c_3 . cxy,$$

where a, b, c are fixed constants, and c_1, c_2, c_3 arbitrary parameters. This is formed as the sum of multiples of the three simplest linearly independent conics of the family, the line-pairs ayz, bzx, cxy.

The equations of the general T_2 with $O_1O_2O_3$ as F-trio are found by putting x', y', z' proportional to any three independent ϕ's chosen as base conics. If we take, as sides of the triangle of reference in S', the lines corresponding to the above line-pairs of (ϕ), we have

$$x' : y' : z' = ayz : bzx : cxy.$$

Then (a, b, c) are the coordinates of the homologue in S' of the unit point $(1, 1, 1)$ of S. If we take this point of S' as unit point, we have $a = b = c$, and the equations of the general T_2 take their simplest, standard form

$$x' : y' : z' = yz : zx : xy. \quad \text{.........................} (1)$$

These are equivalent to the symmetric equations

$$xx' = yy' = zz',$$

and therefore the equations of the reverse transformation T_2^{-1} are of the same form,

$$x : y : z = y'z' : z'x' : x'y', \quad \text{.........................} (2)$$

showing that (ϕ') is of the same nature as (ϕ), and H' as H.

We have not assumed the F-points real. In the case of inversion (p. 53) two are conjugate imaginaries; then the present equations refer to complex frames of reference, and the equations referred to real frames can be obtained by a complex linear substitution.

If the frames of reference are given first, and the F-points $O_1(\xi_1, \eta_1, \zeta_1)$, O_2, O_3 are in any positions with regard to the first, we can change to a new set of coordinates x_1, y_1, z_1 in S, referred to $O_1O_2O_3$ as triangle of reference, by a linear substitution:

$$x_1 : y_1 : z_1 = j_1(x, y, z) : j_2 : j_3,$$

where j_1 is the P-line O_2O_3 referred to the general frame,

$$j_1 \equiv \begin{vmatrix} x, & y, & z \\ \xi_2, & \eta_2, & \zeta_2 \\ \xi_3, & \eta_3, & \zeta_3 \end{vmatrix}.$$

Similarly in S', we change from x', y', z', referred to the given frame, to x_1', y_1', z_1' referred to $O_1'O_2'O_3'$, by another linear substitution:

$$x_1' : y_1' : z_1' = j_1'(x', y', z') : j_2' : j_3'.$$

Then T_2 is expressed by equations of standard form in the new coordinates:

$$x_1' : y_1' : z_1' = y_1z_1 : z_1x_1 : x_1y_1,$$

or, in terms of the given coordinates, by

$$j_1' : j_2' : j_3' = j_2j_3 : j_3j_1 : j_1j_2.$$

Thus with any frames of reference, T_2 can always be expressed by equations of standard form, preceded and followed by a linear substitution (151, 157).

We can solve the last equations for x', y', z' and find

$$x' : y' : z' = \phi_1 : \phi_2 : \phi_3,$$

where the ϕ are general linear functions of $j_2 j_3,\ j_3 j_1,\ j_1 j_2$, and represent any conics through O_1, O_2, O_3, say

$$\phi_1 \equiv a_1 x^2 + b_1 y^2 + c_1 z^2 + 2f_1 yz + 2g_1 zx + 2h_1 xy,$$

where $a_1 \dots h_1$ satisfy the three conditions of incidence

$$a_1 \xi_\alpha^2 + b_1 \eta_\alpha^2 + c_1 \zeta_\alpha^2 + 2f_1 \eta_\alpha \zeta_\alpha + 2g_1 \zeta_\alpha \xi_\alpha + 2h_1 \xi_\alpha \eta_\alpha = 0 \quad (\alpha = 1, 2, 3).$$

3. F- and P-systems.

The only points for which the transformation (1) is not 1, 1 are those for which the ratios $\phi_1 : \phi_2 : \phi_3$ are indeterminate; this occurs when all three functions vanish, and P is one of the three base points of the net.

If P is O_1 (1, 0, 0), equations (2) become

$$1 : 0 : 0 = y'z' : z'x' : x'y',$$

satisfied for every point on $x' = 0$, and for no other point. Hence the line $j_1' \equiv O_2'O_3'$ is the P-element which $\sim O_1$, every point on it having as homologue the same point O_1. Similarly, the F-point $O_2 \sim$ the whole of the P-line $j_2' \equiv O_3'O_1'$ and $O_3 \sim$ the whole of $j_3' \equiv O_1'O_2'$.

These three lines are all the second P-system. For if we substitute from (2) in (1), we obtain

$$x' : y' : z' = z'x' \,.\, x'y' : x'y' \,.\, y'z' : y'z' \,.\, z'x',$$

which becomes identity after rejecting the factor $G' \equiv x'y'z'$.

We can also find the P-elements as the components of the Jacobian of (ϕ'):

$$J' \equiv \begin{vmatrix} \dfrac{\partial \phi_1'}{\partial x'}, & \dfrac{\partial \phi_1'}{\partial y'}, & \dfrac{\partial \phi_1'}{\partial z'} \\ \dfrac{\partial \phi_2'}{\partial x'}, & \dfrac{\partial \phi_2'}{\partial y'}, & \dfrac{\partial \phi_2'}{\partial z'} \\ \dfrac{\partial \phi_3'}{\partial x'}, & \dfrac{\partial \phi_3'}{\partial y'}, & \dfrac{\partial \phi_3'}{\partial z'} \end{vmatrix} \equiv \begin{vmatrix} 0, & z', & y' \\ z', & 0, & x' \\ y', & x', & 0 \end{vmatrix} \equiv 2x'y'z'.$$

By symmetry, the lines $j_1 \equiv O_2O_3,\ j_2 \equiv O_3O_1,\ j_3 \equiv O_1O_2$ stand in the same relation to O_1', O_2', O_3' as do j_1', j_2', j_3' to O_1, O_2, O_3. Thus each vertex O_α of one F-triangle $\sim$ the side of the other F-triangle opposite to the vertex associated with O_α.

Now consider points in the neighbourhood of an F-point. As $\epsilon \to 0$, the point $P(1, \eta\epsilon, \zeta\epsilon) \to O_1$ in the direction $y : z = \eta : \zeta$, and its homologue $P'(\eta\zeta\epsilon^2, \zeta\epsilon, \eta\epsilon) \to Q'(0, \zeta, \eta)$, a definite point of j_1'. Hence the different points of j_1' are associated in a 1, 1 correspondence with the directions at O_1, and similarly for the other P-lines of either plane.

4. Projective pencils.

If a line l passes through an F-point, O_1 say, then ϕ' breaks up into the P-line j_1' which $\sim O_1$, and a line l' through O_1', which $\sim$ all the general points of l, and is its proper homologue.

Let $l \equiv y + az$; then $\phi' \equiv z'x' + ax'y'$, which breaks up into x' and the variable line $l' \equiv z' + ay'$. Thus T_2 transforms the general lines of the pencil (O_1) projectively into those of the pencil (O_1'); in this projectivity (but not in T_2), j_3 corresponds to j_2' and j_2 to j_3'.

The points of the corresponding lines l, l' are related by T_2 in a projectivity in which $O_1 \sim$ the intersection of l', j_1'. If $l \equiv j_3$, $l' \equiv j_2'$, this projectivity is degenerate, every point of $l \sim O_3'$, and every point of $l' \sim O_2$.

In this way, we have three pairs of projective pencils,

$$(O_1), (O_1'); \quad (O_2), (O_2'); \quad (O_3), (O_3').$$

A line l, passing near O_1 but not through it, contains a point near O_1 of every ray of the pencil (O_1) except those nearly parallel to l. The homologue ϕ' therefore has a branch containing a point near j_1' of every ray of the pencil (O_1') except those near the homologue l' of the ray through O_1 parallel to l. This branch approximates to j_1', and only turns away from it near its intersection Q' with l'.

If l moves parallel to itself so as to pass through O_1, this branch of ϕ' ultimately coincides with j_1', and the branch of ϕ' turning away from j_1' becomes a separate component meeting the first in Q'; this is l', the proper homologue of the final position of l; and l', j_1' together form a degenerate homaloid, having a d.p. at Q'. Thus every point of j_1' is a d.p. of some homaloid, and so belongs to the Jacobian.

Conversely, if Q' is a d.p. of a ϕ', this is a pair of lines of which one only, l' say, is the proper homologue of l, and the other must be a P-line j'; every point of J' lies on one of the three P-lines.

5. Homologue of a general curve.

A general curve $f_\mu \equiv (x, y, z)^\mu$ in S is transformed by equations (2) into

$$f_{2\mu}' \equiv (y'z', z'x', x'y')^\mu,$$

that is, a curve of degree 2μ. Since μ is the highest power of any one coordinate, f' has μ-fold points at O_1', O_2', O_3'.

Geometrically, f has 2μ variable intersections with any conic through O_1, O_2, O_3, and μ variable intersections with any line of S, and in particular with each P-line, its homologue has 2μ variable intersections with any line of S', and is of degree 2μ; it has μ variable intersections with any of the conics (ϕ') through $O_1'O_2'O_3'$, and of the remaining 3μ intersections of f', ϕ', there are μ coinciding with each F-point, the directions of the μ tangents to f' at O_1' being those associated with the μ intersections of f with j_1.

If f passes through O_1, the corresponding P-line j_1' is dropped from f', whose degree and multiplicities at O_2', O_3' are each lowered by 1. Generally, if we have

$$f_\mu \equiv O_1^{i_1} O_2^{i_2} O_3^{i_3},$$

we drop $j_1'^{i_1} j_2'^{i_2} j_3'^{i_3}$, and find $f_{\mu'}' \equiv O_1'^{i_1'} O_2'^{i_2'} O_3'^{i_3'}$, where

$$\mu' = 2\mu - i_1 - i_2 - i_3, \quad i_1' = \mu - i_2 - i_3, \quad i_2' = \mu - i_3 - i_1, \quad i_3' = \mu - i_1 - i_2;$$

or in symmetrical form,

$$\mu' - i'_\alpha = \mu - i_\alpha, \quad 3\mu' - \Sigma i'_\alpha = 3\mu - \Sigma i_\alpha, \quad (\alpha = 1, 2, 3).$$

Any point Q of f lying on $j_1 \sim$ the same point O_1'; a point P of f, adjacent to Q but not on j_1, $\sim$ an ordinary point of S', adjacent to O_1' in the direction at O_1' associated with Q. If the μ points Q are distinct from each other and from O_2, O_3, then f' has an ordinary μ-fold point at O_1', with distinct tangents. If two points Q coincide, so that f has either one branch touching j_1, or two intersecting on it, then two tangents to f' at O_1' coincide, and f' has either a cuspidal branch, or two branches in contact, forming a tacnode. But if one Q is at O_2, this $\sim j_2'$; the branch associated with this intersection is the dropped P-line, and not part of the proper homologue, and the multiplicity of O_1' on f' is reduced by 1.

Consider, for example, the transformation of a conic f_2. In general, f' is a quartic with d.p.'s at the three F-points, and the two tangents at $O_1' \sim$ the lines joining O_1 to the intersections of f_2, j_1, and similarly at O_2', O_3'. If f_2 passes through O_1, its homologue breaks up into j_1' and a cubic $f_3' \equiv O_1'^2 O_2' O_3'$. If f_2 passes through O_1, O_2, then j_1', j_2' drop away, and f' is a conic through O_1', O_2'; in particular, if O_1, O_2, O_1', O_2' are the circular points in S, S', then all circles $\sim$ circles (or lines), and the T_2 is called a *circular transformation* (121, 241, 282), of which inversion is the best known variety. Finally, if f_2 passes through all three F-points, its proper homologue is a general line.

Analytically,

$$f_2 \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

$$f_4' \equiv ay'^2z'^2 + bz'^2x'^2 + cx'^2y'^2 + 2fx'^2y'z' + 2gx'y'^2z' + 2hx'y'z'^2.$$

The two intersections of f_2, x, given by putting $x = 0$, are $by^2 + 2fyz + cz^2$; and the two tangents to f_4' at O_1', given by the co-factor of x'^2, are of the same form:

$$bz'^2 + 2fy'z' + cy'^2.$$

If f_2 passes through O_1, then $a = 0$, and x' is a factor of f_4', leaving

$$f_3' \equiv bz'^2x' + cx'y'^2 + 2fx'y'z' + 2gy'^2z' + 2hy'z'^2,$$

which is a cubic with a d.p. at $(1, 0, 0)$.

6. Specialization; two F-points adjacent (404).

The T_2 is specialized if the F-points cease to be distinct, two becoming adjacent; and still further if the third point also becomes adjacent to the other two (p. 36). We can no longer take the F-points as triangle of reference.

If O_2 is adjacent to O_3, then the homaloids have two fixed intersections there, that is, they touch one another and have a fixed common tangent. Thus besides conditions of incidence, such as have hitherto defined (ϕ), there is a contact condition.

The fixed tangent at O_3 is j_1 the limit of O_2O_3, and may be regarded as the path by which $O_2 \rightarrow O_3$. It can have any direction except O_3O_1; for (ϕ) already meets j_1 in two points at O_3, and cannot meet it elsewhere.

Let $A_1 \equiv O_1$, $A_3 \equiv O_3$, and first let O_2 have a general position $(0, \epsilon, 1)$ on the line $j_1 \equiv x$; then $O_2 \rightarrow O_3$ as $\epsilon \rightarrow 0$. In the limit, the pairs of sides of the F-triangle no longer give three independent homaloids; we can take instead O_1O_2, O_1O_3; $O_2O_3A_2$, O_1O_3; O_1A_2, O_2O_3, and the equations are

$$x' : y' : z' = y(y - \epsilon z) : xy : xz.$$

In the limit, when O_2 is adjacent to O_3 and $\epsilon = 0$,

$$x' : y' : z' = y^2 : xy : xz, \quad G \equiv xy^2, \quad J \equiv -2xy^2,$$
$$x : y : z = y'^2 : x'y' : x'z'.$$

The second F-system is therefore specialized in the same way as the first.

This specialized type can be compounded of two general T_2's, for example

$$x' : y' : z' = z''(x'' - y'') : x''z'' : x''(y'' - z''),$$
$$x'' : y'' : z'' = x(y+z) : (x-y)(y+z) : y(x-y).$$

Now G consists of two distinct elements only. One is $y \equiv j_2 \equiv O_1O_3$ reckoned twice, once answering to the condition of incidence of (ϕ') at O_3', and again answering to the condition of contact at the same point. The other P-element is $x \equiv j_1$, the fixed tangent to (ϕ) at O_3, corresponding to the ordinary F-point O_1'. In S' we have a similar P-system G'.

There are now only two pairs of projective pencils of rays, with vertices O_1, O_1' and O_3, O_3'. In the first, $O_1'O_3'$ corresponds to j_2 and in the second, $O_3'O_1'$ to j_1; and the intersection of j_2, j_1, which is the other F-point, is at O_3. There is no pair of projective pencils standing on the points of the P-lines y, y'. The pencil $ax - z$, joining the points of y to A_2, $\sim$ the pencil of conics $ay'^2 - x'z'$ through O_3', all touching x' but with different curvatures; the points of y are associated with these curvatures and conics at O_3', not with different directions.

A general f_μ, meeting y in μ distinct points Q, $\sim$ a curve $f_{2\mu}'$, with μ distinct branches through O_3', all touching x', and each osculating one of the conics associated with the μ points Q. These branches $\sim$ the μ distinct pieces of f near ΣQ. We have

$$f \equiv (x, y, z)^\mu,$$
$$f_{2\mu}' \equiv (y'^2, x'y', x'z')^\mu,$$

with μ-fold points at O_1', O_3'. At O_3' all the μ tangents are x', and the second approximations are of the form $ay'^2 - x'z'$, the values of a being given by $(1, 0, a)^\mu$, answering to the μ intersections of f, y, given by $(x, 0, z)^\mu$.

If f passes through O_3, with a general tangent, one $Q \equiv O_3$, one $a = \infty$, and one conic becomes the P-line y' dropped from f', taken twice. The branch of f through $O_3 \sim$ a branch of f' through O_3', which with the dropped line approximates to a conic with a d.p. at O_3', which we may regard as osculating y'^2. Similarly, if O_3 is i-fold on f, then y'^i is dropped, the degree of f', but not its multiplicity at O_3', is lowered by i. Then i branches of f' at O_3' are in general directions; the remaining $\mu - i$ branches of f' osculate the conics associated with the intersections of f, y other than O_3.

If f satisfies the contact condition, it passes through O_3 and touches x there:

$$f \equiv z^{\mu-1}x + (x, y \overset{2}{\between} x, y, z)^{\mu-2};$$

we drop y' twice, answering once to the incidence and once to the contact:

$$f'_{2\mu-2} \equiv x'^{\mu-1} z'^{\mu-1} + (y', x' \overset{2}{\between} y'^2, x'y', x'z')^{\mu-2},$$

on which O_1' is $(\mu-2)$-fold, and O_3' is $(\mu-1)$-fold, the branches answering as before to the $\mu - 1$ intersections of f, y other than O_3.

If f has an i-fold point at O_3, and α of the tangents are x, we might expect that y'^{i+a} would be dropped, but this is not so in general:

$$f \equiv z^{\mu-i} x^a (x, y)^{i-a} + (x, y \overset{i+1}{\between} x, y, z)^{\mu-i-1};$$

only y'^{i+1} is dropped, the index being governed by the second group of terms in f, and

$$f' \equiv x'^{\mu-i} z'^{\mu-i} y'^{\alpha-1} (y', x')^{i-\alpha} + (y', x' \overset{i+1}{\between} y'^2, x'y', x'z')^{\mu-i-1}.$$

Unless O_3 is a higher singularity than is necessary with these conditions, f has one cycle of multiplicity α at O_3, which $\sim$ a cycle of multiplicity $\alpha - 1$ at O_3', with y' as tangent.

For example, let $i = \alpha = 2$; then f has an ordinary cusp at O_3, and y'^3 is dropped, the same factor as if f had an ordinary d.p. with only one branch touching x; then f' has $\mu - 2$ branches at O_3' touching x' and one touching y'. An additional y' is dropped, and the last branch at O_3' disappears, only if the term $y^3z^{\mu-3}$ is absent from f. This is the condition that at O_3 the cusp is replaced by a tacnode, and the single cycle by two distinct branches in contact.

7. Three F-points adjacent (17).

Now let O_1 be adjacent to both O_2 and O_3; the homaloids have three fixed intersections there, and not only touch but osculate each other. They have a fixed common tangent j and osculate a fixed conic q.

Then $O_1 \rightarrow O_3$ along a path that also touches j and osculates q; this path cannot osculate j, for (ϕ) would have three intersections with j in the limit. And the path cannot touch another line, for then every ϕ would acquire a d.p. at O_3 and would break up into two lines through O_3; two ϕ's would meet in four points at O_3 and none elsewhere, and the transformation would break down entirely. Also (ϕ) would break up if $O_1, O_2 \nrightarrow O_3$ along a cuspidal branch.

We must therefore assume an ordinary curve of approach, which can be replaced by any osculating conic, say

$$q \equiv y^2 - xz.$$

Then $O_1(\epsilon^2, \epsilon, 1) \to O_3$ along q as $\epsilon \to 0$. Here A_1 and the unit point are any points of q, and z is the tangent to q at A_1.

The simplest conics through $O_1O_2O_3$ are the two line-pairs $O_2O_3A_2$, O_3O_1; $O_2O_3A_2$, O_1A_1, and q itself; we can take the equations of transformation to be

$$x' : y' : z' = x(x - \epsilon y) : x(y - \epsilon z) : q.$$

In the limit, when O_1 is adjacent to O_2 and O_3, and $\epsilon = 0$,

$$x' : y' : z' = x^2 : xy : q, \quad G \equiv x^3, \quad J \equiv -2x^3,$$
$$x : y : z = x'^2 : x'y' : q'.$$

This can be resolved into two T_2's of the last type,

$$x' : y' : z' = y''^2 : x''y'' : x''(x'' - z''),$$
$$x'' : y'' : z'' = y^2 : xy : xz,$$

or into four general T_2's, but not fewer.

The P-system G of S now consists of one element only, $x \equiv j \equiv O_2O_3$, reckoned three times, answering first to the condition of incidence of (ϕ') at the F-point O', and then to the conditions of contact and of osculation. The second F- and P-systems are like the first.

There is only one projective pair of pencils of lines (O), (O') and in this x corresponds to x'. The pencil $by - z$, standing on the points Q of x, $\sim$ the pencil of conics $bx'y' - q'$, osculating q' at O' and having different third approximations, each associated with one value of b and one point Q.

The pencil of conics $ay^2 - xz$, touching x at O, but not osculating q, has an intermediate property. It $\sim$ the pencil $(a-1)y'^2 + x'z'$, of like nature; and here x' is the branch through O' of the degenerate conic associated with q, for which $a = 1$, and similarly x is associated with q', for which $a = 0$.

A general $f_\mu \sim f'_{2\mu} \equiv O'^\mu$, all the μ branches osculating q', and having the distinct third approximations associated with the μ intersections Q of f, x. If f passes through O, with a general tangent t say, this absorbs one Q; we drop x', and $f'_{2\mu-1}$ still has μ branches through O'; but now only $\mu - 1$ osculate q', answering to the $\mu - 1$ points Q; the other branch touches t' the homologue of t: this and the dropped component x' together meet q' in three points, and replace one branch osculating q'.

If the branch of f at O touches x and osculates a conic $ay^2 - xz$ different from q, this absorbs two points Q. Then $f'_{2\mu-2}$ has a branch at O' osculating the associated conic, which, with the dropped components x'^2, meets q' in six points, and replaces two of the branches osculating q'. A cycle of f at O of multiplicity α, touching x, absorbs $\alpha+1$ points Q; then $x'^{\alpha+1}$ is dropped, and f' has a cycle of multiplicity α touching x'; these together meet q' in $3(\alpha+1)$ points at O'.

Finally, if the branch of f satisfies the same three conditions as the homaloids, and osculates q at O, the number of points Q is again $\mu-2$. Then the associated branch of $f'_{2\mu-3}$ does not pass through O', but through the point Q' associated with the third approximation to f at O; this, with the dropped components x'^3, meets q' in six points at O', replacing two branches osculating q'.

All this can be proved from the equations. For example, in the last case,

$$f_\mu \equiv z^{\mu-2}q + z^{\mu-2}x(x, y) + (x, y\overset{3}{\between}x, y, z)^{\mu-3},$$

$$f'_{2\mu-3} \equiv q'^{\mu-2}z' + q'^{\mu-2}(x', y') + (x', y'\overset{3}{\between}x'^2, x'y', q')^{\mu-3},$$

on which O' is $(\mu-2)$-fold, all the branches touching x' and osculating q'. The next approximations $q'=b'x'y'$ are given by

$$b'^{\mu-2} + 0 + (0, 1\overset{3}{\between}0, 1, b')^{\mu-3} = 0,$$

and are associated with the intersections of f, x other than O, given by

$$z^{\mu-2}y^2 + 0 + (0, 1\overset{3}{\between}0, y, z)^{\mu-3} = 0.$$

The only intersection of f', x' other than O' is given by

$$z' + (0, y') + y'(0, 1\overset{3}{\between}0, 0, 1)^{\mu-3} = 0,$$

and is associated with the third approximation $q = bxy$ to f at O, given by

$$b + (0, 1) + (0, 1\overset{3}{\between}0, 0, 1)^{\mu-3} = 0.$$

8. Collineation.

We have excluded the case when the F-points are collinear. If O_1, O_2, O_3, whether distinct or adjacent, lie on a line l, then every ϕ breaks up into l and an unrestricted line. The constant factor l can be removed from the equations of transformation, whose degree is reduced to 1; the two nets of lines of S, S' correspond, and T_2 degenerates into a collineation T_1. If the points of reference in the two planes are homologous, the equations are

$$x' : y' : z' = x : y : z,$$

and algebraically the transformation is identity.

Another form of the geometrical condition that T_2 may become T_1 is that j_3' corresponds to j_3 in both the pairs of projective pencils (O_1), (O_1'); (O_2), (O_2').

9. Methods of determination; seven pairs of points (118, 214, 332, 350, 410).

A T_2 depends on 14 parameters. With general frames of reference, the three F-points of the first system are determined by six coordinates; the selection of three base conics from the net $(O_1O_2O_3)$ involves eight more parameters, and T_2 is then determined.

A given pair of homologous points assigns two parameters, the ratios of the coordinates of the homologue P' of a certain point P. Hence T_2 is determined by seven pairs of homologues, or the equivalent.

It is also determined by equations (1) of p. 31, geometrically equivalent to the choice of the two sets of F-points and one pair of ordinary homologues. Since an F-point $\sim$ a line, determined by two points, an F-point and its corresponding P-line are equivalent to two pairs, and the two F-triangles to six pairs.

With sufficient data of any kind, the problem of finding the homologue of a given point P has only one solution, and P' can be constructed with ruler only. If one F-point in S is given, the construction of another has two solutions, and in general requires compasses. If seven general pairs are given, the three F-points in S enter symmetrically, and require a cubic and not a Euclidean construction.

10. Two pairs of projective pencils (17, 174, 359).

We have seen that the lines joining P, P' to F-points of S, S' are corresponding rays of three pairs of projective pencils. If two of these projectivities, say between (O_1), (O_1') and between (O_2), (O_2') are given, then the homologue of any given point P is determined as the intersection of the rays corresponding to O_1P, O_2P. This also determines $O_3'P'$, which $\sim O_3P$ in the third projectivity. Thus any two of the projectivities determine the third, and also determine T_2.

Regarded in turn as belonging to the two pencils, $j_3' \equiv O_1'O_2'$ has two corresponding rays j_2, j_1, whose intersection $\sim$ every point of j_3', and is therefore the third F-point O_3.

If P describes a line l, the pencils $O_1(P)$, $O_2(P)$ are perspective, and $O_1'(P')$, $O_2'(P')$ are projective with these and therefore with each other, and P' describes a conic through O_1', O_2', and also through O_3' corresponding to the intersection of l, j_3.

This construction can give all the cases of T_2 when at least two F-points are distinct, but fails if all three are adjacent, and there is only one pair of projective pencils.

11. Bilinear equations (91, 170, 221, 222, 343, 356, 386, 411).

The symmetric equations of p. 31 are linear in each set of coordinates. Conversely, a general T_2 is given by two general bilinear equations:

$$u_1x' + u_2y' + u_3z' = 0, \quad \text{where} \quad u_\alpha \equiv a_{1\alpha}x + a_{2\alpha}y + a_{3\alpha}z, \quad (\alpha = 1, 2, 3);$$
$$v_1x' + v_2y' + v_3z' = 0, \qquad\qquad v_\alpha \equiv b_{1\alpha}x + b_{2\alpha}y + b_{3\alpha}z.$$

We can solve in the form

$$x' : y' : z' = \phi_1 : \phi_2 : \phi_3,$$

where ϕ_1 is the conic $u_2v_3 - u_3v_2$, etc.; and since

$$u_1\phi_1 + u_2\phi_2 + u_3\phi_3 \equiv v_1\phi_1 + v_2\phi_2 + v_3\phi_3 \equiv 0,$$

we see that ϕ_3 passes through all the common points of ϕ_1, ϕ_2 except the one $u_3 = v_3 = 0$, and the net (ϕ) has three simple base points.

The reverse transformation is given by rearranging the bilinear equations in the form

$$u_1'x + u_2'y + u_3'z = 0, \quad \text{where} \quad u_\alpha' \equiv a_{\alpha 1}x' + a_{\alpha 2}y' + a_{\alpha 3}z', \quad (\alpha = 1, 2, 3);$$
$$v_1'x + v_2'y + v_3'z = 0, \quad\quad v_\alpha' \equiv b_{\alpha 1}x' + b_{\alpha 2}y' + b_{\alpha 3}z'.$$

With special relations between the coefficients, these equations can represent any of the special kinds of T_2. Thus if two F-points are adjacent, the equations of p. 35 are equivalent to

$$xx' - yy' = 0, \quad yz' - zy' = 0;$$

if three F-points are adjacent, the equations of p. 37 are equivalent to

$$xy' - yx' = 0, \quad xz' - yy' + zx' = 0;$$

while if for example

$$u_1 \equiv v_2 \equiv 0, \quad u_2 \equiv v_1,$$

then

$$x' : y' : z' = v_3 : u_3 : - u_2$$

and we have a general collineation.

12. Two correlations (15, 173, 332, 333, 381).

Each bilinear equation by itself expresses a correlation between the points of each plane and the lines of the other. If P is fixed, u_1, u_2, u_3 are given, and P' lies on the fixed line $u_1x' + u_2y' + u_3z'$ of S; similarly if P' is fixed, P lies on $u_1'x + u_2'y + u_3'z$.

Two correlations relate P to two lines in S', whose intersection P' is the one point conjugate to P in both. The 1, 1 relation between P, P' is T_2, being expressed by two bilinear equations.

A given pair of conjugate points of a correlation gives one linear relation between the eight parameters of the bilinear equation. Hence a correlation is determined by eight pairs of conjugate points, a point and its corresponding line being equivalent to two pairs. If seven pairs are given, the correlation has one degree of freedom; the line corresponding to P involves one linear parameter and describes a pencil, whose vertex P' is thereby in 1, 1 relation to P. Any two correlations of the system determine the same T_2, in which $P \sim P'$.

Let one correlation degenerate; it becomes a projectivity between two pencils of rays (O), (O'). Then $P \sim$ the intersection of the ray corresponding to OP in the projectivity with the line corresponding to P in the proper correlation (8). If this degenerates also, we have the construction of T_2 by two pairs of projective pencils.

13. Skew projection (372, 394, 406).

Let l_1, l_2 be two skew lines fixed in space, not lying in either of the planes S, S'. From any point P of S, one and only one line λ can be drawn to meet both l_1 and l_2, namely the intersection of the planes Pl_1, Pl_2. This meets S' in P', which is thus in 1, 1 relation to P. As P describes a line l in S, so λ describes the quadric determined by l, l_1, l_2, and P' describes the conic section of this quadric by S'. The transformation is therefore quadratic.

The construction fails if λ is indeterminate, P being one of the traces O_1, O_2 of l_1, l_2 on S; and $O_1 \sim$ the intersection of S' with the plane O_1l_2. Hence O_1, O_2 are F-points in S, and there are similar F-points O_1', O_2' in S'.

The construction of P' also fails if λ lies in S', being $O_1'O_2'$, and P is O_3, the intersection of $O_1'O_2'$ with the common line l_3 of S, S'. This gives the third F-point in S.

Every point of $l_3 \sim$ itself, and the projective pencils (O_1), (O_1') are perspective with l_3 as axis, and also (O_2), (O_2'). The third pair (O_3), (O_3') are not perspective, and in this projectivity $l_3 \sim$ itself.

There is a simple construction for P' when P is given: join O_1P, O_2P to meet l_3 in X_1, X_2, and join $O_1'X_1$, $O_2'X_2$ to meet in P'.

The construction assumes that S, S' are different planes and that the F-points in each plane are real and distinct. If O_1 were adjacent to O_2, then l_1, l_2 would meet and the construction fail; if O_1 were adjacent to O_3 which lies on $O_1'O_2'$, then $O_1O_2O_1'O_2'$ would be coplanar and again l_1, l_2 would meet.

14. Other space constructions (15, 94, 332).

Another space construction for T_2 uses a quadric q and two fixed points X, X' of q not lying on S, S'. The projections P, P', of a variable point Y of q, from X, X' on to S, S' respectively, are in 1, 1 relation. If P describes a general line l of S, then Y describes the conic section of q by the plane Xl, whose projection from X' on S' is another conic, locus of P': the transformation is quadratic.

Now Y is indeterminate if P lies on one of the generators l_1, l_2 of q through X, at their intersections O_1, O_2 with S. Then P' is any point of the intersections of S' with the planes $X'l_1$, $X'l_2$ respectively. Again, if Y is at X', then P is the intersection O_3 of S, XX', and P' is indeterminate, being any point of the intersection of S' with the tangent plane to q at X'. Hence O_1, O_2, O_3 are the F-points in S, and similarly in S'.

The construction only assumes that, of the F-points, O_3, O_3' are real.

Now O_1 can be adjacent to O_3 if X' lies on l_1; or to O_2 if l_1, l_2 coincide and q is a cone; and all three coincide if XX' is a generator of the cone. If $X \equiv X'$, the T_2 degenerates into a simple projection.

In a somewhat similar construction, a quadric q and two fixed skew lines l_1, l_2 are given; then P' is the intersection of S' with the conjugate, with regard to q, of the line through P meeting l_1, l_2 (54).

II. PLANES SUPERPOSED

15. Cycles (156, 184, 185).

So far, S, S' have been general planes in space. All the properties which have been found, except the space construction of p. 41, § 13, hold also when S, S' coincide, and in addition new properties appear, with which the rest of this chapter is concerned.

Any point P of S may now be considered as a point of S' also, when we call it P_1'; it has two homologues, P' and P_1, in T_2 and T_2^{-1} respectively. In general, the three points P, P', P_1 are distinct from one another and not specially related. We now consider the points for which special relations exist.

A set of points, each of which is transformed by T_2 into a point of the same set, form a *self-corresponding* group, unaltered as a whole by the transformation. The set of all the ∞^2 points of S is such a group; there are also smaller groups with the same property. If one has ∞^1 points, these may form a *self-corresponding curve* of T_2. If the group consists of a finite number of points P, then $P \sim P'$, $P' \sim P''$..., where P', P'' also belong to the group, and after say α applications of T_2, where $\alpha \leqq$ the number of points, we must come back to $P \equiv P^{(\alpha)}$. Then the different points $P, P' \ldots P^{(\alpha-1)}$ form a *cycle* of *index* α of T_2.

Now $P \sim P''$ in a 1, 1 transformation of S into itself, consisting of T_2 carried out twice, called the *square* of T_2 and denoted by T_2^2. In the same way, $P \sim P^\alpha$ in the αth power of T_2.

In particular, if $\alpha = 1$, then P is an *invariant* (or *united*) point, coinciding with both its homologues P_1, P'; it will be denoted by D. There are in general four points D (p. 43); in special cases, there are an infinite number, forming an *invariant curve* Δ, either a line or a conic (pp. 46, 47). Then Δ is necessarily a self-corresponding curve, but a self-corresponding curve need not be invariant.

If P belongs to a cycle of index α of T_2, it is an invariant point of T_2^α.

If $\alpha = 2$, the cycle consists of two distinct points P, P', exchanged by T_2; each is the homologue of the other in both T_2 and T_2^{-1}; and is brought back to itself by two applications of T_2, and is therefore invariant for T_2^2. The pair is called *involutory*. There is in general one such pair (p. 44); in special cases there are an infinite number; then the locus of P, P' may be a single *involutory curve*, which is necessarily self-corresponding, or a pair of curves corresponding to each other in involution; or else the locus may be the whole plane, all of whose points can be grouped into involutory pairs; then T_2 is called an *involution*. P, P' are *conjugate* to each other.

In this connection, the following notation is useful. With a general frame of reference, the same for S as for S', let T_2 be given by two bilinear equations

$$f_a(x, y, z, x', y', z') = 0, \quad f_b = 0,$$

and let each be expressed as the sum of a symmetric and a skew part

$$f \equiv S + \Sigma,$$

where

$$2S \equiv f(x, y, z, x', y', z') + f(x', y', z', x, y, z),$$

$$2\Sigma \equiv f(x, y, z, x', y', z') - f(x', y', z', x, y, z),$$

and say

$$S_a \equiv a_{11}xx' + a_{22}yy' + a_{33}zz' + a_{23}(yz' + zy') + a_{31}(zx' + xz') + a_{12}(xy' + yx'),$$

$$\Sigma_a \equiv a_1(yz' - zy') + a_2(zx' - xz') + a_3(xy' - yx').$$

The equation $S_a = 0$ expresses that P, P' are conjugate with regard to the conic

$$q_a \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{31}zx + 2a_{12}xy;$$

and $\Sigma_a = 0$ expresses that P, P' are collinear with the point A (a_1, a_2, a_3).

If we put $x = x'$ etc., then S_a becomes q_a, and Σ_a vanishes identically. If we interchange x, x' etc., then S_a is unaltered and Σ_a changes sign; thus the equations of the reverse transformation are

$$S_a - \Sigma_a = S_b - \Sigma_b = 0.$$

16. The four invariant points (98, 332).

At an invariant point D we have $x = x'$ etc., and the equations reduce to $q_a = q_b = 0$. Hence the invariant system consists of the common points of these two conics, and the general T_2 has four isolated invariant points $D_1 D_2 D_3 D_4$.

Now OD, $O'D$ are corresponding rays of the projective pencils (O), (O'), and D lies on the conic, locus of intersections of corresponding rays. The equations of T_2 can also be expressed in the form (p. 31)

$$j_1' : j_2' : j_3' = j_2 j_3 : j_3 j_1 : j_1 j_2,$$

where $j_1 \dots j_3'$ are the six P-lines. A general pair of corresponding rays of the pencils (O_1), (O_1') is

$$j_2 + a j_3, \quad j_3' + a j_2',$$

and the conic q_1 which they generate is found by expressing that a point (x, y, z) lies on both, and eliminating a:

$$q_1 \equiv j_2(x, y, z)\, j_2'(x, y, z) - j_3 j_3'.$$

Similarly the other pairs of pencils generate

$$q_2 \equiv j_3 j_3' - j_1 j_1', \quad q_3 \equiv j_1 j_1' - j_2 j_2',$$

where the j' are now functions of x, y, z, not of x', y', z'. In general, these three conics are distinct and proper; since $q_1 + q_2 + q_3 \equiv 0$, they belong to the same pencil, whose base is the set of invariant points ΣD.

If T_2 is given in the form

$$x' : y' : z' = \phi_1 : \phi_2 : \phi_3,$$

the invariant points satisfy

$$x : y : z = \phi_1 : \phi_2 : \phi_3$$

and lie on the cubics

$$y\phi_3 - z\phi_2, \quad z\phi_1 - x\phi_3, \quad x\phi_2 - y\phi_1.$$

The first two have in common O_1, O_2, O_3 and the two points $z = \phi_3 = 0$; their remaining four intersections lie on the third cubic, and are the invariant points. Similarly, ΣD are common points of $y\phi_3' - z\phi_2'$, etc., where the ϕ' are functions of x, y, z.

If P is adjacent to D, then P' is also adjacent, and the directions DP, DP' are related in a homography (44), which in general is neither identity nor an involution, and has no involutory pairs or cycles of directions. It has two double elements, giving two invariant directions at each invariant point; all self-corresponding curves through D have their tangents in these directions.

Let D be $(0, 0, 1)$; then ϕ_1, ϕ_2, ϕ_1', ϕ_2' vanish here, and

$$\phi_1 \equiv zu_1 + u_2, \quad \phi_1' \equiv z'u_1' + u_2',$$

$$\phi_2 = zv_1 + v_2, \quad \phi_2' \equiv z'v_1' + v_2',$$

where the u, v are functions of x, y only.

A line in S through D, say $ax + by$, $\sim$ the conic in S', $a\phi_1' + b\phi_2'$, touching $au_1' + bv_1'$ at D. The invariant directions are given by $x'v_1' - y'u_1'$, and similarly by $xv_1 - yu_1$.

17. The pair of involutory points (113, 156).

An involutory pair K, K' satisfies both the direct and reverse equations, hence
$$S_a = S_b = 0, \quad \Sigma_a = \Sigma_b = 0.$$

Thus in general, K, K' are conjugate with regard to the two conics q_a, q_b and collinear with the two fixed points $A\,(a_1, a_2, a_3)$, $B\,(b_1, b_2, b_3)$. Since K, K' are by definition distinct, they lie on AB, and are the one common pair of the two involutions on AB whose d.p.'s are the intersections with q_a, q_b respectively. Thus the general T_2 has one pair of involutory points.

They also satisfy

$$x' : y' : z' = \phi_1 : \phi_2 : \phi_3 \text{ and } x' : y' : z' = \phi_1'(x, y, z) : \phi_2' : \phi_3'$$

and lie on the quartics

$$\phi_2\phi_3' - \phi_3\phi_2', \quad \phi_3\phi_1' - \phi_1\phi_3', \quad \phi_1\phi_2' - \phi_2\phi_1'.$$

The first two have in common the six F-points, the four points D and the four points $\phi_3 = \phi_3' = 0$. The remaining two lie on the third quartic and are the involutory pair.

If P is adjacent to K, then P' is adjacent to K', and the directions KP, $K'P'$ are related projectively. Also if P is adjacent to K', and P' to K, this gives another projectivity between the pencils (K), (K'). The two have two common pairs, and there are two directions at K each in involutory relation to a direction at K'.

18. Isologues (114, 156, 177, 390).

The locus of points of S', which with their homologues in S are collinear with a given point P, is a curve Ω_P' in S' called the *isologue* of P, and P is the *centre* of the isologue.

A line l in S through $P \sim$ a conic ϕ' in S', meeting l in two points X', Y' of S', which lie with their homologues X, Y on l and are therefore points of the isologue. Hence Ω_P' is the locus of X', Y' as l rotates about P. Then ϕ' describes the pencil of conics $(O_1'O_2'O_3'P')$ projective with (P), and Ω_P' is the product of the two pencils and is therefore a cubic through $O_1'O_2'O_3'PP'$. It also passes through $D_1 \dots D_4$, and these nine points define it completely.

There are four positions of l for which X', Y' coincide, namely the tangents from P to Ω_P': then l, ϕ', Ω_P' all touch. There is one position for which X' coincides with P, lying on the one conic of the pencil (ϕ') through P. Then l is the tangent to Ω_P' at P, but does not touch ϕ'. Also l passes through P_1 where $P_1' \equiv P$.

The nine intersections of any two isologues Ω_{P_1}', Ω_{P_2}' are

$$O_1',\ O_2',\ O_3',\ D_1 \dots D_4,$$

and the two intersections of P_1P_2 and its homologue. Here is a fresh proof of the number of invariant points.

Similarly, the point $P \equiv P_1'$ is the centre of a second isologue Ω_P in S, which is the locus of X when PXX' are collinear, and is the homologue of Ω_P'. It is the cubic through $O_1O_2O_3D_1 \dots D_4PP_1$, product of the projective pencils of rays (P_1') and conics $(O_1O_2O_3P_1)$.

The collinearity which defines the isologue of $P\,(\xi, \eta, \zeta)$ gives its equation; with a general frame of reference,

$$\Omega_P' \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ x', & y', & z' \\ \phi_1', & \phi_2', & \phi_3' \end{vmatrix}, \quad \Omega_P \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ x, & y, & z \\ \phi_1, & \phi_2, & \phi_3 \end{vmatrix}.$$

We verify that both pass through P, where

$$x : y : z = \xi : \eta : \zeta,$$

and through $D_1 \dots D_4$, where

$$x : y : z = \phi_1 : \phi_2 : \phi_3 = \phi_1' : \phi_2' : \phi_3'.$$

Also Ω_P' passes through O_1', O_2', O_3', where

$$\phi_1' = \phi_2' = \phi_3' = 0,$$

and through P', where $\quad \xi : \eta : \zeta = \phi_1' : \phi_2' : \phi_3'$;

similarly Ω_P passes through O_1, O_2, O_3, P_1.

Since Ω_P, Ω_P' are linear in the parameters $\xi : \eta : \zeta$, each set of isologues forms a net. If a pencil is chosen from the net, this establishes a linear relation between ξ, η, ζ, and the locus of P is a line l; the last two base points of the pencil are the intersections of l with its homologue.

For special positions of P, the isologues break up. If $P\,(0, 0, 1)$ is an F-point O, then ϕ_1', ϕ_2' have as a common factor the corresponding P-line j', say $\phi_1' \equiv l_1'j'$, $\phi_2' \equiv l_2'j'$, and

$$\Omega_O' \equiv j'(x'l_2' - y'l_1'),$$

breaking up into the P-line and a conic, while

$$\Omega_O \equiv x\phi_2 - y\phi_1 \text{ has a d.p. at } O.$$

If $P\,(0, 0, 1)$ is invariant, with the notation of p. 44,

$$\Omega_D' \equiv z'(x'v_1' - y'u_1') + x'v_2' - y'u_2',$$

and has a d.p. at D, touching the invariant directions and therefore touching Ω_D also.

19. Invariant line (98).

There can be a line Δ_1 of invariant points, if it is a common part of all the conics q_a, q_b, q_1, q_2, q_3 (p. 43). Since the only homologous lines are rays of the projective pencils, Δ_1 joins an associated pair of F-points, say O_3, O_3', and contains no other F-point of either system. It meets O_1O_2, $O_1'O_2'$ in general P-points, which, being invariant, coincide with their homologues O_3', O_3.

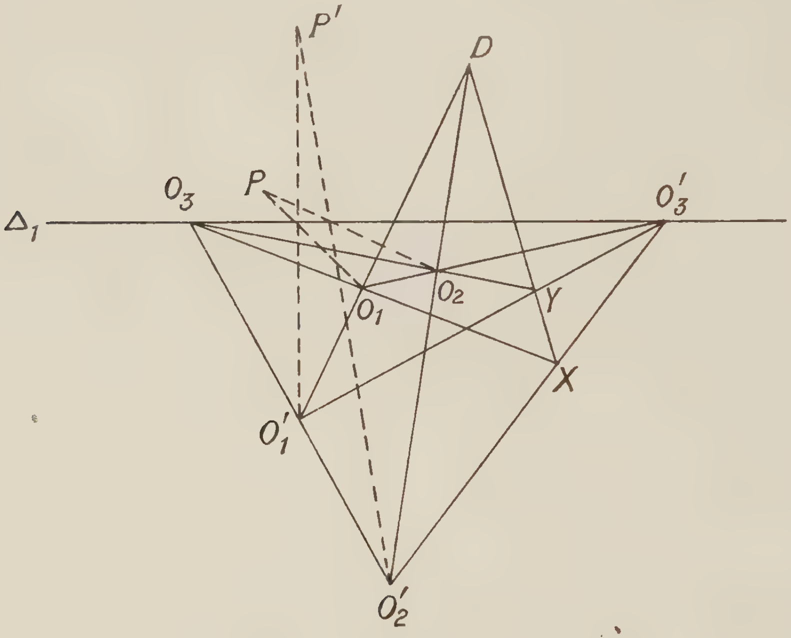

Fig. 1. T_2 with invariant line.

The pencils (O_1), (O_1') are perspective with Δ_1 as axis, and their product q_1 breaks up into Δ_1 and another line which is O_1O_1', since these lie on q_1 and not on Δ_1. Similarly q_2 breaks up into Δ_1 and O_2O_2'. But q_3 consists of $\Delta_1 \equiv O_3O_3'$ and another line, the axis of the perspective pencils (O_3), (O_3'), joining the intersections X, Y of the pairs of corresponding rays O_3O_1, $O_3'O_2'$; O_3O_2, $O_3'O_1'$.

The pencil of conics (q_1, q_2, q_3) breaks up into the fixed component Δ_1 and a pencil of lines (O_1O_1', O_2O_2', XY); these therefore concur at D, the single isolated invariant point of the transformation. The invariant system consists of a line and a point.

If Δ_1 and the F-points are given, P' is simply constructed as the intersection of the lines joining O_1', O_2' to the intersections of Δ_1 with PO_1, PO_2 respectively. The three joins of associated F-points, O_1O_1', O_2O_2', O_3O_3' are each self-corresponding.

Now Δ_1 falls away from each isologue, leaving a conic; in particular, $\Omega_{O_3}' \equiv DXY, j_3'$; Ω_{O_3} is a proper conic touching j_3' at O_3. If P is any point of the self-corresponding line $O_1O_1'D$, this is part of each isologue:

$$\Omega_{O_1}' \equiv O_1O_1'D, j_1', \quad \Omega_{O_1} \equiv O_1O_1'D, j_3; \quad \Omega_D' \equiv \Omega_D \equiv O_1O_1'D, O_2O_2'D.$$

There are no isolated involutory points; for if an involutory pair K, K' are collinear with O_1O_1', then O_1O_1' is an involutory line; if not, K, K' are opposite vertices of the quadrilateral O_1K, $O_1'K$, O_1K', $O_1'K'$, and KK' is divided harmonically, at D_1, D_2 say, by Δ_1 and O_1O_1'. Similarly O_2O_2' passes through $D_2 \equiv D$, and is divided harmonically by D, Δ_1. Now any pair of points of KK' that divide D_1D harmonically are involutory, KK' is an involutory line, and contains two associated F-points, coinciding with O_2O_2'.

20. Invariant conic (98).

If there is an invariant conic Δ_2, it is given by any of q_a, q_b, q_1, q_2, q_3 which does not vanish identically. Then S_b is a multiple of S_a, and we can form one bilinear equation of the form $\Sigma_a = 0$: every pair of homologues are collinear with a fixed point (p. 43), and the transformation is *central* (351).

Conversely, if T_2 is central, every ray through the centre is self-corresponding, and bears a homography whose two d.p.'s are invariant for T_2; their locus is an invariant conic.

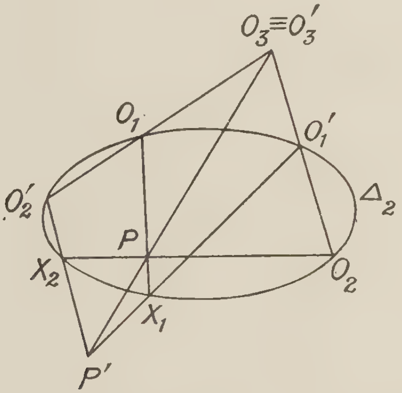

Fig. 2. T_2 with invariant conic.

From each of the homologues of Δ_2 two P-lines are dropped; Δ_2 contains two pairs of associated F-points, say O_1, O_1'; O_2, O_2', and not O_3, O_3'. A general ray l through O_3 meets Δ_2 in two invariant points D_1, D_2 and the homologue is a ray through O_3', also through D_1, D_2, and therefore coinciding with l. Hence $O_3 \equiv O_3'$, and is the centre of the transformation; the projective pencils coincide, $O_3O_1 \equiv O_3'O_2'$, and similarly $O_3O_2O_1'$ are collinear.

If q_1 were also indeterminate, the pencils (O_1), (O_1') would coincide, and the transformation reduce to identity. Hence q_1, q_2 exist, and both $\equiv \Delta_2$.

In the homography on $l \equiv O_3P$, the two homologues of O_3 are the intersections of l with j_3', j_3. If these are distinct, the homography is not involutory, except on the one ray joining O_3 to the intersection of j_3, j_3'; if j_3, j_3' coincide, T_2 is wholly involutory.

Each isologue of P breaks up into Δ_2 and O_3P, and the whole set forms a pencil of lines instead of a net of cubics.

If Δ_2 breaks up, it is into O_1O_1', O_2O_2', since each invariant line contains two associated F-points.

There are no self-corresponding curves except the lines through O_3, and the conic Δ_2. For if such a curve contains any general point P, it contains all its successive homologues in the homography on O_3P, and therefore breaks up, having O_3P as a component.

If Δ_2 is given, and two pairs of its points as associated F-points O_1, O_1'; O_2, O_2', we construct P' by joining O_1P, O_2P to meet Δ_2 again in X_1, X_2, and joining $O_1'X_1$, $O_2'X_2$ to meet in P'. Thus O_3PP' is the Pascal line of the hexagon $O_1X_1O_1'O_2X_2O_2'$ (110).

21. Involutory line (113).

We have seen that the involutory system consists in general of the two points of a line $l \equiv AB$, determined by Σ_a, Σ_b, which are conjugate with regard to two conics q_a, q_b.

If only two of the four functions S_a, S_b, Σ_a, Σ_b are linearly independent, then T_2 is wholly involutory; this case is treated below and excluded here. We assume therefore that S_a, Σ_a are determinate; then A is a definite point.

First, let B be determinate and distinct from A; then l is a definite line, and the involutory system lies on it. It is wholly involutory if the two involutions on it determined by q_a, q_b coincide, then q_a, q_b meet it in the same points and differ by a multiple of l. By replacing f_a by a linear function of f_a, f_b, we can have q_a degenerate with l as a factor, and the first involution on l indeterminate. Now each of S_a, Σ_a, f_a is a sum of multiples of l, l', say

$$f_a \equiv lu_1' + l'v_1;$$

the two associated F-points on the self-corresponding line l are

$$O,\ l = v_1 = 0 \text{ and } O',\ l' = u_1' = 0.$$

If the second involution on l is improper, every point being conjugate to the same point O, then O is a common F-point of T_2, T_2^{-1}, with l as common corresponding P-line. Then l satisfies the definition of an involutory line and is called *improperly involutory.* In this case there is no proper involutory system.

If the second involution is indeterminate, then f_b is also a sum of multiples of l, l', and T_2 degenerates to T_1.

22. Involutory cubic (113).

If B coincides with A, then Σ_b is a multiple of Σ_a, and by subtracting a multiple of f_a, we can replace f_b by S_b and take $\Sigma_b \equiv 0$, and B indeterminate. Then K, K' satisfy only $\Sigma_a = 0$ in addition to the equations of T_2, and every pair of homologues collinear with A is involutory. Their locus, the isologue of A in either plane, is the involutory cubic:

$$\Omega_A \equiv \begin{vmatrix} \dfrac{\partial q_a}{\partial x}, & \dfrac{\partial q_a}{\partial y}, & \dfrac{\partial q_a}{\partial z} \\ \dfrac{\partial q_b}{\partial x}, & \dfrac{\partial q_b}{\partial y}, & \dfrac{\partial q_b}{\partial z} \\ a_2 z - a_3 y, & a_3 x - a_1 z, & a_1 y - a_2 x \end{vmatrix},$$

unless any part of this is invariant.

Since $\Sigma_b \equiv 0$ in the present case, q_a, q_b do not coincide, and if there is an invariant curve, being a common part of q_a, q_b, it is a line Δ_1. Then the join of any involutory pair passes through the intersection X of the other components of q_a, q_b; if this is A, then T_2 is involutory; if not, AX is the only involutory line, and Ω_A breaks up into $\Delta_1{}^2$ and AX.

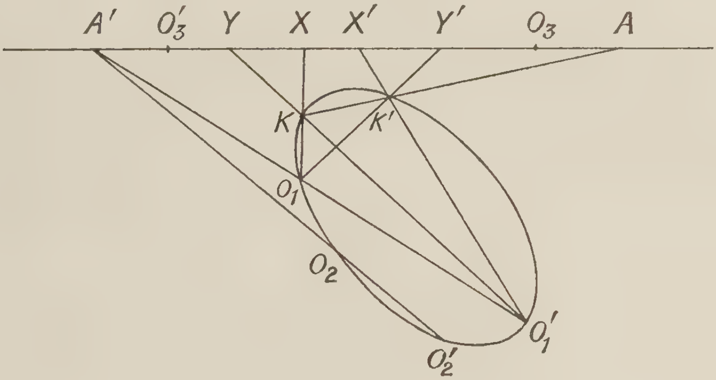

Fig. 3. T_2 with involutory line and conic.

If there is an involutory line l which does not exhaust the proper involutory locus, we can show that the joins of associated F-points are concurrent. One pair, O_3, O_3' say, lie on l. Let K, K' be an involutory pair not lying on l. Then O_1K, $O_1'K'$, and also O_1K', $O_1'K$, meet l in pairs X, X'; Y, Y' of the involution on l; from the property of the quadrangle, O_1O_1', KK' meet l in another pair of the same involution. But KK', l meet in A; therefore O_1O_1' meets $l \equiv O_3O_3'$ in A' the conjugate of A, and similarly O_2O_2' passes through the same point.

A component of Ω_A may be
an involutory cubic through A,
an involutory conic not through A,
a conic through A conjugate to a line not through A,
an involutory line through A,
a common P-line of T_2, T_2^{-1} through A, conjugate to a point of itself,
a line not through A, conjugate to a conic through A,
a line not through A, conjugate to a line not through A,
a common P-line of T_2, T_2^{-1} not through A, conjugate to A,
an invariant line not through A, reckoned twice.

Only certain combinations are possible if T_2 is neither an involution nor degenerate.

III. INVOLUTIONS

23. Two types of I_2 (332, 401).

Now let T_2 be an involution; we denote it by I_2. Then every point has the same homologue, whether it is regarded as belonging to S or S'. The involutory locus is the whole plane, whose points fall into conjugate pairs, homologues in $I_2 \equiv I_2^{-1}$; the ϕ are the same functions as the ϕ'.

The pencil $(O_1) \sim$ the pencil (O_1') in I_2^{-1} as well as in I_2; hence O_1 coincides with an F-point of H', and O_1' with the associated F-point of H. If the two pencils have the same vertex $O_1 \equiv O_1'$, the coincident F-points are an associated pair. If the vertices are distinct, O_1 coincides with another F-point, say $O_1 \equiv O_2'$, and $O_2 \equiv O_1'$. Then O_3 coincides with an F-point of H' which is neither O_1' nor O_2', that is, $O_3 \equiv O_3'$.

Thus there are two types of I_2; in the first, each F-point coincides with its associate; in the second, one F-point coincides with its associate and two coincide each with the associate of the other.

Now the four equations of p. 44,

$$S_a = S_b = \Sigma_a = \Sigma_b = 0,$$

reduce to two (170). If these are $S_a = S_b = 0$, then P, P' are conjugate with regard to q_a, q_b and every conic of the pencil (q_a, q_b). The F-points form the common self-conjugate triangle of the conics, and each vertex coincides with its associate; this is the first type of I_2.

If the equations are $S_a = \Sigma_b = 0$, then P, P' are conjugate with regard to q_a and collinear with B, which is an F-point associated with itself. The two tangents from B to q_a are associated P-lines, each conjugate to its point of contact. This is the second type of I_2.

If the equations are $\Sigma_a = \Sigma_b = 0$, then P, P' are collinear with both A and B, and the transformation is identity.

The *class* of an involution is the number of conjugate pairs of points on a general line l (p. 81). The first I_2 is of class 1, the conjugates on l being the common pair of the two involutions on l of points conjugate with regard to q_a, q_b respectively. The second I_2 is of class 0, since PP' passes through the centre and is not a general line.

24. First type of $\mathbf{I_2}$ (15, 19, 40, 196, 204, 222, 280, 330, 358, 403).

The three coincident pairs of F-points make T_2 involutory without further condition. The projectivity in the pencil $(O_1 \equiv O_1')$ has the involutory pair O_1O_2, O_1O_3, and is an involution; similarly at O_2 and O_3. Then P is the intersection of three rays, one of each pencil, and P' of their three conjugates; the relation between P, P' is symmetrical.

The self-conjugate rays of (O_1) meet those of (O_2) in the four invariant points and these lie also on the self-conjugate rays of (O_3). We may take as collinear:

$$O_1D_2D_3,\ O_1D_1D_4;\quad O_2D_3D_1,\ O_2D_2D_4;\quad O_3D_1D_2,\ O_3D_3D_4.$$

The projectivity of directions at D has now three invariant directions, DO_1, DO_2, DO_3, and is identity: any curve through D touches its homologue there.

Now P, P' are conjugate with regard to the three degenerate conics such as $O_1D_2D_3$, $O_1D_1D_4$, and therefore with regard to the pencil of conics $(D_1D_2D_3D_4)$, of which $O_1O_2O_3$ is the common self-conjugate triangle. The polars of P with regard to the conics of the pencil concur at P'. This type of I_2 cannot have an invariant curve, for it would be a fixed part of every conic of the pencil, and the transformation would break down.

Now D_2, D_3 are the d.p.'s of the involution of points on $O_1D_2D_3$; if P lies on this line, so does P', and $\{PP', D_2D_3\}$ is harmonic. Hence if $\phi_2 \sim l$, it contains, besides $O_1O_2O_3$, the harmonic conjugates of the six intersections of l with the sides of the quadrangle $D_1D_2D_3D_4$.

If the F-points are distinct, take them as triangle of reference, and D_4 as unit point; the equations of I_2 are

$$x' : y' : z' = yz : zx : xy,$$

or in symmetric bilinear form,

$$xx' = yy' = zz'.$$

25. Self-conjugate curves (122, 122·1, 238).

There are the six self-conjugate lines such as $O_1D_2D_3$.

If f_2 is a self-conjugate conic, two P-lines fall away from its homologue, and it therefore contains two F-points, O_1, O_2 say. Then I_2 determines on f_2 an involution, whose d.p.'s are two invariant points of I_2, not collinear with either O_1 or O_2, and therefore either D_1, D_2 or D_3, D_4. Now any conic of the pencil $(O_1O_2D_1D_2)$ meets its corresponding conic in these four points, touching it at D_1, D_2, and therefore coincides with it. There are thus six pencils of self-conjugate conics.

In the same way, a self-conjugate cubic f_3 contains F-points of total multiplicity 3, and if it is rational, it contains two points D. If O_1, D_1 are simple on f_3, then O_1D_1 meets f_3 again in a point whose homologue lies on both O_1D_1 and f_3, and is therefore D_4. Now any cubic of the pencil

$$(O_1O_2^2D_1D_4PP')$$

meets its corresponding cubic in $O_1O_2^4D_1D_4PP'$, touching it at D_1, D_4, and therefore coincides with it. As P varies, these self-conjugate cubics form a net, and there are twelve such nets.

If the d.p. of the rational f_3 is not an F-point, it is invariant, and f_3 passes simply through $O_1O_2O_3$, and can contain no other D without breaking up. As before any cubic of the pencil $O_1O_2O_3D^2PP'$ is self-conjugate, and the curves of this kind form four nets.

If f_3 is not singular, it contains $O_1O_2O_3$. We find a web (∞^3 linear family) of such cubics, and in addition the net $(O_1O_2O_3D_1D_2D_3D_4)$.

Thus there are seventeen nets and one web of self-conjugate cubics.

The two isologues of any point X coincide in a self-conjugate cubic

$$\Omega_X \equiv O_1O_2O_3D_1D_2D_3D_4XX'$$

of the last net, touching XX' at X and XD at D. If an isologue breaks up, one part is either a self-conjugate line on which X lies, or a P-line to which X is the conjugate F-point. If X is a general point of a line OD, the residual is a self-conjugate conic; if $X \equiv O_1$, then $\Omega_X \equiv O_1D_2D_3$, $O_1D_1D_4$, O_2O_3; if $X \equiv D_1$, then $\Omega_X \equiv D_1O_1$, D_1O_2, D_1O_3. Thus the net of isologues contains six pencils of members breaking up into a line and conic, and seven members breaking up into three lines.

26. Specializations.

If the conics of the pencil, with regard to which P, P' are conjugate, touch at $D_1 \equiv D_4$, then O_2, O_3 coincide with D_1, and O_1, D_2, D_3 are all distinct. At O_3, the homaloids touch a different line from the pencil. We can express this I_2 by

$$x' : y' : z' = xy : x^2 : -yz,$$

with $(1, \pm 1, 0)$ as isolated invariant points.

If O_1 is also adjacent to O_2 and O_3, then D_3 coincides with them and D_2 is distinct; we can take

$$x' : y' : z' = x^2 : -xy : y^2 - xz,$$

and D_2 is $(1, 0, 0)$. The homaloids touch the same line as the pencil, and osculate a different conic.

If all the D's coincide, the transformation fails.

The I_2 is determined if the F-triangle is given and one invariant point D_4. The self-conjugate lines are then O_1D_4, O_2D_4, O_3D_4 and their harmonic conjugates with regard to the sides of the F-triangle.

If these are the bisectors of the angles, then D_4, D_1, D_2, D_3 are the in- and escribed centres; OP, OP' are equally inclined to a bisector and to two sides, and P, P' are foci of an inscribed conic. They are sometimes called isogonal conjugates. The circular points I, J are conjugate, and if P describes the line at infinity, the inscribed conic is a parabola, and its focus describes the circumcircle $O_1O_2O_3IJ$ which bisects the line joining any two D's (p. 51), and is the nine point circle of the triangle formed by any three D's (52, 111, 233, 237, 348, 357, 387).

If the medians are self-conjugate lines, the others are parallel to the sides, and conjugate rays through O_1 meet the opposite side in points equidistant from O_2, O_3 (215).

27. Second type of $\mathbf{I}_2$: general inversion (130, 154, 155).

The second type of I_2, when $O_1 \equiv O_2'$, $O_2 \equiv O_1'$, $O_3 \equiv O_3'$, is given by equations of the form $S_a \equiv \Sigma_b = 0$, and consists of pairs of points conjugate with regard to a fixed conic Δ_2 and collinear with a fixed point O_3. The transformation is thus central, with Δ_2 as invariant conic (p. 47). It is called *general inversion*, and becomes ordinary inversion if Δ_2 is a circle centre O_3; then O_1, O_2 are the circular points (16, 19, 59, 92, 93, 171, 172, 212, 240, 241, 242, 329, 371, 377, 388, 389).

The projectivities between the two pairs of pencils (O_1), (O_1'); (O_2), (O_2') are the same, and generate the same conic $q_1 \equiv q_2 \equiv \Delta_2$. Any ray l through O_3 meets Δ_2 in two points D_1, D_2, and $l \sim$ a line through $O_3' \equiv O_3$, D_1, D_2, which coincides with l. The third projectivity is identity, and q_3 is indeterminate.

The involution of points on l has D_1, D_2 as d.p.'s. If l touches Δ_2, the point of contact is $D_1 \equiv D_2 \equiv O_1$ say, and $O_1 \sim$ the whole of l, which is the corresponding P-line $O_2'O_3' \equiv O_1O_3$.

The two invariant directions at D are DO_3 and the tangent to Δ_2. The isologue of any point P breaks up into Δ_2 and PO_3.

The transformation can be expressed by

$$x' : y' : z' = xz : yz : xy.$$

The self-conjugate lines are those of the pencil (O_3). A self-conjugate conic f_2 contains two F-points and two invariant points D_1D_2; it therefore meets Δ_2 again in O_1, O_2, and does not pass through O_3. Any conic through O_1, O_2 and a conjugate pair P, P' meets its homologue in the six points O_1, O_2, P, P', D_1, D_2, and is self-conjugate. There are thus ∞^2 such conics, one through every pair of points of Δ_2, namely the member of the pencil $(O_1O_2D_1D_2)$ that touches the invariant direction D_1O_3. If f_2 passes through O_3, it breaks up into O_1O_2 and a line through O_3.

A self-conjugate cubic f_3 contains F-points of total multiplicity 3, of which O_1, O_2 are of equal multiplicities, hence it passes simply through O_1, O_2, O_3. It meets Δ_2 again in four points $D_1 \dots D_4$, all d.p.'s of the involution of points on f_3, which is therefore of genus 1, and not singular. Now the self-corresponding conic $O_1O_2P_1P_1'P_2$ contains P_2' and with $O_3P_3P_3'$ makes up a cubic through $O_1O_2O_3P_1P_1'P_2P_2'P_3P_3'$. There are three such degenerate cubics, and the nine points are associated, and determine a pencil of cubics, each of which meets its homologue in these nine points and in $D_1 \dots D_4$, and is self-conjugate. There are thus ∞^4 such cubics, one through each set of four points of Δ_2.

This I_2 can be determined by data fixing two pairs of the involution on every ray through O_3, for example, by two self-conjugate conics, or one self-conjugate conic and j_3 (340, 341).

If O_1 is adjacent to O_2 and distinct from O_3, the tangents from O_3 to Δ_2 coincide, and Δ_2 is a line-pair. At O_1 the homaloids touch the fourth harmonic of Δ_2, O_1O_3. We can express this by

$$x' : y' : z' = xz : yz : y^2.$$

If O_3 is adjacent to O_1, it lies on Δ_2 and is also adjacent to O_2. At O_3, the homaloids touch Δ_2 and osculate a different fixed conic. We can take

$$x' : y' : z' = x^2 : xy : y^2 - xz.$$

28. Reduction of I_2 to I_1.

If we have an I_2 in a plane S, and a T_2 between S and another plane S', then the pairs of points of S', homologues by T_2 of conjugate pairs of I_2, are thereby related in a Cremona involution I'. If I' is the linear involution I_1 (harmonic homology), then I_2 is said to be *reduced to* I_1.

In this case, the net of lines of $S' \sim$ itself by I'; and by T_2 this net of lines $\sim$ the homaloidal net of conics of T_2 in S, which $\sim$ the same net by I_2. Conversely, if we take as F-trio of T_2 three base points of a self-conjugate net of conics of I_2, then I_2 is reduced to I_1. In either type of I_2, the two F-points O_1, O_2 and any invariant point D are such a trio.

CHAPTER IV

COMPOSITION AND RESOLUTION OF PLANE TRANSFORMATIONS

I. THE PROBLEMS OF COMPOSITION AND RESOLUTION

1. Definitions.

A Cremona transformation V between two planes S, S', and another W between S' and a third plane S'', together set up a 1, 1 relation between the homologues P, P'' of the same point P' in V^{-1}, W respectively, which is a third Cremona transformation T between S, S''. Then T is *compounded of* and *resolved into* V and W. We express this by $T \equiv W.V$, the product denoting that we first carry out V and afterwards W. The reverse transformation is also resolved, $T^{-1} \equiv V^{-1}.W^{-1}$. We use the symbol $\widetilde{V}$ for correspondence in V, etc.

The first homaloidal net (ϕ) of T $\widetilde{T}$ the lines of S''; now these $\widetilde{W}$ the first homaloidal net of W in S', and this net $\widetilde{V}$ a certain net in S, also homaloidal (p. 5), which is (ϕ). Conversely, if T, V are given, then (ϕ) $\widetilde{V}$ a homaloidal net in S', which determines the F-points in S' for a transformation W which converts it into the lines of S''; and so the single transformation T of (ϕ) into lines is carried out in two stages.

Then T and $W.V$ are two transformations which change (ϕ) into the lines of S''; and $W.V.T^{-1}$ is a transformation of S'' into itself, by which lines $\sim$ lines, that is, a collineation, equivalent to a change of the frame of reference. By properly choosing the base homaloids for W in S', we can make $W.V.T^{-1}$ the identical transformation, $W.V.T^{-1} \equiv 1$.

Thus if any two of T, V, W are given, the third is determined by the equivalent relations

$$T \equiv W.V, \quad V \equiv W^{-1}.T, \quad W \equiv T.V^{-1}.$$

These may equally well be regarded as resolving V or W instead of T. Generally speaking, it gives a useful resolution of T only when both V and W are simpler than T, and we must make clear what we mean by simplicity.

In the first place, a transformation of lower degree is simpler than one of higher degree; after the linear transformation T_1 come the various forms of T_2. This has only one distinct characteristic, $n = 2$; 3^1; but there are varieties in which two or three of the F-points are adjacent, presenting a condition of contact or osculation.

Among transformations having the same characteristic, those with no contact, or contact of lower order, are simpler than those with higher contact;

and those with F-points in general position are simpler than those with F-points specially related, as for example when three lie on a line, or nine on a pencil of cubics; and the simplest has distinct and general F-points.

After T_2 comes the only cubic characteristic, $n=3$; $1^3 4^1$; and next the two quartics, $n=4$; $1^3 6$, $n=4$; $3^2 3^1$, placed in that order by the arbitrary rule that characteristics of given degree n are arranged by descending values of i_1, and for given n and i_1, by descending values of i_2 and so on. Then among those of given degree, the simplest is always T_J, for which i_1 has its maximum value $n-1$, and of which T_2, T_3 and T_4 $1^3 6^1$ are particular cases.

The difference $n-i_1$, $\equiv 2\tau$ say, is a rough estimate of the departure of T_n from the de Jonquières type. It is of some importance in the theory (p. 145) and is called the *complexity* of the transformation (2); some properties of τ are given below (p. 75). In 264 the complexity is called the "class," but this is better avoided, as the latter term has a quite different accepted meaning (p. 81).

This chapter deals with two problems; first that of composition: to determine the transformation T compounded of two given transformations V, W. This is definite, if we are given V, W and also the relative position of their F-systems in the intermediate space, which is an essential feature in the result.

The second problem is that of resolution: given T, to find V and W. This is quite indefinite, for either V or W can be taken arbitrarily; in order that the resolution may be useful, we must arrange that both V and W are simpler than T. This can usually be done in many ways; by further conventions, we define a definite, normal resolution (p. 63).

2. The problem of composition (140, 178, 180, 181, 182, 304, 305).

The simplest example of composition is when both V and W are T_2's. First let the second F-system of V and the first F-system of W, both of which lie in S', be two trios of distinct points in general relation with regard to each other. Let the F-points of V be O_{11}, O_{12}, O_{13} in S and O_{11}', O_{12}', O_{13}' in S'; and those of W, O_{21}', O_{22}', O_{23}' in S' and O_{21}'', O_{22}'', O_{23}'' in S''; let $O_{21}' \,\tilde{V}\, O_{21}$, and so on.

Then lines in $S \,\tilde{V}$ conics $(\phi_2') \equiv O_{11}'O_{12}'O_{13}'$ in S', and these $\widetilde{W}$ a net in S'' of quartics $(\phi_4'') \equiv (O_{21}''O_{22}''O_{23}'')^2\, O_{11}''O_{12}''O_{13}''$, which is the second homaloidal net of T. If to the lines of S'' we apply first W^{-1} and then V^{-1}, we find the first homaloidal net of T, $(\phi_4) \equiv (O_{11}O_{12}O_{13})^2\, O_{21}O_{22}O_{23}$, of the same nature as the other. We have the second quartic transformation of Table I,

$$T_4\ 3^2 3^1, \quad \equiv T_2 . T_2 .$$

It would not be convenient to call this T_2^2, which would mean that V and W were expressed by the same equations, implying certain relations between the F-systems and frames of reference (p. 61).

The P-curves of T_4 are found by tracing the successive homologues of the F-points. Thus the double F-point O_{21}'' $\tilde{W}$ the line $O_{22}'O_{23}'$; and this $\tilde{V}$ the conic $O_{11}O_{12}O_{13}O_{22}O_{23}$, which is therefore a P-curve of T_4, partitioning the group of simple F-points; and O_{21}'' is associated with O_{21}. The simple F-point O_{11}'' of T_4 becomes the point O_{11}' in S' and the line $O_{12}O_{13}$ in S, which is a P-line of T_4; and O_{11}'' is associated with the double F-point O_{11}. The P-system of T_4 consists of three conics and three lines.

If with this T_4 we compound another T_2 between S'', S''' with F-points in general position, the second homaloidal net of the compound transformation T is the homologue of (ϕ_4'') by the last T_2, that is, (ϕ_8''') with three four-fold F-points at the new F-trio, and three double and three simple F-points at the homologues of the F-system of T_4^{-1}; we have T_8 $3^4 3^2 3^1$. By compounding μ T_2's, with general F-systems, we construct a transformation with the general characteristic (38, 143, 156, 381)

$$n = 2^\mu ;\ 3^{2^{\mu-1}} 3^{2^{\mu-2}} \dots 3^1,$$

as can be proved by induction.

Next, let V, W have in S' one common F-point $O_{11}' \equiv O_{21}' \equiv O_1'$ say. Then the lines of S $\tilde{V}$ conics $(\phi_2') \equiv O_1'O_{12}'O_{13}'$, through one F-point of W. On applying W, we drop a P-line, and the second homaloidal net of T is

$$(\phi_3'') \equiv O_1''^2 O_{22}''O_{23}''O_{12}''O_{13}'' ;$$

the characteristic is $n = 3 ;\ 1^2 4^1$.

The P-curves of T_3 are found from the successive homologues of the F-points:

O_1'' $\tilde{W}$ the line $O_{22}'O_{23}'$, and this $\tilde{V}$ the conic $O_1O_{12}O_{13}O_{22}O_{23}$;
O_{22}'' $\tilde{W}$ the line $O_1'O_{23}'$, and this $\tilde{V}$ the line O_1O_{23};
O_{12}'' $\tilde{W}$ the point O_{12}', and this $\tilde{V}$ the line O_1O_{13}.

Thus the first P-system of T_3 consists of a conic and four lines.

The line $O_{12}O_{13}$ is not a P-line of T_3; it $\tilde{V}$ O_1', and O_1' $\tilde{W}$ $O_{22}''O_{23}''$. Though $O_{12}O_{13}$ and $O_{22}''O_{23}''$ are P-lines of the component transformations, corresponding to the common F-point O_1' of V, W, they are a pair of ordinary corresponding lines of T, and their general points are neither fundamental nor principal, those points corresponding in T_3 which are associated, by V and W^{-1}, with the same direction at O_1'.

If with T_3 we compound a T_2 between S'', S''', with one F-point at O_1'' and the other two O_{32}'', O_{33}'' general, then from the homologue of ϕ_3'' we drop twice the P-line which $\sim O_1''$, and

$$(\phi_3'') \sim (\phi_4''') \equiv O_1'''^3 O_{32}'''O_{33}'''O_{22}'''O_{23}'''O_{12}'''O_{13}''',$$

giving the first quartic characteristic of Table I,

$$n = 4 ;\ 1^3 6^1.$$

Going on in this way, we construct a general transformation

$$T_J\ n ;\ 1^{n-1} (2n-2)^1,$$

as may be proved by induction.

If V, W have two common F-points in S', say $O_{11}' \equiv O_{21}'$, $O_{12}' \equiv O_{22}'$, two P-lines fall away from the homologue by W of ϕ_2', and the second homaloidal net of T is $(\phi_2'') \equiv O_{21}''O_{22}''O_{13}''$; then T is a general T_2, like V and W. Finally, if also $O_{13}' \equiv O_{23}'$, then T is of degree 1 (**204**).

The same argument applies in general. If V, W are any transformations, of degrees n_1, n_2, and if the two F-systems in S' are unrelated, we shall show that T is of degree $n = n_1 n_2$, and its first F-system, in S, consists of the first F-system of V, the multiplicity of each point being multiplied by n_2, together with a set of points in S of the same nature as the first F-system of W in S'; and the second F-system of T, in S'', consists of that of W, the multiplicity of each point being multiplied by n_1, together with a set of points in S'' of the same nature as the second F-system of V in S'.

The first homaloidal net (ϕ_n) of T arises from the lines of S'', and therefore $\bar{V}$ the first homaloidal net (ϕ_W') of W. Now this is of degree n_2; it is in general position in relation to V, and has certain simple and multiple base points $\Sigma O'_W{}^{i_W}$ at the F-points of W. The homologue of ϕ_W' by V is therefore of degree $n = n_1 n_2$, and it has the first F-system of V taken n_2 times (p. 4). Now O_W' is an ordinary point for V, neither fundamental nor principal; it follows that its homologue O_W is of the same nature as a point of ϕ_n as O_W' is of ϕ_W'.

3. Regular transformations (44, 209).

This depends on an important property, which holds of any 1, 1·transformation V, that in the immediate neighbourhood of an ordinary pair P, P' of homologues, V is *regular*; that is, if $P \sim$ a unique point P', and $P' \sim$ a unique P, so that P, P' are neither fundamental nor principal, then there exist two finite regions in S, S', with P, P' as interior points, between which V is 1, 1 without exception, and to that extent behaves like a linear transformation, equivalent to a mere change of the frame of reference.

Let the equations of V be

$$x' : y' : z' = \phi_1 : \phi_2 : \phi_3;$$

if O (0, 0, 1) and O' (0, 0, 1) are ordinary homologues, O lies on ϕ_1 and ϕ_2, but not on ϕ_3:

$$\phi_1 \equiv z^{n_1-1}u_1 + \dots, \quad \phi_2 \equiv z^{n_1-1}v_1 + \dots, \quad \phi_3 \equiv z^{n_1} + \dots.$$

In the neighbourhood of these points, V can be replaced to a first approximation by the homography

$$x' : y' : z' = u_1 : v_1 : z.$$

This also sets up a homography between the pencils of rays (O), (O').

Thus if f is any curve through O, and f' its homologue, the points of f adjacent to $O \sim$ points of f' adjacent to O', and the tangents to f at O to a first approximation $\sim$ the tangents to f' at O'. If O is multiple on f, then O' has the same multiplicity on f'; if O is an ordinary singularity

(p. 128), with distinct tangents, so is O'; if O is a higher singularity, the tangents coinciding in any way, O' is also higher, with the same numbers of coincident tangents.

4. Coincident F-points.

Now let the F-points in S' of V, W coincide in any way, those which do not coincide being general, and in particular not lying on any P-curve of the other transformation. The first homaloidal net (ϕ_n) of T in S still $\tilde{V}$ the first homaloidal net (ϕ_W') of W in S'.

Let O' be an ordinary common F-point of multiplicity i for W and i' for V; then from ϕ_n there is separated a P-curve j of V, of degree i', taken i times, and the degree of ϕ_n is lowered by $i'i$. There is no F-point of T in S answering to O'; also the number of branches of ϕ_n, through any F-point O_V of V in S, is lowered by $i \times$ the number of branches of j through O_V.

If we repeat these considerations for each common F-point in S', we find the exact nature of (ϕ_n), with its F- and P-systems. A compact method of tabulation is given in 270; see also 182.

If O' is an F-point without contact for each of V, W, the curve j is not principal for T; for $j\ \tilde{V}\ O'$, and $O'\ \tilde{W}$ a P-curve j'' of W; the points of each curve are associated in a 1, 1 relation with the rays of the pencil (O') (p. 16), and therefore with the points of the other curve. Thus for T, the curves j, j'' are ordinary homologues, and their general points are ordinary points.

But if O' is an F-point of contact for V, but not for W, then all the rays of (O') except one are associated by V with one point O of j, which is an F-point of T and $\tilde{T}$ the whole of j''; and all the general points of j are associated with one ray of (O') and $\tilde{T}$ one point O'' of j''; for T therefore, j, j'' are P-lines and O, O'' F-points.

Thus an ordinary common F-point of V, W disappears from T, but an F-point of contact for one or both of the components survives in general, and disappears only if it presents exactly the same conditions to (ϕ_V') and (ϕ_W').

5. Examples.

(i) Let V be a general T_J and W a T_2,

$$V\ n_1;\ 1^{n_1-1}(2n_1-2)^1, \quad W\ n_2=2;\ 3^1.$$

If all the F-points in S' are distinct, we have

$$n=2n_1;\ 1^{2(n_1-1)}(2n_1-2)^2 3^1,$$

or say

$$C2\ \ n;\ 1^{n-2}(n-2)^2 3^1,$$

provided n is even and $\geqq 4$. It exists also for $n=2$, reducing to T_2.

If one simple F-point is common to V, W, a P-line in S is dropped, joining the (n_1-1)-fold F-point of V to one of the $(2n_1-2)$ simple F-points, reducing by 1 the multiplicities for T of these two points, and also reducing by 1 the degree of T:

$$n=2n_1-1;\quad 1^{2(n_1-1)-1}\ 1^1(2n_1-3)^2\ 2^1,$$

which is of the same form $C2$, and exists provided n is odd and $\geqq 3$. Hence the characteristic exists for all values of $n \geqq 2$; it is the second general type given by Cremona (80), the first being $C1 \equiv T_J$.

In each case, the conjugate characteristic is found by carrying out the reverse combination $V^{-1}.W^{-1}$. Since each of T_2, T_J is self-conjugate, we merely take them in the other order, and find for the conjugate of $C2$, if n is even,

$$C3 \quad n = 2n_2; \quad 3^{n_2}\, 1^{n_2-1}\, (2n_2 - 2)^1;$$

and if n is odd,

$$C5 \quad n = 2n_2 - 1; \quad 1^{n_2}\, 3^{n_2-1}\, (2n_2 - 3)^1.$$

Neither of these exists for all values of n.

(ii) Let V be a T_2, and W a $C2$, its $(n_2 - 2)$-fold F-point being a simple F-point of V. We drop a P-line of V counted $n_2 - 2$ times, and obtain for T

$$n = 2n_2 - (n_2 - 2) = n_2 + 2; \quad 1^{n_2}\, 2^{n_2-(n_2-2)}\, (n_2 - 2)^2\, 3^1,$$

which is $C2$ again, with $n = n_2 + 2$. This furnishes another proof, by induction, of the existence of $C2$; for it shows that if it exists for any value of n, it also exists for $n + 2$. But for $n = 2$ and 3, it is identical with T_J; hence it exists for all higher degrees also.

(iii) This method can be generalized. For example, let there be given a characteristic

$$n = \alpha\mu - \beta; \quad 1^{n-\alpha}\, (2\mu - \gamma)^\alpha\, H_1,$$

where H_1 is some system of multiple and simple points, and the symbols are such as make $n - \alpha$, $2\mu - \gamma \geqq 0$. Then composition with a T_2, whose F-trio consists of the $(n - \alpha)$-fold point and two general points, leads to a characteristic of the same type, with $\mu + 1$ instead of μ. If we can prove the existence of the characteristic for α consecutive values of n, the type exists for all higher values also. For examples, see Table IV and p. 69.

(iv) Let T_μ be any given transformation and H, H' its F-systems; compound it with a T_2 with a general F-trio. We obtain the conjugate characteristics

$$n = 2\mu; \quad H^2\, 3^1, \quad n = 2\mu; \quad 3^\mu\, H'.$$

For example, in Table I there are five characteristics of degree 7, being one conjugate pair and three self-conjugate. These furnish ten of those of degree 14, first by doubling all the multiplicities and adding 3^1 ($n = 14$, nos. 2, 5, 16, 19, 45) and then by prefixing 3^7 (nos. 27—31).

(v) Let V, W be two T_J's of degrees n_1, n_2, with β of their simple F-points in S' coinciding, where we must have

$$0 \leqq \beta \leqq 2n_1 - 2 \text{ and } \leqq 2n_2 - 2.$$

In S there fall away β P-lines of V joining its multiple F-point to β of the simple F-points. The characteristic of T is

$$B \quad n = n_1 n_2 - \beta; \quad 1^{n_1 n_2 - n_2 - \beta}\, (2n_1 - \beta - 2)^{n_2}\, (\beta + 1)^{n_2-1}\, (2n_2 - \beta - 2)^1.$$

This was given by Bianchi (37), and is equivalent to the first of Table IV. It remains valid if $\beta = -1$, and can then be obtained by compounding a T_J of degree n_1 with one of degree $n_2 + 1$, with one of its simple F-points at the multiple F-point of T_{n_1}.

The conjugate characteristic is found by interchanging n_1 and n_2. If then $n_1 = n_2 = \mu$, we have

$$n = \mu^2 - \beta; \quad 1^{\mu(\mu-1)-\beta}\,(2\mu-\beta-2)^{\mu}\,(\beta+1)^{\mu-1}\,(2\mu-\beta-2)^{1}.$$

Here $$0 \leqq \beta \leqq 2\mu - 2; \quad \mu^2 - \beta \geqq (\mu-1)^2 + 1,$$

so there is one such characteristic of each degree, which, for $n > 3$, is different from T_J, and gives a second self-conjugate characteristic of general degree.

If we compound B with another T_J, of degree n_3, its simple F-points coinciding in any way with those of the last three groups in B, we obtain the very general characteristic P due to Palatini, containing seven parameters (305 and Table V).

(vi) Let W be the *same* transformation as V, that is, represented by the same equations, but with the planes S', S'' substituted for S, S' respectively. Then T is called the *square* of V (or in 113, the *derived* transformation)

$$T \equiv V^2.$$

Its degree is in general n^2, but is lower if two F-points of V of the different systems have the same coordinates in S, S', so that when V is repeated, there are coincident F-points in S'.

6. Contact conditions.

So far, we have assumed that the F-points of V, W in S' either coincide or are in general position. They may be related in other ways; these do not affect the characteristic of T, but give special varieties of the transformation.

A certain set of the points may lie together on a curve f' of lower degree than would be possible if they were general; we shall say that they are *related by* f'. If f' is not a P-curve of either V or W, each of the F-systems of T is related by the proper homologues f, f'' of f' in S, S'', from which in each case certain P-curves of V, W fall away; f, f'' are not P-curves of T.

If f' is a P-curve of V, but not of W, it is completely determined by its incidences with $\Sigma O_V'$, and if any point O_W' lies on it, the points are related by f'. As before, the second F-system of T is related by f''. In S, the homologue of f' is now an F-point O of V; if this is ordinary, the point O_W' of f' is associated by V with a certain line through O, and (ϕ_n) touches this line; hence O is an F-point of contact for T. If O_W' is an F-point of contact for W, then O is an F-point of contact of higher order for T than O_W' for W. There is no distinct F-point of T, other than O, answering to O_W'. In S'' both f'' and the P-curve of W that $\widetilde{W}\,O_W'$ are parts of the P-system of T that $\widetilde{T}\,O$, which breaks up further than in the general case (p. 18).

For example, if V, W are T_2's and one F-point of W lies on a P-line of V the characteristic of T is still $n=4$; $3^2 3^1$. In S, T_4 has contact of one branch at O, one of its double F-points, and has only two distinct simple F-points; in S'', two simple F-points are collinear with a double F-point. The P-curve of T which $\sim O$ breaks up into this line and the line through the other two double F-points; the intersection of these two P-lines is not an F-point.

Let the other two F-points of W coincide with two of V; the degree of T is reduced to 2, and it still has an F-point of simple contact, the other being distinct. This specialized T_2 with two adjacent F-points is therefore compounded of two general T_2's (p. 35).

Now let both V and W be special T_2's of this type, the two common F-points being adjacent, so that the two homaloidal families in S' have a common point of contact collinear with the remaining F-points, one of each family. Then T is still quadratic, and has only one distinct F-point of osculation. This further specialized T_2 is therefore compounded of two of the former type, and, if each of these is resolved, of four general T_2's.

7. The problem of resolution.

When T is resolved into the two simpler components V, W, the process is not complete if V or W can itself be resolved; we must carry the resolution as far as possible. By Noether's theorem, proved in Chapter VIII, T can always be resolved into a finite set of general T_2's. These cannot be resolved further: the only transformation simpler than a general T_2 is T_1, and the result of compounding any number of T_1's is always another T_1, and never a T_2.

But a resolution into T_2's is not unique; any one of them can be replaced by two others having two F-points in common (p. 58); hence any set of T_2's is equivalent to an infinite number of other sets. We must make restrictions on each step of the resolution, so as to define a process of *normal resolution*, that gives a unique set of T_2's as the components of a given T.

Now T may require simplifying in two respects. First, if any of its F-points are related, and especially if they are adjacent, T must be resolved into components each of which is of the most general type for its own characteristic. Secondly, if T, or any of its components, is of degree > 2, it must be resolved into general transformations of lower degree.

We considered above a case where T has an F-point O of contact of higher order than W, and V has an ordinary F-point at O, or as we shall say, V is *applied to* O; and the application partly dissipates the special nature of the F-system of T.

We defer to Chapter VII the proof that contact can always be completely dissipated, and the definition of the normal process of resolution of contact. We assume here that this part of the process has been carried out, and that T is general. The rest of this chapter deals with the resolution of characteristics rather than of particular transformations.

8. Normal resolution of a characteristic.

The normal resolution of a characteristic T_n is made unique by two restrictions on each step. First, since we want a final resolution into T_2's, we choose one of the components V to be itself a general T_2; secondly, since we want an immediate reduction of degree, we arrange that the other component $W \equiv T_{n'}$ shall be of as low degree n' as possible.

For this purpose, the positions of the three F-points of V are at our disposal. They must be distinct, since V is general; and, to avoid any speciality in $T_{n'} \equiv T_n V^{-1}$, we must choose them either coincident with F-points of T_n, or else in general positions. If there were no coincidences, the degree of $T_{n'}$ would have its maximum value $n' = 2n > n$, and T_n would not be resolved. If however the three F-points of V are of multiplicities i_1, i_2, i_3 for T_n, the degree of $T_{n'}$ is reduced to

$$n' = 2n - i_1 - i_2 - i_3,$$

and is a minimum when $i_1 + i_2 + i_3$ is a maximum, that is when the three F-points of T_n to which V is applied are the three of highest multiplicities or *top trio* of T_n. Then by Noether's inequality, $i_1 + i_2 + i_3 > n$, we have $n' < n$, and the degree is effectively lowered.

Thus the first step in the normal resolution of T_n is to apply a T_2 to its top trio, this being necessary to secure the minimum degree of the other component. This is always a suitable trio to determine V, with the present assumption that the F-points of T_n are distinct; on account of Noether's inequality $i_1 + i_2 + i_3 > n$, the top trio cannot be collinear, for a line through all three would meet ϕ in more than n points. If they were all adjacent, and not on a linear branch, the method would fail, for every conic through them would have a d.p. and would break up; V would not exist. This is precisely the point of difficulty in Noether's theorem (p. 144). But in dealing with characteristics only, these cases do not arise, and V is always defined.

In the case of F-points of equal multiplicity, it is indifferent to which we apply V, as they lead to identical characteristics for T_n; the choice of base homaloids for V is also indifferent.

The next step is to apply another T_2 to the top trio of $T_{n'}$, and so reduce the degree still further, and so on. The process continues as long as Noether's inequality holds, that is, as long as $n > 1$. We can therefore reach T_1, but at the stage before this, $n' = 2$ and the resolution into T_2's is already complete.

Since the degree is reduced from n to 2, by 1 or more at each stage, T_n is resolved into at most $n - 1$ T_2's.

Let the given characteristic be n; $i_1, i_2, i_3, i_4 \dots i_\sigma$; then that of $T_{n'}$ is obtained by the following operations:

replace n by $n' \equiv 2n - i_1 - i_2 - i_3$;

replace i_1, i_2, i_3 by $i_\alpha' \equiv n - i_2 - i_3$, $i_\beta' \equiv n - i_1 - i_3$, $i_\gamma' \equiv n - i_1 - i_2$;

rearrange $i_\alpha', i_\beta', i_\gamma'$ (omitting any that are 0), together with $i_4 \dots i_\sigma$, in descending order, and call them $i_1', i_2' \dots i'_{\sigma'}$.

If T_n is proper, the sum of any two i's $\leqq n$, hence each of i_α', i_β', $i_\gamma' \geqq 0$.

Thus every T_n belongs to a definite series of transformations

$$T_n,\ T_{n'},\ T_{n''},\ \dots\ T_2,$$

each consecutive pair being connected by a T_2. One set of F-points, of the T_2 connecting T_n and $T_{n'}$, is the top trio of T_n; the other set is a certain trio of F-points of $T_{n'}$, called a *base trio*. This is certainly not the top trio of $T_{n'}$, for that would connect $T_{n'}$ with $T_{n''}$, of lower degree, and not with T_n, of higher; and some or all of the trio may not be F-points of $T_{n'}$ properly speaking, but it is convenient to include them as F-points of multiplicity 0.

9. Test of degeneracy.

Now instead of the characteristic of a known transformation, let there be given one which may be merely arithmetical (p. 6), defining a degenerate family. The degeneracy may be clear on inspection; if $i_1 + i_2 > n$, or $i_1 + i_2 + \dots + i_5 > 2n$, a fixed line or conic is part of every homaloid. But unless the fixed component is of simple type, it may not be obvious, and we want a sure test.

Now the two characteristics T_n, $T_{n'}$ define the families (ϕ_n), $(\phi_{n'})$, homologues in T_2, which are either both proper, or both degenerate; or else the fixed part of one is a P-line of the T_2, when it does not $\sim$ a fixed curve of the other. In the last case, one family has a fixed line as component, and for this $i_1 + i_2 > n$, and consequently for the next, $i_\gamma' < 0$. From either of these facts the degeneracy is apparent.

If it does not appear at this stage, we carry on the process; if any characteristic contains a negative i, the original T_n is arithmetical; otherwise it is geometrical, belonging to the proper transformation compounded of the series of T_2's. This is the sure test of the arithmetical or geometrical nature of any proposed characteristic. For an example, see p. 69.

II. CONSTRUCTION OF THE TABLES

10. Arithmetical and geometrical methods.

To find all the characteristics of a certain degree n, given numerically, there are two methods: one arithmetical, based on the equations of condition, and the other geometrical, based on composition (p. 391).

In the first, we find all the solutions in positive integers of the equations of condition for the characteristic numbers. The advantage is that we can obtain each characteristic once and once only. The disadvantage, from which the method has never been freed, is that it takes no account of degeneracy, and gives all the arithmetical as well as the geometrical characteristics. We must test each solution and reject the degenerate ones, and it is waste of labour to have to consider these (80, 179, 180, 181, 336).

If H_1, H_2 are any two F-systems of transformations of the same degree, we know that Σi, Σi^2 have the same values for both. If H_1 occurs as part of the characteristic of any transformation of higher degree n, we can replace it by H_2, and the result still satisfies the equations of condition for degree n, but may be a merely arithmetical characteristic. For example, the two quartic F-systems are $1^3 6^1$ and $3^2 3^1$; it follows that in any characteristic we can replace $1^3 3^1$ by 3^2, or *vice versa*, and obtain another arithmetical or geometrical characteristic of the same degree. Many such substitutions can be obtained from Tables I—V; a few others are given in **38**.

The second method obtains the characteristics of higher degree by compounding a T_2 with those of lower degree, already tabulated. The advantage is that this leads only to proper transformations, the method itself giving the proof of existence. The drawbacks are that great care is required to ensure completeness, and also to avoid repetitions: the same T_n may arise many times over, corresponding to different resolutions. Also the number of lower characteristics, that have first to be tabulated, becomes very large as n increases. The numbers of different geometrical characteristics for

$n =$	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12,	13,	14,	15,	16
are	1,	1,	1,	2,	3,	4,	5,	9,	10,	17,	19,	29,	34,	51,	63,	88;

they are tabulated below (**80**, **61**, **336**, **264**, **181**, **203**, **267**, **395**, **239·3**).

Consider the single type B, p. 60. If n is given, let n_1 take all integral values such that $3 \leqq n_1 \leqq n^{\frac{1}{2}}$, and let $n_2 - 1$ be the quotient and $n_1 - \beta$ the remainder in $n \div n_1$; then $\beta \leqq n_1 < n_2$. We obtain a set of different characteristics of degree n, whose number is of the same order of magnitude as $n^{\frac{1}{2}}$ when n is large. Thus the total number of characteristics of degree n becomes infinite with n, of order at least $n^{\frac{1}{2}}$, and probably much higher.

11. Properties of base trios.

Since, in its normal resolution, every T_n arises from a definite $T_{n'}$, of lower degree, we are sure of obtaining every possible characteristic of degree n if we apply a T_2 to every trio of F-points of every $T_{n'}$ of degree $n' < n$ in turn. But this involves repetitions, which must be avoided.

Consider a definite characteristic n'; $i_1' \ldots i_\sigma'$, and a definite trio of its F-points i_α', i_β', i_γ', any of which may be 0. Let $i_\alpha' \geqq i_\beta' \geqq i_\gamma'$, and let i_δ' be the greatest of the other multiplicities of $T_{n'}$. A T_2 applied to this trio leads to a new characteristic, by means of the set of operations inverse to those of p. 63:

replace n' by $2n' - i_\alpha' - i_\beta' - i_\gamma'$,

replace i_α', i_β', i_γ' by $n' - i_\beta' - i_\gamma'$, $n' - i_\alpha' - i_\gamma'$, $n' - i_\alpha' - i_\beta'$,

rearrange these, together with $i_\delta' \ldots i_\sigma'$, in descending order, and call them $i_1, i_2, \ldots i_\sigma$.

If this is of degree n, we have the condition

$$n = 2n' - i_\alpha' - i_\beta' - i_\gamma'.$$

But a trio satisfying this need not be a base trio. The resolution of T_n into T_2 and $T_{n'}$, to which it belongs, is not the normal resolution unless the other F-points of T_2 are the top trio of T_n

$$i_1 = n' - i_\beta' - i_\gamma', \quad i_2 = n' - i_\alpha' - i_\gamma', \quad i_3 = n' - i_\alpha' - i_\beta',$$

and each of these $\geqq$ any of the remaining values of i. The latter are the same for T_n as for $T_{n'}$, and it is enough to take i_δ', the greatest of them, $\leqq i_3$ the least of the top trio

$$i_\delta' \leqq n' - i_\alpha' - i_\beta'.$$

If this condition does not hold, the T_n is sure to be obtained again, by the composition that answers to its normal resolution, and we do not want to consider it here. We therefore obtain each T_n once and only once if we consider, for each $T_{n'}$ with $n' < n$, each of the base trios i_α', i_β', i_γ', namely those which satisfy the two conditions

$$i_\alpha' + i_\beta' + i_\gamma' = 2n' - n < n',$$
$$i_\alpha' + i_\beta' + i_\delta' \leqq n'.$$

From the first of these, $n > n' \geqq \frac{1}{2}n$; we need only examine values of n' from $n-1$ to $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$, according as n is even or odd. If $n' = \frac{1}{2}n$, the only base trio is 0, 0, 0; that is, the F-systems of $T_{n'}$ and T_2 are unrelated. Then (p. 58) the characteristic of T_n is obtained from that of $T_{n'}$ by doubling the degree and prefixing $i_1 = i_2 = i_3 = \frac{1}{2}n$. In this case

$$i_\delta' \leqq n' - 1 < \frac{n}{2},$$

and T_n has no other $\frac{1}{2}n$-fold point (p. 74).

In the same way, if $n' = \frac{1}{2}(n+1)$, the only base trio is 1, 0, 0, provided $T_{n'}$ has a simple F-point; otherwise there is none. To obtain T_n, we prefix $i_1 = \frac{1}{2}(n+1)$, $i_2 = i_3 = \frac{1}{2}(n-1)$, and decrease by 1 the number of simple F-points.

If $n' = \frac{1}{2}n + 1$, the base trio can be either 2, 0, 0 or 1, 1, 0, provided $T_{n'}$ has one double or two simple points, and no $(n'-1)$-fold point; and so on, the number of possibilities first increasing with n' and then decreasing.

The case $n' = n-1$ need not be considered separately for each n; for then $i_1 + i_2 + i_3 = n+1$, which shows that T_n is of one of the well-known types T_J or T_{sym} (p. 10).

If n is not specified, there are always at least two base trios of $T_{n'}$; first 0, 0, 0, which gives the maximum value to each of

$$n = 2n', \quad \sigma = \sigma' + 3, \quad i_1 = n',$$

secondly, i_α', 0, 0 is always a base trio, where i_α' is any multiplicity of $T_{n'}$. For T_{sym} there are only the two, 0, 0, 0 and i', 0, 0; and for T_J the three,

$$0, 0, 0; \quad 1, 0, 0; \quad n'-1, 0, 0.$$

Compare the two inequalities

$$i_\alpha' + i_\beta' + i_\gamma' < n', \quad i_\alpha' + i_\beta' + i_\delta' \leqq n',$$

with Noether's inequality for $T_{n'}$

$$i_1' + i_2' + i_3' > n'.$$

It follows that the base trio cannot be the top trio of $T_{n'}$; nor can it contain any two of the top trio, for if so, i_δ' would be the third of the top trio, and the second inequality would fail. Thus a base trio contains at most one of the top trio.

The other two give two of T_n other than the top trio of T_n; each $\geqq$ any remaining i', and they are therefore i_4 and i_5, the next highest for T_n after the top trio. Thus if $n > 2$, there are at least five F-points (p. 72).

There follows an interesting extension of Noether's inequality (267). Each of i_4, i_5, belonging to the top trio of $T_{n'}$, $>$ each of the two lowest of the base trio, which do not belong to the top trio,

$$i_\beta' = n - i_1 - i_3, \quad i_\gamma' = n - i_1 - i_2.$$

Of these four inequalities, the most stringent, involving the other three, is $i_5 > n - i_1 - i_3$, that is,

$$i_1 + i_3 + i_5 > n \text{ provided } n > 2;$$

and we can replace i_3 by i_2, or i_5 by i_4, or both.

If $i_1 = i_2 = i_3$, and therefore $i_\alpha' = i_\beta' = i_\gamma'$, by the same argument, none of them belongs to the top trio of $T_{n'}$, which is i_4, i_5, i_6, and we have

$$i_6 > n - i_1 - i_3,$$

that is

$$2i_1 + i_6 > n \text{ provided } i_1 = i_2 = i_3 \text{ (271).}$$

12. Characteristics of general degree (80).

We have shown how to find all the characteristics when n is given numerically; but at present no method is known of obtaining them all algebraically in terms of n.

But we can obtain any number of general types, each valid for an infinite number of values of n of particular forms. In the characteristic n; $\Sigma\sigma_i^i$, the numbers n, i, σ_i are functions of certain parameters, which can take ∞ sets of values; each of these sets must make all the characteristic numbers integers $\geqq 0$, and there may be other restrictions.

As before, there are two methods of constructing them. By the arithmetical method we make arbitrary assumptions as to how the parameters enter; thence we obtain solutions of the equations of condition, and either verify the existence of the corresponding proper transformations, or reject the solutions (166).

The geometrical method, followed by most writers, is to compound known characteristics; parameters may enter in three ways, being either the numbers of the components, or parameters already contained in the components, or else a third set defining the incidences of the F-systems in the intermediate planes. Then n, i, σ_i are all polynomials in these parameters with integral values. Characteristics found in this way are all proper; but it is difficult to be sure of finding all the different varieties of any particular kind.

We have already had several examples (p. 59). Two of these, T_J and $C2$, are valid for all values of n; by the arithmetical method we can prove that there are no others.

We therefore assume that n itself is an unrestricted integral parameter; then i, σ_i are functions of n, which, since they are always integral, are polynomials, which if linear have integral coefficients. Now (pp. 5, 6, 11)

$$i_1 \geqq \tfrac{1}{3}(n+1), \quad 0 < i \leqq n-1, \quad i_\alpha + i_\beta \leqq n, \quad 0 < \sigma_i \leqq 2n-1.$$

Hence the polynomials are linear, the coefficient of n in i is 1 for one and only one point, with $i_1 = n - \alpha$, $\alpha > 0$, and 0 for every other point, with $i \leqq \alpha$. We can express the characteristic more closely:

$$n; \quad 1^{n-\alpha} \sum_{i=1}^{\alpha} (a_i n + b_i)^i;$$

we also have $a = 0$, 1 or 2.

Now substitute in the equations of condition

$$\Sigma i \sigma_i = 3n - 3, \quad \Sigma i^2 \sigma_i = n^2 - 1,$$

and equate coefficients of the parameter n,

$$1 + \Sigma i a_i = 3, \quad \ldots\ldots(1) \qquad -2\alpha + \Sigma i^2 a_i = 0, \quad \ldots\ldots\ldots(3)$$

$$-\alpha + \Sigma i b_i = -3, \ldots\ldots(2) \qquad \alpha^2 + \Sigma i^2 b_i = -1. \quad \ldots\ldots(4)$$

Now from (1), $\Sigma i a_i = 2$, giving two possibilities:

(i)	$a_1 = 2$, every other $a = 0$.	(ii) $a_2 = 1$, every other $a = 0$.
Then in (3)	$\Sigma i^2 a_i = 2, \quad \alpha = 1,$	$\Sigma i^2 a_i = 4, \quad \alpha = 2,$
(2)	$b_1 = -2,$	$b_1 + 2b_2 = -1,$
(4)	$b_1 = -2,$	$b_1 + 4b_2 = -5,$
		$b_1 = 3, \quad b_2 = -2,$

and the characteristics are

$T_J \quad n; \quad 1^{n-1}(2n-2)^1,$	$C2 \quad n; \quad 1^{n-2}(n-2)^2\, 3^1.$

Thus T_J and $C2$ are the only two characteristics valid for all values of n.

By compounding either with T_2, if it is applied to the highest F-point and any others, we obtain the same types; and if T_2 is applied to F-points of multiplicities 2, 1 or 0, we obtain new types, valid for n even, or n odd, according as the sum of the three multiplicities is even or odd. Several types appear more than once. This is the set marked R in Table II, being given by Ruffini (336). It is remarkable that there are more even types than odd.

Cremona (80) gives 49 general types, of which 44 are essentially distinct; the equations for some of them are given in 141.

In order to prove that these include all characteristics valid for every even n, we now assume $n = 2\mu$, and i, σ_i polynomials in μ, instead of in n, and find all the solutions in positive integers of the equations of condition as before. These include, as they must, all the types C and R for $n = 2\mu$;

and also others, which must be tested for degeneracy. We find they must all be rejected.

An example of such a solution is

$$C24\quad n=2\mu;\quad 1^{n-4}(\mu-2)^4\,1^3 6^1.$$

Resolve it normally, applying a T_2 to the top trio $n-4$, 4, 4; it is reduced to

$$n'=2\mu-4;\quad 1^{n'-4}(\mu-4)^4\,1^3 6^1,$$

of the same type with $\mu-2$ instead of μ (p. 60, iii). There is still no degeneracy apparent, so we repeat the process as long as there remain two 4-fold points, and stop with $n'=4$ or 6 according as μ is even or odd.

If μ is even, the existence of the proposed T_n depends on that of $n=4$; $1^3 6^1$, which is proper; but if μ is odd, on that of $n=6$; $1^2 1^4 1^3 6^1$, which is degenerate, since here $i_1+i_2=4+3>n$. Thus the T_n exists only if μ is even, that is, if n is of the form 4μ, and not merely 2μ, where μ is any positive integer.

The same methods apply when $n=\alpha\mu+\epsilon$, $\epsilon=0, 1, \ldots \alpha-1$, for $\alpha=2, 3, 4\ldots$, and also when α, ϵ are parameters. We find (166 and Table II)

2 solutions for $\alpha=1$,
11 for $\alpha=2$, namely 6 for $n=2\mu$ and 5 for $n=2\mu+1$,
50 for $\alpha=3$, namely 18 for $n=3\mu$ and 16 for each of $n=3\mu+1$, $3\mu+2$.

For $\alpha=4$ the number is much larger. Specimens are given in Table III. There are also 12 types when α is general (Table IV).

In the last degenerate case, by following the series of T_2's in reverse order, we find that, if $\mu>3$, the fixed part of $(\phi_{2\mu})$ is a monoid of degree $\frac{1}{2}(\mu-1)$, determined by the μ highest F-points of $(\phi_{2\mu})$. We verify that the number of intersections of this curve with ϕ is

$$\tfrac{1}{2}(\mu-3)(2\mu-4)+(\mu-2).4+3>\tfrac{1}{2}(\mu-1).2\mu,$$

which proves the degeneracy without having recourse to resolution.

From this a useful test can be constructed (p. 71), to exclude this particular form of degeneracy at the outset; but we cannot test for every possible form. As an example of its use, take Ruffini's more general characteristic (336)

$$R_m\quad n=\alpha\mu+\epsilon;\quad 1^{n-\alpha}(2\mu-2)^{\alpha}\,1^{\alpha+\epsilon-2}\,\{\alpha-\tfrac{1}{2}(\epsilon^2-3\epsilon+4)\}^2(\epsilon^2-2\epsilon+3)^1,$$

and consider for what values of ϵ it is valid. The characteristic numbers are all $\geqq 0$ if α is high enough.

From a line joining the points of multiplicities $n-\alpha$ and $\alpha+\epsilon-2$,

$$n-\alpha+\alpha+\epsilon-2\leqq n;\quad \epsilon\leqq 2.$$

From a monoid of degree $\mu-1$ determined by the $(n-\alpha)$-fold and the $2\mu-2$ points α-fold on ϕ,

$$(\mu-2)(n-\alpha)+(2\mu-2)\alpha\leqq(\mu-1)n,\ \text{where}\ n\equiv\alpha\mu+\epsilon,$$
$$\epsilon\geqq 0.$$

Hence the only values for ϵ are 0, 1, 2. These give the particular cases $\beta=0, -1, -2$ of the second characteristic of Table IV.

R_m arises by compounding T_J of degree μ with $C2$ of degree $\alpha+\epsilon$, the multiple F-point of T_J being of multiplicity ϵ for $C2$. Since ϵ has three values, the numbers of simple and double F-points in the result can be expressed as quadratic functions of ϵ.

13. Symmetric, semi-symmetric and asymmetric characteristics.

A transformation in which all the F-points are of the same multiplicity is called *symmetric* and denoted by T_{sym}. The characteristic consists of a single term $n;\ \sigma^i$.

The equations of condition give

$$i\sigma = 3n-3, \quad i^2\sigma = n^2-1,$$

whence

$$i=\tfrac{1}{3}(n+1), \quad \sigma = 9-6/i.$$

Hence i is a factor of 6, and $i=1, 2, 3$ or 6. There are therefore just four types of T_{sym}:

$$n=2;\ 3^1, \quad n=5;\ 6^2, \quad n=8;\ 7^3, \quad n=17;\ 8^6.$$

The first three were given in Cremona's tables (80) and the last by Bertini (26). See also 74, 264, 267, 336, 366.

A characteristic with just two different multiplicities is called *semi-symmetric* (271, 336). Most of these are compounded of two T_{sym}'s, one group of F-points being either wholly distinct from, or wholly contained in, the other. By compounding with the two alternately, we get a series of semi-symmetric characteristics, which either is ∞, or leads to a T_{sym}.

There are three others of general degree:

$$\begin{array}{lll} T_J & n=\mu+1; & 1^{\mu}(2\mu)^1, \\ R5 & n=2\mu+1; & 4^{\mu}\mu^2, \\ & n=\mu^2+1; & 1^{\mu(\mu-1)}(2\mu+1)^{\mu}, \end{array}$$

the last being compounded of two T_J's of degree $\mu+1$, all the simple F-points coinciding.

Certain particular cases of these, compounded with suitable T_{sym}'s, give other series of semi-symmetric characteristics. The only remaining types are

$$n=29;\ 2^{14}7^8, \qquad n=41;\ 5^{16}4^{10},$$

and the series compounded of the first of these with T_{sym} $n=8;\ 7^3$, and of the second with $R5$ $n=11;\ 5^2 4^5$.

The proofs, given in 271, depend on the limits for the numbers in the two top groups found below (p. 73).

Characteristics in which the multiplicities are all different are called *asymmetric* (187, 264, 267). Then every $\sigma_i=1$, and we do not write them. The simplest is

$$n=23;\quad 12, 10, 9, 8, 7, 6, 5, 4, 3, 2.$$

There are an infinite number. An example where the i are consecutive integers is

$$n=31;\quad 14, 13, 12, 11, 10, 9, 8, 7, 6.$$

III. PROPERTIES OF THE CHARACTERISTIC NUMBERS

14. The highest multiplicities.

The properties collected here have mostly been proved already.

(i) *Definitions.* We take the characteristic in the form

$$n;\quad i_1, i_2, \ldots i_\alpha, \ldots i_\sigma,$$

where σ is the total number of F-points, and the multiplicities are in descending order:

$$i_1 \geqq i_2 \ldots \geqq i_\sigma.$$

(ii) *Equations of condition.*

$$\Sigma i = 3n-3, \quad \ldots\ldots(5)$$

$$\Sigma i^2 = n^2-1. \quad \ldots\ldots(6)$$

(iii) *Degeneracy tests.*

point	$i_1 \leqq n-1,$
line	$i_1+i_2 \leqq n,$
conic	$i_1+\ldots+i_5 \leqq 2n,$
non-singular cubic	$i_1+\ldots+i_9 \leqq 3n,$
nodal cubic	$2i_1+i_2+\ldots+i_7 \leqq 3n,$
monoid	$(\mu-1)i_1+i_2+\ldots+i_{2\mu+1} \leqq \mu n$ if $2\mu+1 \leqq \sigma.$

(iv) *Limits for i_1.* Multiply (5) by i_1 and subtract (6):

$$(n-1)(3i_1-n-1) = \Sigma i_\alpha(i_1-i_\alpha) \geqq 0,$$

$$i_1 \geqq \tfrac{1}{3}(n+1).$$

The lower limit is reached if and only if $i_1 = i_\alpha$ for all values of α, and T_n is T_{sym}.

If the F-system consists of double and simple points only, $i_1 \leqq 2$, $n \leqq 5$ (337 and p. 8).

There is a better lower limit for i_1, in the important cases when O_2, O_3 are both adjacent to O_1, but not on the same linear branch, being either in different directions, or in the same direction lying on one cuspidal branch. These are the cases of failure of the original proof of Noether's theorem (p. 144), when the top trio cannot be used as the F-trio of a T_2.

Apply a T_2 to O_1 and two general points A_2, A_3. Then

$$\phi_n \sim \phi'_{2n-i_1} \equiv (A_2'A_3')^{n-i_1} O_2'^{i_2} O_3'^{i_3},$$

where O_2', O_3' are two points of the P-line $j' \equiv A_2'A_3'$, either distinct or adjacent. From the number of intersections of ϕ', j',

$$2(n-i_1)+i_2+i_3 \leqq 2n-i_1,$$

$$i_1 \geqq i_2+i_3 \geqq n+1-i_1.$$

Hence, in these exceptional cases,

$$i_1 \geqq \tfrac{1}{2}(n+1)$$

(128, 290 and p. 391).

(v) *The top trio.*

Noether's inequality $i_1 + i_2 + i_3 \geqq n + 1$.

Upper limit. If T_n is resolved normally,

$$n' = 2n - i_1 - i_2 - i_3,$$
$$n = 2n' - i_\alpha' - i_\beta' - i_\gamma' \leqq 2n',$$
$$i_1 + i_2 + i_3 \leqq \tfrac{3}{2} n.$$

This limit is reached if and only if $i_\alpha' = i_\beta' = i_\gamma' = 0$, and $i_1 = i_2 = i_3 = \frac{1}{2} n$.

From these we can show that no characteristic has just two multiple F-points. If T_n is

$$n;\quad 1^{i_1}\, 1^{i_2}\, (\sigma - 2)^1,$$

we have

$$i_1 + i_2 \leqq n,$$
$$i_1 + i_2 + 1 \geqq n + 1,$$

whence

$$i_1 + i_2 = n;$$

also

$$i_1 + i_2 + \sigma - 2 = 3n - 3,$$
$$\sigma = 2n - 1;$$

it follows that T_n is T_J, and has at most one multiple F-point (pp. 9, 76).

15. Total number of F-points.

Multiply (5) by n and subtract (6):

$$\sum_{\alpha=1}^{\sigma} i_\alpha (n - i_\alpha) = (n-1)(2n-1).$$

There are σ terms in Σ, and they can vary from $(n-1).1$ to $\frac{n}{2}\cdot\frac{n}{2}$; hence

$$2n - 1 \geqq \sigma \geqq 4(n-1)(2n-1)/n^2,$$

the two limits being reached respectively for T_J and T_{sym}, $n = 2\mu$; σ^μ, that is, T_2.

The first also follows since the first polar of ϕ_n is an $f_{n-1} \equiv \Sigma O^{i-1}$; its genus is

$$\tfrac{1}{2}(n-2)(n-3) - \Sigma \tfrac{1}{2}(i_\alpha - 1)(i_\alpha - 2) = 2n - 1 - \sigma \geqq 0.$$

This genus is the same in both planes (400).

But we can find better limits.

(i) *Lower limit for* σ. Multiply (6) by σ and subtract the square of (5):

$$\sigma(n^2 - 1) - 9(n-1)^2 = \sigma \Sigma i_\alpha^2 - (\Sigma i_\alpha)^2 = \sum_{\alpha, \beta = 1}^{\sigma} (i_\alpha - i_\beta)^2 \geqq 0,$$
$$\sigma \geqq 9 - \frac{18}{n+1}, \text{ since } n > 1,$$

the limit being reached only if $i_\alpha = i_\beta$ for all values of α, β, that is, if T_n is T_{sym}.

Thus if

$$n \geqq 2,\ \ 3,\ \ 4,\ \ 6,\ \ 9,\ \ 18,$$
$$\sigma \geqq 3,\ \ 5,\ \ 6,\ \ 7,\ \ 8,\ \ \ 9.$$

These limits are actually reached by characteristics with $n = 2, \ldots 16$, contained in Table I, and by T_{sym}, $n = 17$; 8^6.

The characteristics with eight or fewer F-points are finite in number, being of degree $n \leqq 17$. They are given in 74, 187, 267.

There are characteristics with just 9 F-points of degrees exceeding any limit; for example, take $n=8$; $3^4\,3^2\,3^1$, and apply T_2 to the lowest trio of this and each succeeding characteristic.

(ii) *Upper limit for* σ (267). The extreme upper limit, $\sigma = 2n-1$, is reached for T_J only; there are also three general types, $C\,2$, $C\,3$, $C\,5$, for which $\sigma = n+2$. We shall show that, except for these, $\sigma \leqq n-1$ if $n \geqq 14$.

In the normal resolution of T_n, the F-points of $T_{n'}$ are the same as those of T_n, except that the top trio i_1, i_2, i_3 of T_n, all > 0, are replaced by a base trio $i_\alpha', i_\beta', i_\gamma'$ of $T_{n'}$ some or all of which may $=0$, and fall out. Hence

$$0 \leqq \sigma - \sigma' \leqq 3.$$

If $\sigma - \sigma' = 0$, $\qquad n - n' > 0$.

If $\sigma - \sigma' = 1$, $\qquad i_\gamma' = 0, \quad n - n' > 1$,

unless $i_1 + i_2 + i_3 = n + 1$, and both T_n and $T_{n'}$ are T_J's.

If $\sigma - \sigma' = 2$, $\qquad i_\beta' = i_\gamma' = 0, \quad n - n' = n' - i_\alpha' > 2$,

unless $T_{n'}$ is either T_J or $C\,2$.

If $\sigma - \sigma' = 3$, $\qquad i_\alpha' = i_\beta' = i_\gamma' = 0, \quad n - n' = n' > 3$,

unless $n' \leqq 3$, and $T_{n'}$ is T_J.

Thus $n - \sigma > n' - \sigma'$ unless $T_{n'}$ is either T_J or $C\,2$, and then every succeeding characteristic in the normal resolution of T_n is of the same type as $T_{n'}$. Conversely, in the normal composition, that leads from T_2 to T_n, as soon as we depart from either of these types, they never recur, and after this, $n - \sigma$ increases at each stage.

Now T_J has three base trios (p. 66), leading to $C\,3$, $C\,5$, T_J respectively; and $C\,2$ has six, leading to $C\,2$ and to the five types R (Table II), for all of which $n > \sigma$ if $n > 10$. In the same way, $C\,3$, $C\,5$ can only occur once in the series, and they lead at the next stage to other general types, for all of which $n > \sigma$ if $n > 14$.

Thus as soon as we depart from the four general types for which $n - \sigma \leqq -2$, we have $n - \sigma$ increasing, and $\geqq 1$ for all except certain low values of n, for which it $=0$ or -1; and becomes $\geqq 1$ after one or two more stages. The highest of these exceptional cases is

$$C\,13 \quad n = 13; \quad 1^9\,5^4\,7^1 \quad \sigma = 13,$$

and they all occur in Table I.

16. The highest groups.

Let the numbers of F-points in the highest groups be

$$\sigma_\alpha, \quad \sigma_\beta, \quad \sigma_\gamma \ldots,$$

of multiplicities $\qquad i_\alpha > i_\beta > i_\gamma \ldots.$

Then $\qquad i_\alpha \equiv i_1, \quad i_\beta \equiv i_{\sigma_\alpha + 1}, \ \ldots; \quad \sigma_\alpha + \sigma_\beta + \ldots \equiv \sigma.$

(i) *Limits for* σ_α (195). (1) If $i_1 > \frac{1}{2}n$, then $\sigma_\alpha = 1$; for

$$i_2 \leqq n - i_1 < \tfrac{1}{2}n < i_1.$$

(2) If $i_1 = \frac{1}{2}n$, then $\sigma_\alpha \leqq 3$ (p. 66); for if $n = 2$, then $\sigma_\alpha = 3$; and if $n \geqq 3$, then $\sigma \geqq 5$, $i_5 \geqq 1$, and

$$4i_1 = 2n \geqq i_1 + i_2 + i_3 + i_4 + i_5 > i_1 + 3i_4, \quad i_1 > i_4.$$

The seventh in Table III, $R4$ and $C3$ are examples of the three values of σ_α.

(3) If $i_1 < \frac{1}{2}n$, then $\sigma_\alpha \leqq 8$; for if $i_9 \geqq 1$, then

$$i_1 \geqq \tfrac{1}{3}(n+1); \quad 9i_1 \geqq 3(n+1) > 3n \geqq i_1 + i_2 + \ldots + i_9 \geqq i_1 + 8i_9; \quad i_1 > i_9.$$

Or thus $\qquad i_1\sigma_\alpha \leqq \Sigma i$ which $= 3(n-1)$,

$$\sigma_\alpha \leqq 9\frac{n-1}{n+1}, \text{ which } = 9 - \frac{18}{n+1},$$

$$\leqq 8, \text{ and } = 8 \text{ only if } n \geqq 17,$$

for example, $T_{\text{sym}}\ n = 17$; 8^6. All lower values of σ_α occur in Table I.

Hence in every case $\sigma_\alpha \leqq 8$.

(ii) *Limits for* σ_β (195, 271). (1) The types $C2$, $R5$ show that if $\sigma_\alpha = 1$ or 4, then σ_β can take all values. We shall show that in all other cases $\sigma_\beta \leqq 8$.

(2) If $\sigma_\alpha = 2$,

$$i_1 = i_2 > i_3, \quad i_\beta = i_3, \quad 2i_1 + \sigma_\beta i_3 \leqq \Sigma i \text{ which } = 3n - 3.$$

This can be arranged

$$(\sigma_\beta - 8)\,i_3 + (n - 2i_1) + 4\,(i_1 + 2i_3 - n) < 0.$$

But now $\qquad n \geqq 2i_1, \quad i_1 + 2i_3 > n \quad$ (p. 67),

$$\sigma_\beta \leqq 7.$$

(3) If $\sigma_\alpha = 3$, 6, 7 or 8, apply to the group a T_{sym}, μ; $\sigma_\alpha{}^i$, for which $3i = \mu + 1$, $i\sigma_\alpha = 3\mu - 3$, and $J_{3\mu-3}$ passes $3i - 1 = \mu$ times through each point of the group (p. 17). Then $J^{i_\alpha}_{3\mu-3}$ drops away from the homologue of ϕ_n, and we obtain $\phi_{n'}$ on which the group is i'-fold, where

$$n' = n\mu - i_\alpha(3\mu - 3), \quad i' = ni - i_\alpha\mu, \quad n' - 3i' = 3i_\alpha - n > 0.$$

Hence i' is not the highest multiplicity for $T_{n'}$; the second group for T_n is the highest for $T_{n'}$, and $\sigma_\beta = 1, 2 \ldots$ or 8. By the converse process we can construct a T_n combining any of these values of σ_α, σ_β.

(4) If $\sigma_\alpha = 5$, the number of intersections of ϕ_n with an $f_5\ 5^2\ 5^1$, determined by the ten highest points, gives

$$5n \geqq 10i_1 + i_6 + \ldots + i_{10} \geqq 10i_1 + 5i_{10}, \quad n \geqq 2i_1 + i_{10}.$$

But now $i_1 = i_2 = i_3$, hence (p. 67) $2i_1 + i_6 > n$;

$$i_6 > i_{10}, \quad \sigma_\beta \leqq 4.$$

Examples of all possible values of σ_β with $\sigma_\alpha = 2$ or 5 are given by Table I together with the semi-symmetric types.

17. The complexity. (i) *Properties of* 2τ (264).

The complexity $2\tau \equiv n - i_1$ (p. 56) has its minimum 1 for T_J and its maximum $\frac{1}{3}(2n-1)$ for T_{sym}.

If $\alpha \geqq 2$, $$i_\alpha \leqq 2\tau;$$

hence if there are γ points, different from O_1, of multiplicity 2τ, either $i_1 = \frac{1}{2}n = 2\tau$, and the highest group consists of these $\gamma + 1$ points, or $i_1 > \frac{1}{2}n > 2\tau$ and the highest group consists of one point, and the second of these γ points.

We can find limits for σ in terms of γ and τ. Multiply the first equation of condition by 2τ and subtract the second,

$$\sum_{\alpha=1}^{\sigma} i_\alpha (2\tau - i_\alpha) = 6\tau(n-1) - (n^2 - 1).$$

In Σ, the first term is

$$i_1(2\tau - i_1) = (n - 2\tau)(4\tau - n);$$

the next γ terms vanish. Each of the $\sigma - 1 - \gamma$ other terms $\geqq 1\,(2\tau - 1)$ and $\leqq \tau^2$; hence

$$(\sigma - 1 - \gamma)(2\tau - 1) \leqq (2\tau - 1)(4\tau - 1) \leqq (\sigma - 1 - \gamma)\,\tau^2.$$

If T_n is not T_J, we have

$$\tau \geqq 1, \quad 2\tau - 1 > 0,$$
$$3 \leqq \sigma - 1 - \gamma \leqq 4\tau - 1,$$
$$4 + \gamma \leqq \sigma \leqq 4\tau + \gamma,$$

provided T_n is not T_J. The lower limit is reached if and only if $\tau = 1$, and T_n is $C\,2$, and the upper limit if and only if i_α has no value other than i_1, 2τ, $2\tau - 1$, 1.

We have also a relation between n, γ, τ. If $\gamma \geqq 2$, apply a T_2 to the top trio; we have

$$n' = n - 2\tau; \quad n - 4\tau,\ 2\tau, \ldots 2\tau,\ i_{\gamma+2} \ldots i_\sigma,\ 0,\ 0.$$

The number of 2τ-fold points is lowered by 2; as long as two or more remain, we repeat the process. The number of T_2's is the integral part of $\frac{1}{2}\gamma$, and is at least $\frac{1}{2}(\gamma - 1)$. The multiplicity of the first point must remain $\geqq 0$; hence

$$n - 2\tau - 2\tau \cdot \tfrac{1}{2}(\gamma - 1) \geqq 0,$$
$$n \geqq \tau(\gamma + 1).$$

The proof fails, but the result holds, if $\gamma = 0$ or 1.

(ii) *Properties of* τ (2, 67, 128, 239·1, 279, 290).

First we have $$\tfrac{1}{2} \leqq \tau \leqq \tfrac{1}{6}(2n - 1),$$

the limits being reached for T_J, T_{sym} respectively.

Also since $$i_1 \geqq \tfrac{1}{3}(n + 1),$$
$$i_1 > \tau.$$

Let β be the number of F-points, after O_1, of multiplicities $> \tau$; these are called *major* F-points. Let $\sum\limits_{(\beta)}$ refer to them, and Σ_0 to the remaining

$\sigma - \beta - 1$ F-points. We shall now prove $\beta \geqq 2$. This is a more precise relation than Noether's inequality, which follows at once; for

$$i_1 + i_2 + i_3 > n - 2\tau + \tau + \tau, \text{ which } = n.$$

Multiply the first equation of condition by τ and subtract from the second:

$$i_1(i_1 - \tau) + \sum_{(\beta)} i_a(i_a - \tau) - \Sigma_0 i_a(\tau - i_a) = n^2 - 1 - 3\tau(n-1)$$
$$= (i_1 + 2\tau)(i_1 - \tau) + 3\tau - 1.$$

Now in Σ_0, every term $\geqq 0$, and $3\tau - 1 > 0$,

$$\sum_{(\beta)} i_a(i_a - \tau) > 2\tau(i_1 - \tau).$$

Now $2\tau \geqq i_a$, and $i_1 \geqq i_a$; hence the R.H.S. $\geqq$ any one term in $\sum_{(\beta)}$; it follows that $\sum_{(\beta)}$ has more than one term,

$$\beta \geqq 2.$$

If $\tau \geqq 1$, the major F-points are multiple; hence every T_n has at least three multiple F-points, except T_J, for which $\tau = \frac{1}{2}$ (pp. 9, 72).

Further, since $$2\tau \geqq i_2 \geqq i_a,$$

$$(i_2 - \tau)\sum_{(\beta)} i_a \geqq \sum_{(\beta)} i_a(i_a - \tau) > 2\tau(i_1 - \tau)$$

which $$\geqq i_1(i_2 - \tau) + (2\tau - i_2)(i_1 - \tau) + \tau(i_1 - i_2)$$
$$\geqq i_1(i_2 - \tau),$$

and since $$i_2 - \tau > 0,$$
$$\sum_{(\beta)} i_a > i_1.$$

We shall use this result later (p. 151) in the form: if any set $\sum_{(\gamma)} O$ of γ F-points is such that $\sum_{(\gamma)} i < i_1$, then $\sum_{(\gamma)}$ does not include all the major F-points.

Also

$$2\tau \sum_{(\beta)} (i_a - \tau) \geqq \sum_{(\beta)} i_a(i_a - \tau) > 2\tau(i_1 - \tau), \text{ which } = 2\tau(n - 3\tau),$$
$$\sum_{(\beta)} i_a > n - 3\tau + \beta\tau$$
$$> n \text{ if } \beta \geqq 3;$$

hence if there are three or more major F-points, they are not collinear.

18. Properties of $\mathbf{T}_J$ and $\mathbf{T}_{\text{sym}}$.

Lower limit reached for	Inequality	Upper limit reached for
T_{sym}	$\frac{1}{3}(n+1) \leqq \quad i_1 \quad \leqq n - 1$	T_J
T_J, T_{sym}	$n + 1 \leqq i_1 + i_2 + i_3 \leqq \frac{3}{2}n$	
T_{sym}	$0 \leqq \quad i_1 - i_\sigma \quad \leqq n - 2$	T_J
T_{sym}	$9 - \dfrac{18}{n+1} \leqq \quad \sigma \quad \leqq 2n - 1$	T_J
T_J	$1 \leqq \quad 2\tau \quad \leqq \frac{1}{3}(2n - 1)$	T_{sym}

CHAPTER V

TRANSFORMATIONS IN ONE PLANE

I. PLANES SUPERPOSED

1. Definitions (114, 144).

When the planes S, S' are superposed, we become interested in invariant, involutory and self-corresponding elements, and especially in transformations that are wholly involutory. All these terms are defined exactly as for T_2 (p. 42).

While S, S' are distinct, it is natural to choose the frame of reference in each plane with special regard to the F-system in that plane, so as to have the equations of T in their simplest form. When S, S' are superposed, if we keep the same form of equations, x, y, z are the coordinates of P referred to one triangle $A_1A_2A_3$, and x', y', z' are those of the homologue P', which is a different point, referred to a different triangle $A_1'A_2'A_3'$. The two sets of coordinates are neither those of different points referred to the same frame, nor of the same point referred to different frames.

It is usually more convenient to sacrifice some of the simplicity of the equations, and to choose a common frame of reference for both S and S', either that suited to one F-system, or one symmetrical with regard to both, with points of reference either in general position or else at common F-points of H, H', if any, or at invariant or involutory points. The equations referred to the new frame are obtained from the old ones by making proper linear substitutions for both sets of coordinates. Thus the equations of T, when S, S' are superposed, are always compounded of the simplest form for separate planes, together with one or two linear transformations. We assume therefore that all the coordinates refer to the same frame.

Any point $P(x, y, z)$ of S coincides with a point P_1' of S'; by T_n it has a homologue P' in S', given by

$$x' : y' : z' = \phi_1(x, y, z) : \phi_2 : \phi_3,$$

and also, by T_n^{-1}, a second homologue P_1 in S, given by

$$x_1 : y_1 : z_1 = \phi_1'(x, y\ z) : \phi_2' : \phi_3'.$$

If there is a self-corresponding line, it contains F-points of either system of total multiplicity $n-1$. If there is a self-corresponding pencil of lines, the vertex is a common $(n-1)$-fold F-point of both systems, and the transformation is T_J, with coincident vertices.

Any pencil of curves (f) and the pencil of homologues are projective; their product contains the invariant system and therefore has any invariant curve as a part. The two pencils, homologues of (f) in T_n, T_n^{-1}, are projective with each other, and their product contains all the invariant and involutory systems.

2. Invariant system (80, 122·1, 177).

An invariant point D coincides with its homologue: the conditions are

$$x : y : z = \phi_1 : \phi_2 : \phi_3,$$

and D is an intersection of the two $(n+1)$-ics

$$f_1 \equiv x\phi_3 - z\phi_1, \quad f_2 \equiv y\phi_3 - z\phi_2.$$

The intersections of these which are not invariant are the F-points of H, common to ϕ_1, ϕ_2, ϕ_3, of equivalence $n^2 - 1$, and the n intersections of z, ϕ_3. In general, therefore, the number of isolated points D is

$$(n+1)^2 - (n^2 - 1) - n = n + 2.$$

If the two $(n+1)$-ics have a common part, every point of it is invariant, and it is the invariant curve, say

$$\Delta_\mu \equiv \Sigma O_\alpha{}^{\delta_\alpha} \, \Sigma O'_\beta{}^{\delta_\beta};$$

the number of isolated invariant points is reduced (p. 87).

Let D be an invariant point which is not an F-point and does not lie on z. If f is any curve containing the whole invariant system, then in the neighbourhood of D it contains all the intersections of f_1, f_2 and behaves like a member of the pencil $c_1 f_1 + c_2 f_2$.

If D is a simple intersection of f_1, f_2, it is a simple base point of the pencil, and f passes simply through D with a variable tangent. If D is a simple point of s-point contact of f_1, f_2, then f has s-point contact with either, and D is equivalent to s adjacent simple invariant points.

If D is a common i-fold point of f_1, f_2, it is of the same multiplicity on f, and is an *i-fold invariant point* of T; this is essentially different from any set of adjacent simple invariant points. We shall show that it cannot occur if T is involutory (p. 92).

3. Invariant directions (25).

Let f be any curve through an invariant point D, either isolated or lying on Δ. Then f' also passes through D, and the tangents are projective; for if another curve touches f at D, its homologue touches f'.

In general, this homography of the pencil of directions at D is proper, and has two double elements, the *invariant directions* at D, such that if f touches either, it also touches f'. Then D is an *invariant point of first kind*. If P is adjacent to D, so is P', in general in a different direction. If DP is an invariant direction, it $\equiv DP'$, but P, P' are distinct unless there is another invariant point adjacent to D in this direction.

If the homography is identity, every direction at D is invariant, and every curve through D touches its homologue; then D is *of second kind.*

If the homography is improper, every direction corresponding to a fixed direction t, then the pencil of lines $(D) \sim$ a pencil of homaloids (ϕ') all touching t, one of which therefore has a d.p. at D. Hence D lies on J' and is a P-point of S' which $\equiv$ its homologue, and is an F-point of S lying upon its corresponding P-curve, whose tangent is t.

Conversely, if D is an F-point O_α of H, it $\equiv$ one of its homologues and lies on j_α', being a P-point of S'. If D is a P-point of S, lying on j_β, it is either an F-point of S, or it $\equiv$ its unique homologue O_β'; hence it is an F-point of one or other plane, or of both.

If D lies on Δ, the tangent to Δ is one invariant direction. If the general points of Δ are of first kind, a line in the second invariant direction at D is called the *H-line* of D and its envelope the *H-curve*. If the general points of Δ are of second kind, there is an invariant curve to which every point of Δ is adjacent, that is, Δ may be regarded as two adjacent invariant curves. The definition of the H-line is extended to this case on p. 86. In general, we shall assume there are no such curves.

In either case, a finite number of points of Δ may be of opposite kind.

4. F-points on Δ (30, 50).

If a point O_α of Δ is an F-point of S but not of S', it lies on j_α'. Now j_α' is rational by virtue of its multiple points at $\Sigma O'$, and cannot have another one at O_α, but passes simply through it. If $P \rightarrow O_\alpha$ along a line l, so $P' \rightarrow$ a definite point Q' of j_α'; as l rotates about O_α, and $\rightarrow$ the tangent to Δ, so $Q' \rightarrow O_\alpha$ along j_α', and conversely. Hence Δ has only one branch through O_α; it does not touch j_α' in general, unless T_n is an involution. Any other intersection of j_α', Δ is an F-point of S', for it $\sim O_\alpha$ and also $\sim$ itself.

If O_α is multiple on Δ, it is an F-point $O_\alpha' \equiv O_\alpha$ of S'. As $l \rightarrow$ the different tangents to Δ, so $Q' \rightarrow O_\alpha$ along different branches of j_α', which has at least as many branches as Δ, hence $\delta_\alpha \leqq i_{\alpha\alpha} \leqq i_\alpha$.

Since $\Delta \sim$ itself, its degree μ and multiplicities δ_α, δ_β satisfy

$$\sum_\alpha \delta_\alpha i_\alpha = \sum_\beta \delta_\beta i_\beta' = \mu n - \mu,$$

$$\sum_\beta \delta_\beta i_{\beta\alpha} = \mu i_\alpha - \delta_\alpha, \quad \sum_\alpha \delta_\alpha i_{\beta\alpha} = \mu i_\beta' - \delta_\beta.$$

These express the numbers of intersections of Δ with ϕ_n, ϕ_n', j_α', j_β respectively.

5. Limit for the degree of Δ (114).

The pencil of conics through O_1, O_2, O_3, O_4 is projective with the corresponding pencil of degree $2n - i_1 - i_2 - i_3 - i_4$; unless each of these conics is self-corresponding, the product of the two pencils has Δ as a part:

$$\mu \leqq 2n + 2 - i_1 - i_2 - i_3 - i_4.$$

Since $$i_1 + i_2 + i_3 \geqq n + 1, \text{ and } i_4 \geqq 1 \text{ if } n \geqq 3,$$

$$\mu \leqq n \text{ if } n \geqq 3,$$

and $\mu = n$ only if $i_1 + i_2 + i_3 = n + 1$ and $i_4 = 1$, that is, for T_J.

If the conics are self-corresponding, a T_2 between S, S'', applied to O_1, O_2, O_3, reduces T to a transformation of S'' into itself, for which the lines of the pencil (O_4'') are self-corresponding, and which is therefore a central T_J vertex O_4''; and it converts Δ into the invariant curve of this T_J. Whether or not any of O_1'', O_2'', O_3'' are simple F-points of T_J, we still find $\mu < n$ for T, if $n > 3$.

Other properties of Δ are given in I, 56.

6. Involutory system (113, 185, 307).

The two homologues of an involutory point K coincide in the conjugate point $K' \equiv K_1$, but do not coincide with K. Thus $K \sim K'$ and $K' \sim K$, both in T_n and in T_n^{-1}. The pair form a cycle of index 2; each is invariant for T_n^2 (p. 42), and so are the invariant points of T_n. In general, T_n^2 is of degree n^2 and has $n^2 + 2$ invariant points, while T_n has $n + 2$ (p. 78). The number of involutory points of T_n is $n^2 + 2 - (n - 2)$ and the number of pairs is $\frac{1}{2}n(n-1)$.

This number can also be found from the equations expressing $K' \equiv K_1$:

$$\phi_1 : \phi_2 : \phi_3 = \phi_1'(x, y, z) : \phi_2' : \phi_3',$$

for the $4n^2$ intersections of the two $(2n)$-ics

$$\phi_1\phi_3' - \phi_3\phi_1', \quad \phi_2\phi_3' - \phi_3\phi_2'$$

consist of the $n(n-1)$ involutory points, $n+2$ invariant, $2(n^2-1)$ fundamental of the two systems, and n^2 intersections of ϕ_3, ϕ_3'.

If any F-point of H' coincides with one of H, the degree of T_n^2, and the number of involutory pairs of T_n, are lowered. A curve invariant for T_n^2 but not for T_n is an involutory curve of T_n. Since the first $(2n)$-ic contains at least one general point, the free intersection of ϕ_1, ϕ_3, the involutory curve is of degree $\leqq 2n - 1$; this limit is only reached for T_J (p. 102).

Involutory pairs can be classified according to kinds and multiplicities, defined as those of the invariant points of T^2 that arise from them.

If T_n^2 is identity, its invariant locus is the whole plane; this is then the involutory locus of T_n, which is an involution. In every case, the involutory system of T_n is found by removing the invariant system of T_n from that of T_n^2.

In almost all investigations about the involutory system, the invariant system occurs along with it, for an invariant point satisfies the involutory definition except for the negative condition of not coinciding with its homologue.

The involutory definition is also satisfied by a common F-point $K \equiv O_\alpha \equiv O_\beta'$ of H and H', and a point K' of intersection of the two corresponding P-curves j_α', j_β; such points are *improperly conjugate* and do not occur in general. If $j_\alpha' \equiv j_\beta$, this curve and O_α form an improperly involutory system, and j_α' is part of the involutory locus (p. 48).

The two homologues of a line through K meet in K' but do not touch in general. Their tangents set up two homographies between the pencils of directions (K), (K'), which either have just two common pairs, or coincide, according as K, K' are of first or second kind. If any curve touches one of these common directions at K, its two homologues touch at K', and the tangents are said to be in *conjugate directions*. If K, K' are multiple involutory points, the two homologues of any point adjacent to K not only are collinear with K', but also coincide with each other to a higher approximation.

7. Class (50).

The *class* ν of a transformation is the number of distinct pairs of homologues lying on a general line l. If T_n is not an involution, l does not contain any involutory pair; if a special line contains a pair K, K', this must be reckoned twice. But for an involution, the homologues lying on l all fall into involutory pairs, and the *class of an involution* is defined as the number of such pairs, each reckoned once.

These two definitions are inconsistent, except for $\nu = 0$, and confusing, and we shall as far as possible avoid using the former.

Now l meets Δ in μ invariant points lying on ϕ' the homologue of l; the $n-\mu$ other intersections of l, ϕ' are the points P' of pairs P, P' on l. Thus for a T_n which is not an involution,

$$\nu = n - \mu,$$

but for an involution $$2\nu = n - \mu.$$

As P describes l, so PP' envelopes a curve, the tangents to which, from any point X of l, are XX' and l counted ν times (2ν for an involution). The class of the curve is therefore $\nu + 1$ (or $2\nu + 1$). If $\nu = 0$, this $= 1$, and PP' passes through a fixed point: the transformation is central, and is T_J. This also follows since if $\nu = 0$, then $\mu = n$.

All non-involutory transformations of classes 0, 1 and 2 are obtained in 114, and those of class 3 in 1.

8. Isologues (25, 80, 177).

The homologues, in T_n^{-1} and T_n, of a pencil of lines (l) through a point X, are two pencils (ϕ), (ϕ'), selected from the nets of homaloids by the additional base points X_1, X' respectively. The three pencils are projective, and their products are certain interesting curves. The most important are the *isologues* Ω_X, Ω_X', defined as being Ω_X the locus of P, and Ω_X' of P', where P, P' are collinear with X, which is called the *centre* of the isologues. These are generated by (l), (ϕ) and (l), (ϕ') respectively; it follows that $\Omega_X \sim \Omega_X'$, and it is natural to think of Ω_X as lying in S, and Ω_X' in S'. The idea is extended to repetitions of T_n in 184.

If l contains an invariant point, this lies on ϕ and on ϕ'; hence Δ is part of the products, and Ω_X is of degree $n+1-\mu$, and passes through all the isolated invariant points. The intersections, other than invariant points, of l with its homologues ϕ, ϕ' form the $n-\mu$ pairs of homologues P, P' on l.

Considering the base points of the pencils, we see that $\Delta\Omega_X \equiv \Sigma O_a^{i_a} XX_1$. If $P \equiv O_a$, then XP contains i_a points P' on j_a'; thus O_a is i_a-fold on the product. Since $\Delta \equiv \Sigma O_a^{\delta_a}$, and does not contain X, X_1, we have

$$\Omega_X \equiv \Sigma O_a^{i_a-\delta_a} XX_1, \text{ and similarly } \Omega_X' \equiv \Sigma O'_\beta{}^{i_\beta'-\delta_\beta} XX'.$$

One position of P is at X, if and only if $l \equiv XX'$; hence Ω_X passes simply through X and touches XX'; similarly Ω_X' touches XX_1.

9. M-, N- and L-curves (114, 144).

The projective pencils (ϕ), (ϕ') generate a curve M_X called the *M-curve* of X, locus of points P such that P_1, P' are collinear with X; it belongs equally to S, S'. The product contains all the invariant and involutory points and curves, and these curves fall away from the net of proper M-curves. If there is no involutory curve, the degree is $2n-\mu$, and

$$M_X \equiv \Sigma O_a^{i_a-\delta_a} O'_\beta{}^{i_\beta'-\delta_\beta} (X_1, X')^1,$$

but does not in general contain X.

The points N common to Ω_X, Ω_X' are such that N, N_1, N', X lie on a line L. Thus with every point X is associated a set of points N and lines L; as X varies, these describe the *N-curve* and envelope the *L-curve* respectively. The N-curve is the locus of points common to a line and both its homologues, and also the locus of a point that lies on its own M-curve; the L-curve is the envelope of a line containing a point and both its homologues.

Each set of isologues forms a net. If T_n is an involution, Ω_X, Ω_X' coincide. If not, as X describes a line l, so Ω_X, Ω_X' describe projective pencils, said to be *related* to l, and generate N. The product is of degree $2(n+1-\mu)$; of this, l is a part, and also Δ, for any point of Δ lies on both isologues when X is the intersection of l with the H-line of the point. The degree of N is therefore $2n+1-3\mu$; the isolated invariant points are double on N, being simple base points of each pencil, and it also contains the involutory points. The degree cannot < 0, hence for a non-involutory transformation

$$3\mu \leqq 2n+1, \quad n \leqq 3\nu+1,$$

giving a limit for the degree of a transformation of given class. Thus if $\nu = 1$, $n \leqq 4$; if $\nu = 2$, $n \leqq 7$. It is found that a T_4 of class 1 exists (p. 119) but no T_7 of class 2 (114).

For an involution this limit must be replaced by the higher one of p. 87,

$$n \leqq 6\nu + 2.$$

If N vanishes identically, every point is collinear with both its homologues, every L is self-corresponding, and T_n is a central T_J.

If X is (ξ, η, ζ), the conditions of collinearity give the equations

$$\Delta\Omega_X \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ x, & y, & z \\ \phi_1, & \phi_2, & \phi_3 \end{vmatrix}, \quad \Delta\Omega_X' \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ x, & y, & z \\ \phi_1'(x, y, z), & \phi_2', & \phi_3' \end{vmatrix},$$

$$\Delta M_X \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ \phi_1, & \phi_2, & \phi_3 \\ \phi_1', & \phi_2', & \phi_3' \end{vmatrix}, \quad \Delta^3 N \equiv \begin{vmatrix} x, & y, & z \\ \phi_1, & \phi_2, & \phi_3 \\ \phi_1', & \phi_2', & \phi_3' \end{vmatrix},$$

where M_X, N include any involutory curve.

The two curves f_1 and f_2, used on p. 78 to determine the number of points D when there is no Δ, are simply Ω_{A_2} and Ω_{A_1}; and those used on p. 80 to determine the involutory points are M_{A_2} and M_{A_1}.

If O (0, 0, 1) is an F-point of S but not of S' lying on Δ, it lies on ϕ_1, ϕ_2, ϕ_3, ϕ_1', ϕ_2', but not on ϕ_3'. Both $\Delta\Omega_X$ and ΔM_X touch $\xi\phi_2 - \eta\phi_1$; hence Ω_X, M_X touch each other at these points, but do not touch Ω_X'.

10. Behaviour at an invariant point (167·1).

Let D (0, 0, 1) be invariant but not fundamental; it lies on ϕ_1, ϕ_2, ϕ_1', ϕ_2', but not on ϕ_3, ϕ_3'. Let x, y be invariant directions, of which there are at least two. Then

$$\begin{aligned} \phi_1 &\equiv axz^{n-1} + \dots, & \phi_1' &\equiv a'xz^{n-1} + \dots, \\ \phi_2 &\equiv byz^{n-1} + \dots, & \phi_2' &\equiv b'yz^{n-1} + \dots, \\ \phi_3 &\equiv cz^n \quad + \dots, & \phi_3' &\equiv c'z^n \quad + \dots. \end{aligned}$$

From the leading terms in the identities

$$\frac{1}{x}\phi_1'(\phi_1, \phi_2, \phi_3) \equiv \frac{1}{y}\phi_2'(\phi_1, \phi_2, \phi_3) \equiv \frac{1}{z}\phi_3'(\phi_1, \phi_2, \phi_3) \equiv G,$$

and the similar set for G', we find

$$aa' = bb' = cc',$$

and relations between the other coefficients in $\phi_1 \dots \phi_3'$.

A curve touching $c_1x + c_2y \sim$ one touching the corresponding direction $c_1ax + c_2by$. If $a \neq b$, these are different unless either c_1 or $c_2 = 0$, and there are only the two invariant directions x, y. If $a = b$, every direction is invariant, and D is of second kind.

If $P(\xi, \eta, 1)$ is adjacent to D, then P' is approximately $(a\xi, b\eta, c)$, and is adjacent to D in the corresponding direction. In general, P, P' are distinct even if they lie in the same, invariant direction. There is an adjacent invariant point, on x say, if $b = c$; this is either isolated, or a point of Δ, which then touches x at D.

This is the condition that f_1 touches $c_1f_1 + c_2f_2$, which we have seen to ensure that D is equivalent to two adjacent simple invariant points (p. 78). For an example see p. 101.

If there are two invariant points adjacent to D in different directions, we must have $a = b = c$. These conditions make D an invariant point of second kind, and also a d.p. on f_1 and f_2 and therefore a double invariant point. In general, this is not to be considered as having any adjacent invariant points, except in the same sense as that in which a curve touches any line through a d.p. (p. 12).

In general, if D is of second kind, $a = b$; we can write

$$\phi_1 \equiv xz^{n-1} + u_2 z^{n-2} + \dots, \quad \phi_2 \equiv yz^{n-1} + v_2 z^{n-2} + \dots, \quad \phi_3 \equiv cz^n + u_1 z^{n-1} + \dots.$$

The leading terms in the identities give $cc' = 1$ and

$$\frac{1}{x}(cu_2 + u_2') \equiv \frac{1}{y}(cv_2 + v_2') \equiv u_1 + cu_1'.$$

Every curve through D touches its homologue; they osculate only if they have a certain definite curvature depending on the direction of the tangent. In particular, there are three special lines through D, given by $yu_2 - xv_2$, or equally by $yu_2' - xv_2'$, that osculate their homologues; these in a sense replace the invariant directions.

If D is double and $a = b = c$, then a curve osculates its homologue if and only if it touches one of the special lines.

In any case,

$$\Delta\Omega_X \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ x, & y, & z \\ axz^{n-1} + \dots, & byz^{n-1} + \dots, & cz^n + \dots \end{vmatrix},$$

$$\Delta M_X \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ axz^{n-1} + \dots, & byz^{n-1} + \dots, & cz^n + \dots \\ a'xz^{n-1} + \dots, & b'yz^{n-1} + \dots, & c'z^n + \dots \end{vmatrix}.$$

In general, $\Delta\Omega_X$ has one branch through D, touching

$$t \equiv (a - c)\,\eta x - (b - c)\,\xi y,$$

which is definite and variable with X, and therefore a tangent to Ω_X, not to Δ; and D is a simple base point of (Ω_X) and similarly of (Ω_X'), (M_X).

If DX is an invariant direction, either X lies on $\xi = 0$ say, and $t \equiv x$, or D is of second kind and $a = b$, $t \equiv \eta x - \xi y$; in either case $t \equiv DX$, and the line t touches Ω_X, Ω_X', M_X for any centre on t. The point P of t adjacent to D is such that P, P' are collinear with both X and D.

If t is definite and fixed for all centres, it is one of the invariant directions, x say, and $b = c \neq a$; there is an adjacent invariant point and D is either a point of Δ or a base point of contact of (Ω_X).

If $t \equiv 0$, then D is a multiple base point of $\Delta(\Omega_X)$, and $a = b = c$; if these $= 0$, then D is an F-point of S.

Similarly, the leading term of $\Delta^3 N$ is

$$xyz^{2n-1}\begin{vmatrix} 1, & 1, & 1 \\ a, & b, & c \\ a', & b', & c' \end{vmatrix};$$

if D is isolated and of first kind, the N-curve has a d.p. and touches the invariant directions. If D is isolated of second kind, the N-curve has a triple point.

Now let D be a point of Δ. The equations of transformation are satisfied if $\Delta=0$, $x' : y' : z' = x : y : z$; hence

$$\phi_1 \equiv x\phi_{n-1}+\Delta\psi_1, \quad \phi_2 \equiv y\phi_{n-1}+\Delta\psi_2, \quad \phi_3 \equiv z\phi_{n-1}+\Delta\psi_3 \quad (114),$$

where D lies on Δ but not on ϕ_{n-1}.

First let the general points of Δ be of first kind. Let x be the tangent to Δ at D, and y the H-line:

$$\Delta \equiv xz^{\mu-1}+\ldots, \qquad \phi_{n-1} \equiv z^{n-1}+\ldots,$$

$$\psi_1 \equiv a_1 z^{n-\mu}+\ldots, \qquad \psi_2 \equiv (b_2 x+b_3 y)\, z^{n-\mu-1}+\ldots, \qquad \psi_3 \equiv c_3 z^{n-\mu}+\ldots,$$

where $a=1+a_1$, $b=c=1$, and D is of second kind if $a_1=0$.

Now $$\Delta\Omega_X \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ x, & y, & z \\ x\phi_{n-1}+\Delta\psi_1, & y\phi_{n-1}+\Delta\psi_2, & z\phi_{n-1}+\Delta\psi_3 \end{vmatrix},$$

and we have the equation of the isologue alone,

$$\Omega_X \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ x, & y, & z \\ \psi_1, & \psi_2, & \psi_3 \end{vmatrix}, \text{ and similarly } \Omega_X' \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ x, & y, & z \\ \psi_1', & \psi_2', & \psi_3', \end{vmatrix}, \ \Delta N \equiv \begin{vmatrix} x, & y, & z \\ \psi_1, & \psi_2, & \psi_3 \\ \psi_1', & \psi_2', & \psi_3' \end{vmatrix}.$$

The leading term in Ω_X is $a_1\eta z^{n+1-\mu}$; if D is of first kind, $a_1 \neq 0$, and Ω_X does not pass through D unless $\eta=0$, when X lies on the H-line.

If D is of second kind, $a_1=0$, and D is a double invariant point. Now D lies on every Ω_X; being a base point of the net of isologues, it must be regarded as an isolated invariant point of first kind, adjacent to Δ. Since $\Delta \equiv$ its homologue, it touches one of the special lines at D, and we verify that x is a factor of u_2 and of $yu_2 - xv_2$.

Conversely, if a simple point $D \rightarrow$ a point of Δ, by a path not touching Δ, any curve containing D and Δ acquires a d.p., and D is a double invariant point in the limit. Any curve through D also contains a point of Δ, and has two points in common with its homologue, which it touches; hence D is of second kind. This is a case when D has another invariant point adjacent to it in every direction.

Now $$\Delta M_X \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ x\phi_{n-1} + \Delta\psi_1, & y\phi_{n-1} + \Delta\psi_2, & z\phi_{n-1} + \Delta\psi_3 \\ x\phi'_{n-1} + \Delta\psi_1', & y\phi'_{n-1} + \Delta\psi_2', & z\phi'_{n-1} + \Delta\psi_3' \end{vmatrix},$$

$$M_X \equiv \phi_{n-1}\,\Omega_X' - \phi'_{n-1}\,\Omega_X + \Delta \begin{vmatrix} \xi, & \eta, & \zeta \\ \psi_1, & \psi_2, & \psi_3 \\ \psi_1', & \psi_2', & \psi_3' \end{vmatrix};$$

the leading term is $(a_1' - a_1)\,\eta z^{2n-\mu}$, and in general M_X does not pass through D unless X lies on the H-line.

If however $a_1 = a_1'$, then M_X passes here for all positions of X. Then $a = a' = \pm 1$; with the upper sign, D is of second kind. With the lower sign, it is a point of Δ where the homography of directions is involutory; M_X touches the direction conjugate to DX.

The tangent to ΔN is $$\begin{vmatrix} 0, & y, & 1 \\ a_1, & b_2x + b_3y, & c_3 \\ a_1', & b_2'x + b_3'y, & c_3' \end{vmatrix};$$

by virtue of the identities between the coefficients, this reduces to a multiple of x the tangent to Δ, and D does not lie on the N-curve. If D is of second kind, $a_1 = a_1' = 0$; then ΔN has a d.p. and D lies on the N-curve.

If the general point D of Δ is of second kind, and is therefore double, then after dropping Δ once from the original determinants, every D still lies on the residuals; hence Δ can be dropped again, and the proper isologues are of degree $n + 1 - 2\mu$; and Δ also drops away from M_X. We can show that the valid forms are $\phi_1 \equiv x\phi_{n-1} + \Delta^2\chi_1$ etc.; that two of the special directions coincide with the tangent to Δ, and that if X lies on the third, which replaces the H-line, then Ω_X touches Ω_X' at D. It follows that Δ drops away twice from the pencils (Ω_X), (Ω_X') that generate the N-curve, whose degree is $2n + 1 - 6\mu$. In all these ways, Δ behaves like two adjacent invariant curves. For an example, see p. 103.

11. Isologues of invariant and fundamental points (30, 50).

If X is an isolated invariant point D (0, 0, 1), then

$$\Delta\Omega_D \equiv x\phi_2 - y\phi_1, \qquad \Delta M_D = \phi_1\phi_2' - \phi_2\phi_1',$$

and the leading terms are

$$(b - a)\,xyz^{n-1}, \qquad (ab' - ba')\,xyz^{2n-2},$$

and both Ω_D and M_D have d.p.'s at D, touching the invariant directions, if D is of first kind, or triple points if D is of second kind.

If $X \equiv O\,(0, 0, 1)$ is an F-point of H but not of H', it is i-fold on ϕ_1, ϕ_2, ϕ_3, but does not lie on $\phi_1', \phi_2', \phi_3'$; and ϕ_1', ϕ_2' have j' as a factor. Then $\Delta\Omega_O \equiv x\phi_2 - y\phi_1$ has O as an $(i + 1)$-fold point, and Ω_O has one more branch through O than a general isologue; $\Delta j'\Omega_O' \equiv x\phi_2' - y\phi_1'$ has j' as a component; the residual $\Delta\Omega_O'$ is generated by the pencil of lines (O) and their homologues (ϕ'_{n-i}). Since O is not an F-point of H', it is not a base point of (ϕ') and is simple on Ω_O'. If this vanishes identically, every line through O is self-corresponding, and the transformation is a central T_J. If not, Ω_O' is of degree $n+1-i-\mu$; we have another limit for μ:

$$\mu \leqq n+1-i \leqq \tfrac{2}{3}(n+1).$$

For a non-involutory transformation, this is a worse limit than that of p. 82; but it is the best available for an involution, giving

$$n \leqq 6\nu + 2.$$

12. Incidences and intersections (50, 114).

We have the following table of multiplicities, assuming no invariant points of second kind, no involutory curve, and no coincident F-points of S, S'.

Curve	Degree	O_α	O_β'	D	K, K'	X	X_1	X'
Ω_X	$n+1-\mu$	$i_\alpha-\delta_\alpha$		1		1	1	
Ω_X'	$n+1-\mu$		$i_\beta'-\delta_\beta$	1		1		1
M_X	$2n-\mu$	$i_\alpha-\delta_\alpha$	$i_\beta'-\delta_\beta$	1	1		1	1
Δ	μ	δ_α	δ_β					
N	$2n+1-3\mu$	$i_\alpha-3\delta_\alpha$	$i_\beta'-3\delta_\beta$	2	1			

Other singularities of Ω, Ω', L are discussed in 142.

From the numbers of intersections of these curves we can prove some important relations.

(i) The other intersections of Ω_X, Ω_Y are the $n-\mu$ points P of the pairs P, P' on XY; hence the number of isolated invariant points D is

$$(n+1-\mu)^2-\Sigma(i_\alpha-\delta_\alpha)^2-(n-\mu)=n+2+\mu^2-3\mu-\Sigma\delta_\alpha^2,$$

by the relations of p. 79. This reduces to $n+2$ if $\mu=0$, as it should (p. 78), and shows that $\Sigma\delta_\alpha^2=\Sigma\delta_\beta^2$.

(ii) The other intersections of Ω_X, Δ are the points of Δ whose H-lines contain X; hence their number, which is the class of the H-curve, is

$$\mu(n+1-\mu)-\Sigma\delta_\alpha(i_\alpha-\delta_\alpha)=\Sigma\delta_\alpha^2-\mu(\mu-2).$$

(iii) These also lie on Ω_X', and the other intersections of Ω_X, Ω_X' are the points N associated with X. Their number is also that of L's through X, the class of the L-curve, and is

$$\begin{aligned}(n+1-\mu)^2-(n+2+\mu^2-3\mu-\Sigma\delta_\alpha^2)-1-\{\Sigma\delta_\alpha^2-\mu(\mu-2)\}\\ =n^2+n-2-\mu(2n+1-\mu).\end{aligned}$$

(iv) The variable intersections of M_X, Δ are the points (ii) where the H-line contains X. The fixed intersections which are simple base points of (M_X) are points of Δ where there is an involution of directions (p. 86); the number of these is

$$\mu(2n-\mu)-\Sigma\delta_\alpha(i_\alpha-\delta_\alpha)-\Sigma\delta_\beta(i_\beta'-\delta_\beta)-\{\Sigma\delta_\alpha^2-\mu(\mu-2)\}=\Sigma\delta_\alpha^2.$$

(v) The other intersections of M_X, M_Y, besides the $n^2-\mu$ intersections that do not lie on Δ of the two homologues of XY, are the involutory points. The number of these is

$$\begin{aligned}(2n-\mu)^2-\Sigma(i_\alpha-\delta_\alpha)^2-\Sigma(i_\beta'-\delta_\beta)^2-\{n+2+\mu^2-3\mu-\Sigma\delta_\alpha^2\}-\Sigma\delta_\alpha^2-(n^2-\mu)\\=n(n-1)-2\Sigma\delta_\alpha^2,\end{aligned}$$

reducing to $n(n-1)$ if there is no Δ as well as no involutory curve (p. 80).

(vi) The intersections of Ω_X, M_X are those of Ω_X, Ω_X', with X_1 in place of X, and also the F-points, those on Δ being points of contact, with $\delta_\alpha=1$. If $\underset{\Delta}{\Sigma}$ refers to these:

$$(2n-\mu)(n+1-\mu)-(n+1-\mu)^2=\Sigma(i_\alpha-\delta_\alpha)^2+\underset{\Delta}{\Sigma}(i_\alpha-\delta_\alpha)$$

$$\underset{\Delta}{\Sigma}(i_\alpha-\delta_\alpha)(\delta_\alpha-1)=0.$$

13. Other properties of N- and L-curves (114, 142, 144).

Since Ω_X touches $t\equiv XX'$ at X, and Ω_X' touches $t'\equiv XX_1$, if X is a point N_O of the N-curve, both t and $t'\equiv L_O$. Hence the N-curve is the locus of a centre whose two isologues touch at the centre, and the L-curve is the envelope of the common tangent. If X describes L_O, both (Ω_X), (Ω_X') have N_O as a base point, and the tangents t, t' to Ω_X, Ω_X' at N_O describe projective pencils. One coincidence occurs when $X\equiv N_O$, $t\equiv t'\equiv L_O$, and the other for a different position Y of X and t_Y of $t\equiv t'$. Now another of the N's associated with Y is adjacent to N_O on t_Y, and another of the L's is adjacent to L_O, meeting it in Y. But N_O, L_O are ordinary elements of their curves, hence t_Y is the tangent at N_O to the N-curve and Y is the point of contact of L_O with the L-curve. Hence the L-curve is the locus of centres whose two isologues touch, but not at the centre, and the N-curve is the envelope of the common tangent.

If N_O is an isolated invariant point D of first kind, then L_O is either of the invariant directions, and if X is any point of it, both Ω_X, Ω_X' pass through D and touch L_O. The homography (t, t') is improper, and Y coincides with D, which is the point of contact of each L_O. The L-curve has a d.p. at D, touching the invariant directions, and therefore touching the N-curve and Ω_D, Ω_D'.

If two distinct points N associated with X are collinear with it, they both lie on $L\equiv NX$ and on both its homologues, and are either invariant or involutory. A line D_1D_2 is an improper double tangent of the L-curve, joining two double points, and a line KK' is a proper double tangent.

14. Space construction for isologues (80, 177).

Project S, S' from any two points Y, Y' of space, and let YY' meet S in X. There is a T_n between the rays YP, $Y'P'$, of the two stars. Corresponding rays meet if and only if P, P' are collinear with X, and then they generate a twisted curve through Y, Y' whose degree is in general $n+2$. It meets S in the isolated invariant points of T_n, and its projections from Y, Y' on S are Ω_X, Ω_X' respectively.

II. INVOLUTIONS

15. Definitions (50).

There remains the important case when the involutory locus is the whole plane. Then T_n is an *involution*, and is denoted by I_n. Every point P belongs to one and only one involutory pair, and its conjugate $P' \equiv P_1$ is the homologue of P both in the direct transformation I_n and in its reverse I_n^{-1}, which coincides with I_n. If I_n is repeated, it brings every point back to its original position, and I_n^2 is identity:

$$I_n \equiv I_n^{-1}, \qquad I_n^2 \equiv 1.$$

The term *involution* has often a wider meaning, being defined as a set of groups of α points such that any point P of S belongs to one and only one group, and is thereby set in an $\alpha-1$, $\alpha-1$ relation to any other point of the same group. Then α is called the *degree* of the involution. We are only concerned with the case $\alpha=2$, when the relation is a Cremona transformation, and we shall continue to use *degree* to mean that of the homaloids.

16. Coincidence of F- and P-systems.

By I_n, an i-fold F-point O_α of $H \sim$ a P-curve j' of degree i; by I_n^{-1}, it has the same homologue, and is therefore also an i-fold F-point of H'. Thus H, H' consist of the same points with the same multiplicities, and coincide; I_n has only one F-system and one P-system, and its characteristic is self-conjugate.

Two coincident F-points of H, H' may or may not be associated (p. 24). If O_α is the only i-fold point, it is associated with itself, by the convention of associating single-member groups by order of multiplicity. If O_α is one of a group of several members, it is associated with the point O_β selected by j_α', both in I_n and in I_n^{-1}. Thus the F-points are either associated with themselves or with each other in pairs.

We order H, H' so that each F-point has the same suffix in each. Then $O_\alpha \equiv O_\alpha'$, and $i_{\alpha\alpha}$ is the multiplicity of O_α on its homologue j_α.

Thus when the characteristic is given, there may be different types of involution given by different associations of F-points. When the F-points and their associations are given, I_n may or may not be fully determined, as for example in the different types of I_2.

The class ν of an involution is the number of pairs P, P' on a general line (p. 81). The degree of the invariant curve Δ is $\mu = n - 2\nu$, and an involution of odd degree must have an invariant curve.

If $\nu = 0$, the transformation is a de Jonquières involution I_J, treated in Chapter VI and excluded here.

Expressed in terms of the class, the relations of p. 79 are

$$\sum_{\alpha} \delta_{\alpha} i_{\alpha} = (n-1)(n-2\nu), \quad \sum_{\gamma} \delta_{\gamma} i_{\alpha\gamma} = (n-2\nu)\, i_{\alpha} - \delta_{\alpha}.$$

17. F-points on Δ (30).

The δ_{α} branches of Δ through O_{α} all touch branches of j_{α}; for if Q is a point of Δ adjacent to O_{α}, its homologue coincides with Q and lies on j_{α}, and $O_{\alpha}Q$ is a common tangent to Δ, j_{α}.

Conversely, if Q is a point of j_{α} adjacent to O_{α}, then Q' is adjacent to O_{α} and lies on j_{α}, and $O_{\alpha}Q'$ touches one of the branches of j_{α}, either the same as or different from that touched by $O_{\alpha}Q$. If it is the same, Q is invariant and lies on a branch of Δ touching j_{α}; if it is different, the two branches are an associated pair. Let there be e_{α} such pairs, then

$$i_{\alpha\alpha} - \delta_{\alpha} = 2e_{\alpha} \geqq 0.$$

18. Isologues (30, 34).

The two isologues of X coincide in one self-conjugate curve

$$\Omega_X \equiv \Sigma O_{\alpha}^{\,i_{\alpha}-\delta_{\alpha}} \Sigma D,$$

of degree $n + 1 - \mu \equiv 2\nu + 1$. Thus every involution contains a net of self-conjugate curves.

Now (Ω_X) touches at O_{α} all the branches of j_{α} that do not touch Δ; for if O_{α} is $(0, 0, 1)$, then j_{α} is a common factor of ϕ_1, ϕ_2, and $\Delta\Omega_X$ touches $\xi\phi_2 - \eta\phi_1$, which has j_{α} as a factor. Thus the $2e_{\alpha}$ branches of j_{α} that do not touch Δ touch Ω_X, which also has $i_{\alpha} - \delta_{\alpha} - 2e_{\alpha} \equiv i_{\alpha} - i_{\alpha\alpha}$ tangents variable with X. Now unless j_{α} is a line through O_{α}, we have $i_{\alpha\alpha} \leqq i_{\alpha} - 1$; hence Ω_X has one or more branches through O_{α}, and all the F-points are effective base points of (Ω_X).

The excepted case, when $i_{\alpha\alpha} = i_{\alpha} = 1$, only occurs in I_J. For the P-line j_{α} joins two F-points the sum of whose multiplicities is n; one of them $\sim j_{\alpha}$, and is simple, hence the multiplicity of the other is $n - 1$.

We verify that the free intersections of Ω_X, j_{α} are the $i_{\alpha} - i_{\alpha\alpha}$ intersections, other than O_{α}, of $O_{\alpha}X$, j_{α}, while $i_{\alpha\alpha} - \delta_{\alpha}$ are due to contact at O_{α}:

$$\begin{aligned} \sum_{\gamma} i_{\gamma\alpha}(i_{\gamma} - \delta_{\gamma}) &= (2\nu+1)\, i_{\alpha} - (i_{\alpha} - i_{\alpha\alpha}) - (i_{\alpha\alpha} - \delta_{\alpha}) \\ &= 2\nu i_{\alpha} + \delta_{\alpha}. \end{aligned}$$

If X moves on a line l, then (Ω_X) describes the related pencil. If l is self-conjugate, it is part of Ω_X, and the whole pencil breaks up into the fixed part l and a pencil of self-conjugate curves of lower degree.

Conversely, if a pencil of isologues breaks up and has a fixed part, this is a line; for if P is a general point of the fixed part, neither fundamental nor invariant, PP' contains the centres of all the pencil of isologues, and the locus of P is the related line, which is the locus of P' also, and is self-conjugate. For example, the net of cubic isologues of the general I_2 has six pencils that break up into a line and a pencil of conics (p. 52).

The general isologue cannot break up; for if it does, all the parts are variable, and therefore belong to a certain pencil (f) (28). Every member of this is the fixed part of some pencil of isologues. Hence (f) is a pencil of self-conjugate lines, and I_n is central; but then each isologue consists of one line only.

A self-conjugate line l cannot meet a general Ω_X in a variable point P, unless X lies on $l \equiv PP'$, and Ω_X has l as a part. Conversely, a curve f meeting the net (Ω_X) in base points only is a set of self-conjugate lines; for if P is a general point of f, the pencil of isologues through P all break up, and have as a fixed part a branch of f through P.

19. Self-conjugate families (26).

We saw on p. 26, that there is an $\infty^{i_{\alpha\beta}}$ linear family of curves of degree i_β', defined by the same multiplicities at ΣO as j_β, except that they pass only $i_{\alpha\beta} - 1$ times through O_α; and that this family $\sim$ one of the same degree and multiplicities at $\Sigma O'$ as j_α', except that the curves pass only $i_{\alpha\beta} - 1$ times through O_β'.

Now for I_n, if $O_\alpha \equiv O_\beta'$, and therefore $j_\alpha' \equiv j_\beta$, the two families are the same, and it is self-conjugate; the individual curves either correspond in pairs, with $i_{\alpha\beta} + 1$ self-conjugate members, or are all self-conjugate.

One such family arises from each F-point for which $i_{\alpha\alpha} \geqq 1$, that is, which lies on its corresponding P-curve. Now we know (p. 21) that this holds for one of the three highest F-points; hence there is always at least one self-conjugate family of this sort. The only exception is the first I_2.

Any member of it cuts Δ in

$$i_\alpha (n - 2\nu) - \sum_\gamma \delta_\gamma i_{\alpha\gamma} + \delta_\alpha = 2\delta_\alpha$$

points other than F-points, and therefore meets its homologue in fundamental and invariant points of equivalence

$$\sum_\gamma i_{\alpha\gamma}^2 - (2i_{\alpha\alpha} - 1) + 2\delta_\alpha = i_\alpha^2 + 2 - 4e_\alpha.$$

Hence if $e_\alpha = 0$, the curve and its homologue coincide, and each member of the family is self-conjugate. For an example see p. 110.

20. Isolated invariant points.

At O_α, two general isologues Ω_X, Ω_Y have $2e_\alpha$ branches in contact, touching j_α. On account of this, the number of isolated invariant points of T_n, given by the formula of p. 87, must be reduced for I_n by $2\Sigma e_\alpha$. The result can be put in the form

$$2p_\Delta + n - \Sigma i_{\alpha\alpha},$$

where

$$p_\Delta \equiv \tfrac{1}{2}(\mu - 1)(\mu - 2) - \tfrac{1}{2}\Sigma\delta_\alpha(\delta_\alpha - 1)$$

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and is the genus of Δ when this has no multiple points except F-points.

The homography of directions at D is now an involution. As on p. 83, we take

$$\phi_1 \equiv \phi_1' \equiv axz^{n-1} + \dots, \quad \phi_2 \equiv \phi_2' \equiv byz^{n-1} + \dots, \quad \phi_3 \equiv \phi_3' \equiv cz^n + \dots$$

with the conditions $a^2 = b^2 = c^2$.

If D is of first kind, we can take $-a = b = c$. The two invariant directions are not on the same footing; there is an adjacent invariant point on x, but none on y, and Ω_X touches x, so that D is a base point of contact of (Ω_X).

If $a = b = -c$, then D is of second kind; Ω_X touches DX and D is an ordinary simple base point of (Ω_X).

If $a = b = c$, then I_n is identity; for, assume

$$\phi_1 \equiv x\phi_{n-1} + u_{\alpha+1} z^{n-\alpha-1} + \dots, \quad \phi_2 \equiv y\phi_{n-1} + v_{\alpha+1} z^{n-\alpha-1} + \dots,$$
$$\phi_3 \equiv z\phi_{n-1} + u_\alpha z^{n-\alpha} + \dots.$$

The leading terms in the identities give

$$\frac{1}{x} u_{\alpha+1} \equiv \frac{1}{y} v_{\alpha+1} \equiv u_\alpha,$$

and the same forms for ϕ are also valid for the next higher value of α, with $\phi_{n-1} + u_\alpha z^{n-\alpha-1}$ in place of ϕ_{n-1}. If $a = b = c$, they hold for $\alpha = 1$ and therefore for $\alpha = n$, and the equations of transformation reduce to

$$x' : y' : z' = x : y : z.$$

Thus an involution cannot have a multiple invariant point.

21. Isologue of an F-point (30).

If $X \equiv O_\alpha$, the isologue has an additional branch through O_α and j_α is dropped from it (p. 86), leaving the proper isologue of degree $2\nu + 1 - i_\alpha$:

$$\Omega_O \equiv O_\alpha^{i_\alpha - \delta_\alpha - i_{\alpha\alpha} + 1} \sum_{\gamma \neq \alpha} O_\gamma^{i_\gamma - \delta_\gamma - i_{\gamma\alpha}}.$$

Each of these multiplicities $\geqq 0$; hence

$$i_\alpha - \delta_\alpha - i_{\alpha\alpha} + 1 \geqq 0,$$
$$i_\gamma - \delta_\gamma - i_{\gamma\alpha} \qquad \geqq 0, \quad \gamma \neq \alpha.$$

Sum the second and add to the first:

$$3(n-1) - \Sigma\delta_\alpha - (3i_\alpha - 1) + 1 \geqq 0.$$

But $3i_\alpha$ can be $3i_1 \geqq n + 1$, hence

$$\Sigma\delta_\alpha \leqq 2n - 2,$$

giving an upper limit for the sum of the multiplicities of the F-points on Δ.

We have also

$$i_\alpha - \delta_\alpha - i_{\alpha\alpha} + 1 \leqq 2\nu - i_\alpha + 1,$$

that is,

$$\nu - e_\alpha \geqq i_\alpha - i_{\alpha\alpha}$$
$$\geqq 1, \text{ except for } I_J.$$

The nature of O_a as a base point of (Ω_X) is given by the multiplicity

$$i_\Omega \equiv i_a - \delta_a \equiv i_a - i_{aa} + 2e_a,$$

together with the number $2e_a$ of branches that touch fixed tangents, where

$$0 \leqq e_a \leqq \nu - 1; \quad 2e_a + 1 \leqq i_\Omega \leqq \nu + e_a.$$

22. Classification of involutions (30).

From the degree of Ω_O, we have already found a limit for the degree of an involution of given class (p. 87),

$$n \leqq 6\nu + 2.$$

$$\begin{array}{lllll}
\text{Thus for} & \nu = 1,\ \ 2, & 3,\ \ 4, & 5, & \nu \leqq 5 \\
 & n \leqq \underbrace{8,\ \ 14,} & \underbrace{20,\ \ 26,} & 32; & \\
\text{these are studied in} & \quad 30, & 226,\ 227, & 34,\ 300, & 275.
\end{array}$$

Each class is divided into *species* according to the nature of the net of isologues. The degree $2\nu + 1$ of the net is fixed, and the base consists of $\Sigma O_a^{i_\Omega}(2e_a)^1$, with $2e_a$ fixed tangents at O_a, and a number of simple points D, isolated invariant points of I_n. These may not completely define the net, and may have to be specially related.

Since ν is given, by the above inequalities, there are only a finite number of possible sets of values of i_Ω, e, and of possible types of base points of (Ω_X); also the total postulation and equivalence are limited, two general isologues having 2ν free intersections. Thus we can enumerate all possible bases for (Ω_X).

When the base is determined, the isologue of each F-point is a member having higher multiplicity there, and it contains the corresponding P-curve; hence the whole transformation is determined. If the base points are in as general position as possible, I_n has the maximum degree for the species. Other involutions of lower degree arise from special positions of the F-points, which cause (ϕ_n) to break up, but not (Ω_X).

The case $\nu = 0$ is I_J (p. 81).

If $\nu = 1$, then $2\nu + 1 = 3$, $e = 0$, $i_\Omega = 1$. The isologues are cubics passing simply through all the F-points. The cubic net must have 7 simple base points, and in general a proper cubic has a d.p. at each. The transformation exchanges the two free intersections of any two curves of the net, and is Geiser's $I_8\ 7^3$, or one of its degenerations (p. 123).

If $\nu = 2$, then $2\nu + 1 = 5$, and we can have

$$e = 0,\ i_\Omega = 1; \text{ or } e = 0,\ i_\Omega = 2; \text{ or } e = 1,\ i_\Omega = 3,$$

and the total equivalence is 21. The base of the quintic net consists of ordinary simple and double points, and triple points with two fixed tangents, of equivalences 1, 4, 11 respectively. The various possibilities are discussed in 30. The only sets that do not lead to some sort of contradiction are

$$1^3\,(2)^1\,10^1 \text{ and } 4^2\,5^1.$$

The first is shown to lead only to certain cases of I_J of degree ≤ 6. The second can be constructed from the pencils of cubics and conics with four common base points (275 and p. 126) and is an $I_{11}\, 4^5\, 5^2$ or a degeneration.

23. Construction by pencils (68, 366).

It has been shown that every plane involution, whether 1, 1 or not (p. 89), can be constructed as the set of groups of free intersections of two pencils of curves. Every plane involution, but not every multiple space involution, is rational (33, 57, 219, 292, 345, 367), that is, its groups can be set in 1, 1 relation to the points P_1 of a plane S_1. The equations expressing this give the coordinates a, b of P_1 explicitly as rational functions of the coordinates of any point of the corresponding group, and if a, b are given, these equations are satisfied by the points $P, P'\ldots$ of one group of the involution, and by no other points that vary with a, b. They represent two pencils of curves which generate the involution. Since we can effect any Cremona transformation of S_1 into itself, this is possible in ∞ ways.

For a Cremona involution I_n, there are just two free intersections; every curve of either pencil has a linear series of pairs of points cut out by the other pencil, and is rational, elliptic or hyperelliptic.

Let I_n be defined by the two pencils

$$f_\mu \equiv f_{\mu,1} + a f_{\mu,2}, \quad f_\nu \equiv f_{\nu,1} + b f_{\nu,2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The base points of the pencils are in general F-points of I_n. A common base point $O \sim$ the locus of the remaining intersection of curves f_μ, f_ν that touch at O, which is in general a P-curve, but may be a fixed point O', the ordinary homologue of O. If O is a base point of (f_μ) but not of (f_ν), then P' is any point of the one f_ν through O; but it may be that as $P \rightarrow$ a general point of f_ν, so $P' \rightarrow$ another definite point of the same curve, which is an ordinary self-conjugate curve of I_n, and O is not an F-point.

The other P-curves, if any, are common parts of curves of the two pencils, called *singular curves of the pair of pencils*. Let

$$f_{\mu,1} \equiv f_\delta f_{\mu-\delta}, \quad f_{\nu,1} \equiv f_\delta f_{\nu-\delta}.$$

According to the geometrical definition, if P belongs to the set made up of the points of f_δ and any free intersections X of $f_{\mu-\delta}, f_{\nu-\delta}$, then P' is any other point of the same set. Such a correspondence cannot exist in I_n; it can be shown that there is at most one point X in the set; if it exists, it is an F-point and $\sim f_\delta$; if there is no X, but $f_{\mu-\delta}, f_{\nu-\delta}$ meet at base points only, then f_δ is an ordinary self-conjugate curve of I_n.

In either case, f_δ drops away, from the homologue of any line l, once for each of the δ intersections of l, f_δ, and the degree of I_n is reduced by δ^2.

Further details are given in **68**. The degree of I_n is

$$n = 2\mu\nu - 1 - \Sigma\delta^2,$$

and the multiplicity as an F-point of a base point O is

$$i = \mu\beta + \nu\alpha - \Sigma\gamma\delta,$$

where α, β, γ are the multiplicities of f_μ, f_ν, f_δ at O, and the summations extend to all singular curves. There is also a δ-fold F-point which $\sim$ any f_δ whose residuals have a free intersection.

We can also obtain these expressions by the principle of correspondence, as is done for Geiser's I_8 on p. 124.

Examples are studied in 34, 275, 302, 303, 339, 341, 366, 409. One example of general degree is given by I_J, and another, with the characteristic $R5$, in 34. Details of many of these will be found in Chapter VI.

In other examples (39, 302, 303), the involutory pairs are the free intersections of curves $f_\mu \equiv f_{\mu,1} + af_{\mu,2}$ of a fixed pencil with those of a variable pencil, some of whose base points vary with a, usually the intersections of f_μ with a fixed curve. By eliminating a between the equations of f_μ and of the variable pencil, we deduce a fixed pencil cutting out the same involution on (f_μ), and the previous theory applies.

24. Construction by a net (68, 346, 375).

We can also construct I_n by a net of curves such that the general members have just two free intersections. This method reduces to the last, since P, P' are the free intersections of the two curves through P of the pencils selected from the net by arbitrary base points X, Y respectively.

Conversely, the pencils (1) are the variable parts of degenerate pencils of the same net, for example, the net of degree $\mu+\nu$,

$$c_1 f_{\mu,1} f_{\nu,1} + c_2 f_{\mu,2} f_{\nu,1} + c_3 f_{\mu,1} f_{\nu,2}, \qquad \text{.................}(2)$$

where
$$a = c_2/c_1, \quad b = c_3/c_1.$$

If there is a singular curve f_δ of the two pencils, a common factor of $f_{\mu,1}, f_{\nu,1}$, it drops away from the net, whose degree is reduced. Only if $\mu=\nu$, and the pencils have a common curve, is the degree of the net as low as that of each component. Hence though the general formulae of the net construction are simpler, the pencil construction is more manageable in most particular cases.

A singular curve of the net is the fixed part of any degenerate pencil in the net. Any curve singular for the pair of pencils (1) is also singular for the net (2). The net also has $f_{\mu,1}$, $f_{\nu,1}$ as singular curves; these are not P-curves, being general curves of the pencils. Conversely, the two pencils selected from the net by X, Y have the one curve through X, Y as an additional singular curve, for the pair of pencils but not for the net, which again is not a P-curve.

Change the notation, and let μ be the degree of the net and α, γ the multiplicities of f_μ, f_δ at a base point O; then the degree of I_n is

$$n = \mu^2 - 1 - \Sigma\delta^2,$$

and the multiplicity of O as an F-point is

$$i = \mu\alpha - 1 - \Sigma\gamma\delta,$$

the sums extending to all singular curves of the net. There is also a δ-fold F-point which $\sim$ each f_δ whose residual pencil has a free base point.

Let T be a 2, 1 transformation between two planes S, S''. Then a point P'', of the simple plane S'', $\tilde{T}$ a pair of points P, P' of the double plane S, which are thereby related in a Cremona involution I in S, associated with T; and P, P' are a pair of free intersections of the net of curves of S which $\tilde{T}$ the lines of S'' (101, 102, 103).

In 137, involutions are regarded as cut out on a family of lines by a family of conics, and an attempt is made to classify involutions according to the freedom of these families.

25. Reduction of involutions (25, 26, 293).

Let there be a T_2 between the plane S of I_n and another plane S_1. A general pair P, P' of points of S, conjugate in I_n, are transformed by T_2 into a pair P_1, P_1' of points of S_1, which are conjugate in the involution in that plane

$$I_{n_1} \equiv T_2 . I_n . T_2^{-1};$$

we verify that $\quad I^2_{n_1} \equiv T_2 . I_n . T_2^{-1} . T_2 . I_n . T_2^{-1} \equiv 1.$

To find its homaloids, start with a line l_1 in S_1; by T_2^{-1}, this $\sim$ a conic in S through the first F-trio of T_2; by I_n, this $\sim$ a curve f, of degree $\leqq 2n$, depending on the multiplicities for I_n of the F-points of T_2; and by T_2 finally, $f \sim \phi_{n_1}$, the homologue of l_1 by the compound transformation I_{n_1}, of degree $n_1 \leqq 4n$.

We also have $I_n \equiv T_2^{-1} . I_{n_1} . T_2$; if either of I_n, I_{n_1} is of lower degree or simpler type than the other, this is said to be *reduced* to that, by the auxiliary T_2.

If T_2 is applied to the top trio of I_n, then f is

$$f_{2n-i_1-i_2-i_3} \equiv O_1^{2i_1-i_{11}-i_{12}-i_{13}} \; O_2^{2i_2-i_{21}-i_{22}-i_{23}} \; O_3^{2i_3-i_{31}-i_{32}-i_{33}},$$

and the degree of the reduced involution is

$$\begin{aligned} n_1 = n - 3\,(i_1 + i_2 + i_3 - n) - (i_1 - i_{11} - i_{12} - i_{13}) - (i_2 - i_{21} - i_{22} - i_{23}) \\ - (i_3 - i_{31} - i_{32} - i_{33}) \\ = n - \sum_{\alpha=4}^{\sigma} (i_{\alpha 1} + i_{\alpha 2} + i_{\alpha 3} - i_\alpha). \end{aligned}$$

Now $i_{\alpha 1} + i_{\alpha 2} + i_{\alpha 3}$ is either the sum of the three highest multiplicities of j_α, or is less than this by 1; each term in $\Sigma \geqq 0$, and unless each $= 0$, the degree of I_n is reduced.

This process of reduction is different from that of normal resolution. The homaloidal net (ϕ_n) of I_n is transformed by T_2 not into (ϕ_{n_1}), but into a different homaloidal net, (ψ_1) say, in S_1, the homologue by I_{n_1} of the net of conics through the second F-trio of T_2. Then (ψ_1) determines a Cremona transformation $V \equiv T_2 . I_n$ between S and S_1, and I_n is resolved into V and T_2^{-1}; but V is not an involution. We have to distinguish between the resolution of an involution, regarded as a Cremona transformation between superposed planes, and its reduction, regarded as an involution in one plane.

Any system self-conjugate for I_n reduces to another system self-conjugate for I_{n_1}, and one involution cannot be reduced to another unless the linear systems of self-conjugate curves are of the same genera and degrees of freedom.

We have seen (p. 26), that arising from any F-point of I_n for which $i_{\alpha\alpha} \geqq 1$, there is a self-conjugate family

$$(f_{i_\alpha}) \equiv O_\alpha{}^{i_{\alpha\alpha}-1} \Sigma O_\beta{}^{i_{\alpha\beta}},$$

of genus $p \equiv i_{\alpha\alpha} - 1$ and freedom $q \equiv i_{\alpha\alpha}$. By the condition of p. 11, the degree of f can be reduced by a T_2, provided the highest value of $i_{\alpha\beta} \geqq i_{\alpha\alpha}$.

There is always some F-point for which $i_{\alpha\alpha} > 0$ (p. 91), and (f) exists.

(i) If $i_{\alpha\alpha} = 1$ for any F-point, then (f) is a rational pencil, reducible to a pencil of lines; then I_n is reducible to have a self-conjugate pencil of lines, and becomes I_J.

(ii) If $i_{\alpha\alpha} = 2$, then (f) is an elliptic net, reducible till all its base points are simple; then it must be cubic with seven base points. The involution consists of the pairs of free intersections of cubics of the net, and is Geiser's I_8 or one of its degenerations.

(iii) If $i_{\alpha\alpha} = 3$, then (f) is a web of genus 2, reducible till all its base points are double or simple. The only irreducible webs of this kind are

$$(f_4) \equiv 1^2 8^1 \text{ and } (f_6) \equiv 8^2.$$

In the first case, since the homologue of f_4 is also of degree 4,

$$4n - 4 = 2i_1 + i_2 + \dots + i_9 \leqq i_1 + 3n - 3,$$
$$i_1 \geqq n - 1,$$

and the type is I_J; similarly in the second case, $6n - 6 = 2(i_1 + \dots + i_8)$, and there are no more F-points. Since $\sigma = 8$, we have $n \leqq 17$ (p. 72), and the involution of highest degree of this species is Bertini's $I_{\text{sym}}\, n = 17; 8^6$ (p. 127).

(iv) If $i_{\alpha\alpha} > 3$, the argument is more laborious (26); it is shown that in every case I_n can be reduced to one of the four types:

$$I_1; \quad \text{central } I_J; \quad \text{Geiser's } I_8; \quad \text{Bertini's } I_{17}.$$

These are considered in the next chapter.

In particular, any I_n with no invariant curve can be reduced to I_1; it follows that I_n has just four isolated invariant points. For in the reducing transformation, the centre O of I_1 is not an F-point and the axis is a P-line, joining two F-points O_α, O_β; for if not, these would give rise to invariant curves of I_n. The only invariant points of I_n are the homologues of the centre and axis of I_1 and of points adjacent to O_α, O_β in the direction of O (30).

In this case therefore, $n-2$ of the $n+2$ isolated invariant points are absorbed by the F-points,

$$\Sigma i_{\alpha\alpha} = n - 2 \text{ (231)}.$$

CHAPTER VI

SPECIAL PLANE TRANSFORMATIONS

I. THE DE JONQUIÈRES TRANSFORMATION (80, 112, 114, 175, 177)

1. Definitions.

The transformation with the characteristic

$$n; \quad 1^{n-1}\,(2n-2)^1$$

is called, after its discoverer, the de Jonquières transformation, and we denote it by T_J or T_{Jn}. It exists for all values of n; the equations of condition are satisfied, and if the F-points are in sufficiently general position, they define a proper net of curves (ϕ_n), each having an $(n-1)$-fold point and therefore rational. Such a curve is called a *monoid*, and the multiple point its *vertex*. All lines, conics and nodal cubics are monoids; for degree $\geqq 4$, there are rational curves and homaloidal nets which are not monoidal.

Any homaloidal net of monoids defines a T_J: for since a linear family of curves cannot have a variable multiple point, the vertex is fixed, and is an $(n-1)$-fold F-point; this makes the curves rational, they have no other multiple F-point, and by the condition $\Sigma i = 3n-3$, the number of simple F-points is $2n-2$, and the transformation is T_J.

A ray of the pencil through the vertex O_1 meets the lines and also the homaloids of S in one variable point only. Its homologue is therefore a curve of a pencil meeting the homaloids and lines of S' in one variable point only, that is, a pencil of lines through an $(n-1)$-fold F-point O_1' of H'; hence (ϕ') also consists of monoids, the reverse transformation is a T_J, and the characteristic is self-conjugate.

Thus T_J transforms two pencils of lines into each other. Conversely, this property can define T_J; for since each of the lines $\sim$ a line, it meets the homaloids in one variable point only, and the vertices of the pencils are $(n-1)$-fold F-points of H, H'.

2. The P-system.

Let the first F-system be $H \equiv O_1^{n-1} O_2 \ldots O_{2n-1}$. The lines $O_1 O_\alpha$, joining the vertex to the simple F-points, meet (ϕ_n) in F-points only, and are the P-lines which $\sim$ the simple F-points $O_2' \ldots O'_{2n-1}$ of H'.

Now the whole Jacobian is

$$J_{3n-3} \equiv O_1^{3n-4}\,(O_2 \ldots O_{2n-1})^2;$$

the only remaining element is the P-curve which $\sim O_1'$; hence this is

$$j_{n-1} \equiv O_1^{n-2} O_2 \ldots O_{2n-1};$$

and we verify that this monoid is exactly determined, and meets (ϕ) in F-points only.

Thus H consists of two groups, one of the single point O_1, and the other of the $2n-2$ simple F-points. The first is associated with the single P-monoid, which does not partition any group, and with the vertex of (ϕ'). The second group is partitioned by the P-line O_1O_a into one F-point O_a lying on it and $2n-3$ not lying on it; the determination is positive, and O_a is associated with this P-line, and with its corresponding simple F-point O_a' of the second group of H'.

No line through O_1 can contain more than one simple F-point; for it would meet (ϕ_n) in more than n points.

3. Equations of $\mathbf{T}_J$.

Let O_1, O_1' be the points (0, 0, 1) in S, S'. For x, x' and y, y', take corresponding lines of the projective pencils (O_1), (O_1'), and let the unit points also lie on corresponding lines. Then

$$c_1x + c_2y \sim c_1x' + c_2y';$$

one of the equations of transformation is

$$x' : y' = x : y,$$

and we have

$$x' : y' : z' = xj : yj : \phi.$$

Since these are rationally reversible, ϕ and j are linear in z, and are monoids of degrees n, $n-1$ respectively:

$$\phi \equiv zu_{n-1} + u_n, \quad j \equiv zv_{n-2} + v_{n-1},$$

where u, v are functions of x, y only. We see again that if there are two corresponding pencils of lines, then (ϕ) is monoidal.

The reverse transformation is

$$x : y : z = x'j' : y'j' : \phi',$$

where

$$\phi' \equiv -z'v'_{n-1} + u_n', \quad j' \equiv z'v'_{n-2} - u'_{n-1},$$

and u', v' are the same functions of x', y' that u, v are of x, y. We find

$$J \equiv nj\,(u_{n-1}v_{n-1} - u_nv_{n-2}),$$

and J' has the same form.

The simple F-points in S are the intersections, other than O_1, of j, ϕ. The set of P-lines joining them to O_1 is

$$u_{n-1}v_{n-1} - u_nv_{n-2}.$$

Those in S' are given by an equation of the same form.

4. Constructions.

If the F-points are in general position, we resolve T_J normally by applying a T_2 to $O_1O_{2n-2}O_{2n-1}$. This transforms (ϕ_n) into another homaloidal net (ϕ') of degree $2n-(n-1)-2=n-1$, on which the multiplicities of the second F-points of T_2 are $n-2$, 0, 0; the other $2n-4$ simple F-points of (ϕ) give simple F-points of (ϕ'). The new characteristic is $1^{n-2}(2n-4)^1$, and (ϕ'_{n-1})

is of the same nature as (ϕ_n) and of degree lower by 1. We repeat the process, and finally transform (ϕ_n) into a net of lines, resolving T_{Jn} into $n-1$ T_2's.

By the converse process, we can construct a T_J of any degree (p. 57).

This normal resolution fails if all the simple F-points are adjacent to O_1, and no two lie on a linear branch, so that we cannot apply the T_2 (293 and p. 63). The simplest example is given on p. 113. Alternative processes for resolving T_J in all cases are discussed in Chapter VIII.

The following construction (80, 111·1) for a particular T_J, in which $n-1$ of the simple F-points in each plane are collinear, is a generalization of the construction of T_2 by skew projection.

Let k_{n-1} be a twisted curve of degree $n-1$, with $n-2$ intersections with a given line l_1. Then corresponding points P, P' are the intersections with S, S' of lines meeting both k and l_1. Since the plane l_1P meets k in one point not lying on l_1, this construction determines P' uniquely.

If P, P' lie on two given lines l, l' of S, S' respectively, then PP' lies on the quadric through l, l', l_1, and passes through one of its intersections with k not lying on l_1. There are $2(n-1)-(n-2)=n$ of these; hence the homologue of l meets l' in n points, and the transformation is of degree n.

Let l_1 meet S, S' in O_1, O_1'. If $P \equiv O_1$, then P' is any point of the projection of k from O_1 on S', which is of degree $n-1$; hence O_1 is an $(n-1)$-fold F-point of H and similarly O_1' of H', and the transformation is T_J.

The $n-1$ intersections $O_2 \dots O_n$ of k, S are simple F-points, for $O_\alpha \sim$ the line of intersection of the planes $O_\alpha l_1$, S'; similarly k meets S' in $n-1$ simple F-points $O_2' \dots O_n'$ of H'. Let $O_1'O_\alpha'$ meet S in $O_{n+\alpha-1}$, which lies on the line l_2 of intersection of S, S'; it is another simple F-point, and $\sim O_1'O_\alpha'$. This gives the other $n-1$ simple F-points of H, all lying on l_2, and similarly $O'_{n+\alpha-1}$ is the intersection of l_2, O_1O_α. Thus l_2 contains $n-1$ simple F-points of each system, associated with the remaining simple F-points of the other system.

5. T_J in one plane, vertices distinct (114).

Let the planes be superposed, and first let the two vertices be distinct. As sides z, x, y of the triangle of reference for both planes, take O_1O_1' and its two homologues when regarded as a ray of the two pencils (O_1), (O_1') respectively; for the present we assume these three lines distinct. Take the unit point at an intersection of corresponding rays. Then

$$c_1y + c_2z \sim c_1z' + c_2x',$$

and the equations are

$$z' : x' : y' = yj : zj : \phi,$$

$$z : y : x = x'j' : z'j' : \phi',$$

where

$$\phi \equiv x u_{n-1}(y, z) + u_n(y, z), \qquad j \equiv x v_{n-2}(y, z) + v_{n-1}(y, z),$$
$$\phi' \equiv - y' v_{n-1}(z', x') + u_n(z', x'), \quad j' \equiv y' v_{n-2}(z', x') - u_{n-1}(z', x').$$

The invariant points are given by

$$\frac{yj}{z} = \frac{zj}{x} = \frac{\phi}{y},$$

and are the intersections, other than O_1, O_1', of the conic $q \equiv xy - z^2$, through O_1, O_1', generated by the projective pencils, and the $(n+1)$-ic $x\phi - yzj$ through $O_1^{n-1} O_1'$. There are therefore $2(n+1) - (n-1) - 1 = n+2$ isolated invariant points, and no invariant curve.

The involutory points satisfy

$$\frac{yj}{xj'} = \frac{zj}{\phi'} = \frac{\phi}{zj'},$$

and are the intersections of the two corresponding $(n+1)$-ics

$$x\phi - yzj, \quad y\phi' - xzj'$$

other than O_1^{n-1}, $O_1'^{n-1}$, A_3 and the $n+2$ invariant points; their number is therefore $n(n-1)$.

With the present assumption, q cannot break up, but it may be invariant; then it absorbs all the isolated invariant points. Since each homologue of q is of degree 2, it contains F-points of either system of total multiplicity $2n-2$, namely the vertex and $n-1$ simple F-points. Also q falls away from each of the $(n+1)$-ics that give the involutory system, leaving two $(n-1)$-ics meeting in A_3, in n points of q and in $n(n-3)$ involutory points.

Next let $O_1 O_1'$ be a self-corresponding line; it bears a homography with two d.p.'s, isolated invariant points of T_J. The pencils (O_1), (O_1') are perspective; take A_3 and the unit point on the axis of perspective $x - y$. Then

$$c_1 y + c_2 z \sim c_1 x' + c_2 z',$$
$$x' : z' : y' = yj : zj : \phi,$$
$$y : z : x = x'j' : z'j' : \phi',$$

where j, ϕ are as before, and now

$$\phi' \equiv - y' v_{n-1}(x', z') + u_n(x', z'), \quad j' \equiv y' v_{n-2}(x', z') - u_{n-1}(x', z').$$

The invariant system consists of two points D_1, D_2 on the line of vertices z, given by $\Lambda_{n+1} \equiv x\phi - y^2 j$, and n points $D_3 \dots D_{n+2}$ on the axis $x - y$, given by $\Lambda_n \equiv \phi - yj$, where

$$\Lambda_{n+1} = x\Lambda_n + y(x-y)j.$$

Let $D_1 \rightarrow$ the intersection $D(1, 1, 0)$ of the line of vertices and the axis, which we suppose not to lie on j. Then Λ_{n+1} contains D; let it touch $t \equiv a(x-y) + bz$; then Λ_n also contains D, and touches $t - c(x-y)$, where $c = j(1, 1, 0)$, and $D_3 \rightarrow D$ also. Near D, to a first approximation, we find

$$t' : z' : x' = \left(1 - \frac{a}{c}\right) t : z : x.$$

This shows that D is an invariant point of first kind, with z, t as invariant directions, and with an adjacent invariant point on t, that is, on Λ_{n+1} (p. 83).

If also $D_2 \rightarrow D$, then Λ_{n+1} has two intersections with z at D, and touches z; and $a=0$, $t \equiv bz$. Then D is still of first kind, and both invariant directions coincide in z. But if we keep the direction D_1D_3 fixed, and therefore a/b fixed, as $D_2 \rightarrow D$, then in the limit $a=b=0$, and D is a double invariant point.

Either or both of the lines z, $x-y$ may be invariant, containing F-points of each system of total multiplicity $n-1$, namely, the two vertices O_1, O_1', or $n-1$ simple F-points of each system on the axis. If $x-y$ is invariant, it is a component of Λ_n, and its $n-1$ intersections with j lie on ϕ and are fundamental. Its intersection with O_1O_1' is one of the d.p.'s of the homography on O_1O_1', and T_J has only one isolated invariant point. If O_1O_1' is invariant, T_J has only $n-1$ isolated invariant points on the axis, one being absorbed by the intersection with the invariant line.

6. Vertices coincident (114).

Now let the vertices coincide; we have a homography in the self-corresponding pencil ($O_1 \equiv O_1'$).

(i) First assume this proper, with two distinct double rays x, y; these are self-corresponding lines of T_J, and each contains two isolated invariant points. Now

$$c_1x + c_2y \sim c_1x' + ac_2y',$$

where $a \neq 1$,

$$x' : ay' : z' = xj : yj : \phi,$$
$$x : y : z = x'j' : ay'j' : \phi',$$

where

$$\phi \equiv zu_{n-1}(x, y) + u_n(x, y), \qquad j \equiv zv_{n-2}(x, y) + v_{n-1}(x, y),$$
$$\phi' \equiv -z'v_{n-1}(x', ay') + u_n(x', ay'), \quad j' \equiv z'v_{n-2}(x', ay') - u_{n-1}(x', ay').$$

The invariant system is given by

$$x = a\phi - zj = 0, \text{ and } y = \phi - zj = 0;$$

in general there are four isolated invariant points, two on each self-corresponding ray. The other $n-2$ isolated points are absorbed by O_1, which is an F-point of each system through which pass $n-2$ branches of the corresponding P-curves.

The involutory system is given, with the invariant points, by

$$(1-a^2)\,xyjj' = j\phi' - a^2j'\phi = 0,$$

and, if $a^2 \neq 1$, is absorbed at O_1, except for the $2n-3$ other common points of j, j'; each of these forms with O_1 an improperly involutory pair (p. 80).

If however $a=-1$, there is an involutory locus

$$f_{2n-1} \equiv O_1^{2n-3}\,O_2 \ldots O_{2n-1}O_2' \ldots O'_{2n-1},$$

given by $j\phi' - a^2j'\phi$, of maximum degree for a transformation of degree n (113 and p. 80). Then the homography of rays is an involution, and each ray meets f in two points conjugate respectively to two points on the conjugate ray.

The case in which f has a cubic component is studied in **228**.

Either or both of x, y may be invariant, each absorbing two isolated invariant points and dropping away from any involutory locus. If x is invariant, and $D(0, 1, 0)$ is a general point of it, D lies on ϕ but not on j, say

$$j \equiv y^{n-1} + \dots, \quad \phi \equiv y^{n-1} t_1 + \dots;$$

the homography of directions at D is given by

$$x' : z' = x : t_1,$$

and is in general proper, and D is of first kind.

(ii) To consider the case when the two self-corresponding rays coincide, let one be x and the other $ax + by$; we pass to the limit as $b \to 0$. Now the equations are

$$x' : y' : z' = xj : \{ax + (b+1)y\} j : \phi.$$

The invariant points are

$$D_1, D_2, \quad x = \phi - (b+1) zj = 0; \quad D_3, D_4, \quad ax + by = \phi - zj = 0,$$

and $D_3 \to D_1$ and $D_4 \to D_2$ along $\phi - zj$. In the limit, D_1 or D_2 is an isolated invariant point of first kind, with x and the tangent to $\phi - zj$ as invariant directions, and with an adjacent isolated invariant point on the latter only.

If also $D_2 \to D_1$, then $\phi - zj$ touches x, and D is still of first kind, with three adjacent invariant points, all in the same direction, which absorbs both invariant directions.

If x is an invariant line, but not the other, then

$$\phi - (b+1) zj \equiv x\Lambda \text{ say,}$$

and D_3, D_4 lie on

$$y(\phi - zj) - zj(ax + by) \equiv x(y\Lambda - azj);$$

since, before the limit, they do not lie on x, they $\to x$, always lying on $y\Lambda - azj$, which does not touch x. The general points of x are of first kind, both invariant directions coinciding with x; and D_3, D_4 are of second kind, and are double invariant points.

If both lines are invariant, we have also

$$\phi - zj \equiv (ax + by)\psi,$$

and therefore $$zj - y\psi \equiv \frac{x}{b}(a\psi - \Lambda),$$

and has x as a factor, whence

$$j \equiv x\chi_2 - y\chi_1,$$
$$\psi \equiv x\chi_3 - z\chi_1, \text{ say.}$$

In the limit, when $b = 0$,

$$\phi \equiv zj + ax\psi,$$

and we find

$$x' : y' : z' = x\phi_{n-1} + ax^2\chi_1 : y\phi_{n-1} + ax^2\chi_2 : z\phi_{n-1} + ax^2\chi_3,$$

where $$\phi_{n-1} \equiv j - ax\chi_1,$$

which is the form given on p. 86, and shows that two invariant lines coincide in x; every point of either has an adjacent invariant point in every direction. If $D(0, 1, 0)$ is a general point of x, it lies on ϕ, ψ, but not on j; near D, to a first approximation,

$$x' : y' : z' = x : y : z,$$

and D is a double invariant point.

The only exceptional point on x is O_1; if $P(x, y, z)$ is near O_1, not in the direction of a tangent to j, then P' is also near O_1, and

$$x' : y' = x : ax + y,$$

which is a proper homography, with both invariant directions coinciding with x. Regarded as an invariant point, the F-point O_1 is of first kind.

(iii) If the homography in the pencil (O_1) is identity, lines joining corresponding points all pass through a fixed centre, and we have a central transformation. Every central transformation is a T_J, since there is a pencil of self-corresponding lines (351). The equations of transformation are

$$x' : y' : z' = xj : yj : \phi,$$

as for the general case with distinct planes.

The invariant system is the curve

$$\Delta_n \equiv O_1^{n-2} O_2 \ldots O_{2n-1} O_2' \ldots O'_{2n-1},$$

given by $\phi - zj$, meeting each ray through O_1 in the two d.p.'s of the homography which T_J sets up on it. There are no isolated invariant points.

The involutory system is given, together with Δ, by

$$\phi j' - j\phi' \equiv -(v_{n-1} + u_{n-1})\,\Delta,$$

and consists of $n-1$ rays, for each of which the homography is an involution. If this expression vanishes identically, the T_J is wholly involutory (113).

For a quite general T_J, if we take x, y as two general lines through O, and if l_1', l_2' are their homologues and l_3' a general line,

$$\begin{aligned} l_1' : l_2' : l_3' &= \quad xj \quad : \quad yj \quad : \quad \phi, \\ x : y : z &= l_1'j'(l_1', l_2', l_3') : l_2'j'(l_1', l_2', l_3') : \phi'(l_1', l_2', l_3'). \end{aligned}$$

Hence any T_J is compounded of a central transformation in S, between P, P'' and a linear transformation between P'' in S and P' in S', given respectively by

$$x'' : y'' : z'' = xj : yj : \phi,$$

and

$$x'' : y'' : z'' = l_1' : l_2' : l_3' \text{ (351)}.$$

Any variety of T_J which has an invariant conic or line can be constructed by a succession of T_2's with the same invariant curve and with O_1, O_1' as F-points.

If O_1, O_1' are distinct and there is a Δ_2 passing through them, we have for T_2 the construction of p. 48. Carry this out $n-1$ times, keeping Δ_2 the same, interchanging O_1, O_1' and letting O_2, O_2' vary on Δ, and O_3, O_3'

in S. We obtain T_{Jn} with an invariant conic passing through O_1, O_1' and $n-1$ simple F-points of each system, one arising from each component T_2.

If $O_1 \equiv O_1'$, and Δ_2 passes through $O_2O_3O_2'O_3'$, then repetition leads to a central T_{Jn}, with all the simple F-points on a cónic, and Δ_n breaks up into this and $n-2$ rays through O_1.

A T_2 with $\Delta_1 \equiv O_1O_1'$ leads in the same way to a T_J with the line of vertices invariant, and a T_2 with $\Delta_1 \equiv O_3O_3'$ to a T_J with an invariant axis through half the simple F-points.

A special construction by a succession of T_3's is given on p. 115.

7. **General involutory case** (25, 30).

If the transformation is wholly involutory, we denote it by I_J. The multiple F-points of H, H' coincide, and also the groups of simple F-points; this is therefore a case of coincident vertices.

The homography in the pencil (O_1) is now an involution, and there are two types of I_J according as this is proper or identity.

In the general case, the simple F-points fall into associated pairs, say O_2, O_2'; ... O_n, O_n', the points of one pair lying on conjugate rays of the involution. This has two double rays, self-conjugate lines of I_J; the invariant system lies on them, and can have three forms.

(i) In general, each double ray bears a proper involution with two d.p.'s, and I_J has four isolated invariant points D_1, D_2, D_3, D_4, lying by pairs on two self-conjugate lines OD_1D_2, OD_3D_4.

(ii) One ray is an invariant line Δ_1, and there are two isolated invariant points D_1, D_2 on the other self-conjugate line.

(iii) Both rays are invariant and there are no isolated invariant points.

For example, let conjugate rays be the pairs of tangents from O_1 to conics touching four given lines, and let the homography of conjugate points on each pair be cut out by the other tangents to the same conic. The two double rays are the tangents at O_1 to the two conics through O_1 and each is an invariant line. The transformation is I_{J4}, and the simple F-points are the vertices of the quadrilateral (376 and p. 126).

If ν is the class of the involution, a general line meets its homologue in ν conjugate pairs and $n-2\nu$ invariant points. Hence n is even in cases (i) and (iii) and odd in case (ii).

Now $O_1 \sim$ the P-monoid $j \equiv O_1^{n-2}$. A point Q of j adjacent to $O_1 \sim$ a point Q' adjacent to O_1 and lying on j; and O_1Q, O_1Q' are conjugate rays. If these coincide, Q, Q' lie on the same branch of j, and this touches one of the self-corresponding lines, and if this is not invariant, one of the invariant points on it is adjacent to D. Conversely, the invariant lines touch j, but the self-corresponding lines do not, except in the special case when a D coincides with O_1. The other branches of j through O form $\nu-1$ associated pairs, touching conjugate rays.

In the notation of p. 102 with $a=-1$, the isologue Ω_X, locus of conjugate pairs collinear with $X(\xi, \eta, \zeta)$ is given by

$$\Omega_X \Delta_{n-2\nu} \equiv \begin{vmatrix} \xi, & \eta, & \zeta \\ x, & y, & z \\ xj, & -yj, & \phi \end{vmatrix}.$$

At O_1 (0, 0, 1), this touches $j(x\eta + y\xi)$, the only variable tangent being conjugate to O_1X. The isologues are of degree $2\nu+1$, and the base points of the net are: the $(2\nu-1)$-fold point O_1, with $2\nu-2$ adjacent simple base points on j, the $2n-2$ simple F-points and $4\nu+4-2n$ isolated invariant points. The free intersections of two isologues Ω_X, Ω_Y are the ν pairs of conjugates on XY.

8. Self-conjugate curves (30).

In all three cases, there are a certain number of self-conjugate ν-ics and pencils of self-conjugate $(\nu+1)$-ics.

Let $O_2 \dots O_n$ be $n-1$ simple F-points, one from each associated pair; there are 2^{n-1} such sets. The family of monoids $(f_\mu) \equiv O_1^{\mu-1} O_2 \dots O_n$ is $\infty^{2\mu-n+1}$, and f exists if $2\mu \geqq n-1$. Any member $\sim$ a curve of the same family which meets it in $2\mu-n$ points other than F-points, and if it contains $2\mu-n+1$ or more invariant points, it either is self-conjugate, or else breaks up and has a self-conjugate part. Now f contains at most two invariant points, one on each self-conjugate ray. We therefore take μ so that

$$n-1 \leqq 2\mu \leqq n+1.$$

Case i. Here $n=2\nu$, $\mu=\nu$, and $f_\nu \equiv O_1^{\nu-1}(O_2 \dots O_n D_1)^1$ is determinate and self-conjugate. It therefore meets OD_3D_4 in an invariant point, D_3 say.

If we make the monoid pass through D_2 instead of D_1, we obtain a different self-conjugate curve, meeting f_ν in ν^2 points at ΣO, and therefore not meeting it at D_3, but passing through D_2, D_4. If, instead, we exchange any F-point, O_2 say, for its associate, we obtain a pair of self-conjugate curves through $O_1^\nu O_2' O_3 \dots O_n$ and either D_1D_4 or D_2D_3.

Since $n+1$ is odd, the set $O_2' \dots O_n' D_2D_4$ does not lie on any of these curves. But the monoids $(f_{\nu+1}) \equiv O_1^\nu O_2' \dots O_n' D_2D_4$ form a pencil which $\sim$ itself in involution; each of the members through $O_2, \dots O_n$, D_1 or D_3 is degenerate and self-conjugate; hence the involution in the pencil is identity, and every member is self-conjugate. The pencil determined by

$$O_1^\nu O_2 \dots O_n D_1D_2$$

is wholly degenerate.

Thus we have $2^{2\nu}$ self-conjugate ν-ics, and $2^{2\nu}$ pencils of self-conjugate $(\nu+1)$-ics, each set falling into four groups, through D_1D_3, D_1D_4, D_2D_3 or D_2D_4 respectively.

Case ii. Here $n = 2\nu + 1$, $\mu = \nu$ or $\nu + 1$; each monoid contains one invariant point on Δ_1. Then $f_\nu \equiv O_1^{\nu-1}(O_2 \ldots O_n)^1$ is determinate and self-conjugate, and meets OD_1D_2 in an invariant point, D_1 say. We obtain 2^{n-1} such curves by exchanging associated pairs of F-points and also D_1, D_2. Now $O_1^\nu O_2 \ldots O_n D_1$ determines a degenerate pencil, but the general member of the proper pencil $(f_{\nu+1}) \equiv O_1^\nu O_2 \ldots O_n D_2$ meets its homologue in $(\nu + 1)^2$ base points and in one point of Δ_1, and is self-conjugate. Again we have $2^{2\nu}$ self-conjugate ν-ics, and $2^{2\nu}$ pencils of $(\nu + 1)$-ics; half pass through D_1 and half through D_2.

Case iii. Here $n = 2\nu + 2$, $\mu = \nu + 1$; each monoid contains two invariant points on Δ_2. Then $(f_{\nu+1}) \equiv O_1^\nu O_2 \ldots O_n$, if it is proper, is a pencil of self-conjugate curves; the member through $O_2' \equiv$ its homologue, which breaks up into the P-line O_1O_2 and a self-conjugate $f_\nu \equiv O_1^{\nu-1} O_2' O_3 \ldots O_n$. This is already determined without O_2'; hence of the pair O_2, O_2', one O_2' is specially related to the set $O_3 \ldots O_n$, by lying on f_ν, and O_2 is not so related; there are 2^{n-2} such special sets. The pencil $(f_{\nu+1}) \equiv O_1^\nu O_2' O_3 \ldots O_n$ breaks up into f_ν and the pencil of lines (O_1).

Thus in all three cases we have $2^{2\nu}$ self-conjugate ν-ics, and $2^{2\nu}$ pencils of $(\nu + 1)$-ics.

In each case, two of the self-conjugate pencils which have all their base points common except two determine the I_J as the set of pairs of their free intersections. It is also determined by any one of the pencils and the involution of lines (O_1).

9. Reduction (25, 30).

A T_2 between S, S'', applied to $O_1O_2O_3$ reduces I_{Jn} to an involution $I'' \equiv T_2 . I_J . T_2^{-1}$ in S'' (p. 96). By T_2^{-1}, a line of S'' $\sim$ a conic through $O_1O_2O_3$, by I_J this $\sim$ an $f_{n-1} \equiv O_1^{n-2} O_2 O_3 O_4 O_4' \ldots O_n O_n'$, and by T_2 this $\sim$ an $f''_{n-2} \equiv O_1''^{n-3} O_4'' O_4''' \ldots O_n'' O_n'''$. Hence I'' is an $I_{J(n-2)}$; its invariant system is the homologue by T_2 of that of I_J, and I'', I_J are in the same case.

The process repeated reduces I_J to I_2 or I_1 according as n is even or odd; and I_2 can always be reduced to I_1. Hence every non-central I_J can be reduced to I_1.

We can always reduce the different cases to each other. If I_J is in case ii or iii, it has an invariant line Δ_1; let D be any point of this, and apply T_2 to O_1O_2D. Then I_{Jn} is reduced to an $I''_{J(n-1)}$. Since Δ_1 is a P-line of T_2, it $\sim$ a point D'', isolated invariant for I''. Of the other points of the P-line of T_2 which $\sim D$, one only is invariant for I'', arising from the invariant direction at D, other than DO_1 which is principal for T_2. Hence Δ_1 is replaced by a pair of isolated points, and I'' also has invariant the homologue of the rest of the invariant system of I_J, unaltered in nature. Thus case iii is reduced to ii, and ii to i, each time with a reduction of degree by 1.

We can also reverse this process, applying T_2 to O_1O_2 and an isolated invariant point D; and thus reduce i to ii or ii to iii, decreasing the degree by 1. In particular, a T_J of even degree can be reduced, one way or the other, to odd degree.

These reductions fail if all the invariant and simple F-points are adjacent to O_1, and no pair O_2, O_3 or O_2, D lie on a linear branch. The alternative processes can be applied here also, and the work on linear systems based on the earlier method has been completely established (p. 390).

10. Central $\mathbf{I}_J$ (25, 30, 42, 357).

Now let the involution of rays (O_1) be identity; then the join of any conjugate pair PP' passes through O_1. The transformation is central, and of class 0, and has an invariant curve Δ of maximum degree n, containing all the intersections of any line with its homologue. There are no isolated invariant points.

Each ray bears an involution of conjugate points. The two d.p.'s D_1, D_2 are invariant for I_J, and are the only free intersections of the ray with Δ. The rest are at O_1, hence $\Delta \equiv O_1^{n-2}$, and there are $n-2$ rays touching Δ at O_1, for which D_1 is adjacent to O_1 and therefore lies on j; thus all the branches of Δ at O_1 touch j.

There are $2n-2$ rays touching Δ elsewhere; on one of these, $D_1 \equiv D_2$, the involution is improper and the ray is a P-line and the point of contact an F-point of both systems, associated with itself. Thus $\Delta \equiv O_1^{n-2} O_2 \ldots O_{2n-1}$, and every branch through an F-point touches the corresponding P-curve there.

On a general ray, any conjugate pair divides D_1D_2 harmonically: this determines the transformation when O_1, Δ are given. The simplest examples are general inversion, when $n=2$, and the Arguesian transformation when $n=3$ (p. 116).

No ray l can be an invariant line; for the residual of Δ would completely determine the I_J, independently of l.

Since this is a central T_J which coincides with its reverse, it is represented by (pp. 99, 104)

$$x' : y' : z' = xj : yj : \phi,$$

where

$$\phi \equiv zu_{n-1} + u_n, \quad j \equiv zu_{n-2} - u_{n-1},$$

$$\Delta \equiv zj - \phi \equiv z^2 u_{n-2} - 2zu_{n-1} - u_n,$$

and the P-rays are

$$u_n u_{n-2} + u^2_{n-1}.$$

Now $j \equiv \frac{1}{2}\frac{\partial \Delta}{\partial z}$, and is the first polar of O_1 with regard to Δ. This also follows because j is the locus of Q the homologue of O in the involution on each ray, and $\{OQ, D_1D_2\} = -1$; or again because j touches each branch of Δ at O and passes through the points of contact of all tangents from O to Δ. This accounts for all the intersections of j, Δ, which are all at F-points.

Any multiple point of Δ lies on the first polar, and absorbs at least two intersections of Δ, j. Hence if the F-points are distinct, Δ has no multiple point other than O_1 and is of genus

$$p = \tfrac{1}{2}(n-1)(n-2) - \tfrac{1}{2}(n-2)(n-3) = n-2,$$

and of class $$n(n-1) - (n-2)(n-3) = 2(2n-3).$$

If the F-points coincide, p may be lower, but not higher; hence if p is given, the degree of $I_J \geqq p+2$.

11. Singularities of Δ (25, 112).

There can be no triple or higher point D on Δ other than O_1; for Δ would break up, with $l \equiv O_1 D$ as a component, and we have seen that no ray is invariant; l would be a factor of Δ, and therefore of u_{n-2}, u_{n-1}, u_n and of j, ϕ, and I_J would degenerate.

If Δ has a d.p. other than O_1, it reduces the genus by 1 and the class by 2, and absorbs two intersections with j, and two F-points, which are adjacent on a common fixed tangent to (ϕ), j.

At a general point D of Δ, the two invariant directions are DO_1 and the tangent to Δ. If D is an F-point O_2, these directions coincide, and the involution is improper, and every direction $\sim O_2O_1$. For the homologue of any line through O_2 breaks up, and the only branch through O_2 is the P-line O_1O_2. But if D is a d.p. of Δ and a pair of F-points O_2O_3 are adjacent here, a general line through $O_2 \sim$ a homaloid through O_3, and both pass through D in general directions. The involution of directions at D is again proper, with the two tangents to Δ as double elements; the direction $O_2O_1 \sim O_2O_3$, which is the tangent to (ϕ) and j, and these directions are harmonically conjugate with regard to the tangents to Δ, as may also be seen from the harmonic range (OQ, D_1D_2) on an adjacent ray.

Conversely, if O_2O_3 are adjacent, the point is double on Δ; for if Δ had only one branch, it would touch both the P-line O_1O_2 and also O_2O_3 which is a different line (p. 99).

Now Δ may also be specialized by having O_1 as an $(n-1)$-fold instead of an $(n-2)$-fold point. Then no tangent from O_1 to Δ touches it elsewhere, and all the simple F-points are adjacent to O_1. Each ray l contains only one other point of Δ, and one of the d.p.'s of the involution on l is at O_1. If l touches Δ at O_1, the other d.p. also coincides with O_1, the involution is improper and l is a P-line which $\sim O_1$, or rather the point adjacent to O_1 on l. Thus j breaks up into these $n-1$ rays, fixed tangents to (ϕ) and Δ at O_1; and $n-1$ simple F-points are adjacent to O_1 in the first neighbourhood, one on each branch of (ϕ), and the remaining $n-1$ F-points on some or all of the branches, in the second or higher neighbourhoods (p. 131).

12. Reduction of central $\mathbf{I}_J$ (25).

The genus of the invariant curve is unaltered by any reduction of the involution: if Δ is of genus $p \geqq 1$, we cannot reduce I_J to I_1. The lowest degree to which we can reduce Δ is $p+2$, when it has a p-fold point and no other singularity. Hence the utmost reduction possible for a central I_J with an invariant curve of genus p, is to an I_J of degree $p+2$, for which Δ has no multiple point except the vertex, and therefore all the F-points are distinct.

If Δ_n has two d.p.'s, say $O_2 \equiv O_3$ and $O_4 \equiv O_5$, a T_2 applied to $O_1 O_2 O_4$ reduces Δ_n to the type $\Delta_{n-2} \equiv O_1^{n-4} O_6 \ldots O_{2n-1}$, and no other invariant elements arise. The pencil $(O_1) \sim$ another pencil of self-conjugate lines of the reduced involution, which is another central I_J, with the degree and the number of d.p.'s of Δ each reduced by 2.

We repeat the process till Δ has either one or no d.p. at $O_2 \ldots O_{2n-1}$. If there is only one, and there is another F-point O_4, simple on Δ, a T_2 applied to $O_1 O_2 O_4$ eliminates the last d.p. of Δ. If there is no O_4, then I_J has only two F-points other than O_1, and is an I_2, reducible to I_1.

If $\Delta \equiv O_1^{n-1}$, it is rational, and can be transformed into a line by an auxiliary T_J with O_1 as vertex and general points of Δ as simple F-points, or by the equivalent series of T_2's. The reduced involution has a pencil of self-conjugate lines, with Δ of degree 1, and is I_1.

If Δ breaks up, since no ray is invariant, Δ consists of two monoids Δ_1, Δ_2, both rational, and the only d.p.'s of Δ are their intersections. A T_2 applied to O_1 and two of these reduces the degree of each of Δ_1, Δ_2 by 1, and eventually makes one or both disappear, unless there is only one intersection, when each is a line and I_J is quadratic, reducible to I_1.

Thus in every case, a central I_{Jn} with Δ of genus p can be reduced to I_1 if $p=0$; and if $p \geqq 1$, to $I_{J(p+2)}$ with distinct F-points.

13. Self-conjugate curves for the central $\mathbf{I}_J$ (42).

Since each F-point lies on its corresponding P-curve, it gives rise to a self-conjugate family (p. 91), a simple F-point to the pencil of rays (O_1), and O_1 to $(f_{n-1}) \equiv O_1^{n-3} O_2 \ldots O_{2n-1}$. Any member of this meets Δ in $2n-4$ general points, which lie on the homologue f', and f, f' meet in $(n-1)^2+2$ points and coincide.

This is one of a series of families of self-conjugate curves of the type

$$(f_\mu) \equiv O_1^{\mu-2} O_2 \ldots O_{\alpha+1}$$

which exists if

$$\mu \geqq 2, \quad \alpha \leqq 2n-2, \quad \alpha \leqq 3\mu-1$$

and has at least $3\mu - 1 - \alpha$ degrees of freedom. Any member meets Δ in $2n + 2\mu - 4 - \alpha$ general points ΣD; these lie on the homologue

$$f'_{\mu'} \equiv O_1^{\mu'-2} O_2 \ldots O_{\alpha+1} (O_{\alpha+2} \ldots O_{2n-1})^2 (\Sigma D)^1,$$

where

$$\mu' = 2n + \mu - 2 - \alpha,$$

and the number of free intersections of f, f' is $2(n+\mu-2-\alpha)$. This is negative if

$$\alpha \geqq n+\mu-1;$$

then f, f' coincide or have a self-conjugate curve as common part. With the former conditions for α, this gives

$$\tfrac{1}{2}n \leqq \mu \leqq n-1.$$

The family arising from O_1 is the highest of these.

From the next highest family we obtain a relation between the F-points. Let $\mu=n-2$, and therefore $n \geqq 4$, $\alpha=2n-3$ or $2n-2$. There is an ∞^{n-4} family,

$$(f_{n-2}) \equiv O_1^{n-4} O_2 \dots O_{2n-2} \sim (f'_{n-1}) \equiv O_1^{n-3} O_2 \dots O_{2n-2} O^2_{2n-1}.$$

Unless the F-points are in such special positions that f_{n-2} breaks up, it is part of f'_{n-1}, which breaks up into $O_1 O_{2n-1}$ and f_{n-2} which must pass through O_{2n-1}. Thus any one of the F-points belongs to the base of the (f_{n-2}) determined by all the others.

If $n=4$, the six simple F-points lie on a conic. If $n=5$, the nine F-points are the base of a pencil of cubics (302, 303).

This also follows since the F-points lie on

$$v_{n-4}(\phi+zj)+v_{n-3}j \equiv zu_{n-2}(zv_{n-4}+v_{n-3})+v_{n-4}u_n-v_{n-3}u_{n-1},$$

from which the factor u_{n-2} can be rejected if the $2n-6$ parameters contained in v_{n-4}, v_{n-3} satisfy $n-2$ conditions; there remains the ∞^{n-4} family (f_{n-2}).

The lowest value of μ is $\tfrac{1}{2}(n+\epsilon)$, where $\epsilon=0$ or 1 according as n is even or odd. Then the lowest value of α is $3\mu-1-\epsilon$ and the freedom of the family is ϵ.

Thus for n even, there is a determinate $f_{\frac{1}{2}n}$, and for n odd, a pencil $(f_{\frac{1}{2}(n+1)})$, all self-conjugate and passing through all the F-points. In any case there is a pencil of degree $\tfrac{1}{2}(n-\epsilon)+1$, and the I_J can be generated by the intersections of the pencil of lines (O_1) with this or any other pencil selected from any of these families of self-conjugate $(f_\mu) \equiv O^{\mu-2}$.

14. **Constructions** (275, 302, 303, 357).

Conversely, any pencil $(f_\mu) \equiv O_1^{\mu-2}$, and therefore with $4\mu-4$ simple base points, generates, by its intersections with the pencil of lines (O_1), a transformation whose characteristic is $n=2\mu-1$; $1^{2\mu-2}(4\mu-4)^1$, and which is a central I_J of odd degree. If any line l of (O_1) is part of an f_μ, it is a singular line of the pair of pencils, and the degree of I_J is reduced by 1. The two intersections of l with a general f_μ are base points of the pencil (f_μ) but not F-points of I_J.

The I_J is determined if we have any means, such as the intersection with (f_μ), of determining the involution of conjugate points on each ray through O_1. One pair is determined by any given self-conjugate $f_\mu \equiv O_1^{\mu-2}$, which may consist of a given line together with its homologue. Or we may be given the two d.p.'s of the involution, by means of their locus Δ; or if Δ breaks up, we may be given the locus of one d.p., which is a monoid vertex O_1, and one pair of conjugates. An example is given in 108, where one d.p. is at O_1, and a given monoid $f_n \sim$ the line at ∞; then (ϕ_n) osculates all the branches of f_n at O_1 (p. 109).

II. OTHER SPECIAL TRANSFORMATIONS

15. Examples of $\mathbf{T}_3\ 1^2 4^1$ (80).

For $n=3$, the two equations of condition for the characteristic numbers become

$$2\sigma_2+\sigma_1=6, \quad 4\sigma_2+\sigma_1=8,$$

with only one solution $\sigma_2=1$, $\sigma_1=4$. The only cubic characteristic is

$$n=3;\ 1^2 4^1,$$

and it is self-conjugate.

Since the second F-system also consists of $1^2 4^1$, the first P-system consists of one conic and four lines. We determine these by inspection; they are the four lines joining the double F-point O_1 to each of the simple F-points $O_2 \dots O_5$, and the conic $O_1 \dots O_5$. These all meet (ϕ_3) in F-points only. Let the associated F-points of H' be $O_2', \dots O_5', O_1'$ respectively.

We have seen (p. 57) that the general T_3 can be compounded of two T_2's having one common F-point in the intermediate plane. Conversely we can in general resolve T_3 into a T_2 applied to $O_1O_2O_3$ and another T_2. There is, however, a case of failure, discussed below, when all the F-points are adjacent.

Consider the example (183):

$$x' : y' : z' = xz(x+y) : z(x+y)^2 : x^2(x+z),$$
$$x : y : z = x'(y'z'-x'^2) : (y'-x')(y'z'-x'^2) : x'^3.$$

The numbers of *distinct* F-points are different in the two planes. This is no exception to Clebsch's theorem which deals with the numbers of points in the characteristics, and takes no account of adjacence.

In S, the double F-point is (0, 0, 1); one simple F-point is distinct at (−1, 1, 1), and the other three adjacent at (0, 1, 0) where (ϕ_3) osculates z. In S', the double F-point is (0, 0, 1), with one simple F-point adjacent to it, one tangent to (ϕ_3') being the fixed line y', and the other variable; the other three simple F-points are adjacent at (0, 1, 0), where (ϕ_3') osculates $y'z'-x'^2$. The first P-conic breaks up, but not the second (p. 19).

The T_2 with the three distinct F-points of (ϕ_3) as F-trio:

$$x'' : y'' : z'' = x(x+y) : (x+y)(x+z) : x(x+z),$$
$$x : y : z = x''z'' : x''(y''-z'') : z''(y''-x''),$$

converts (ϕ_3) into a net of conics in S''; and this T_3 is compounded of the T_2 just given and another:

$$x' : y' : z' = z''(y''-x'') : y''(y''-x'') : z''^2,$$
$$x'' : y'' : z'' = y'z'-x'^2 : y'z' : x'z'.$$

The next example is the simplest case of failure of a T_2 to reduce the degree of a homaloidal net (**354**):

$$x' : y' : z' = xy^2 : y^3 : y^2z + x^3,$$
$$x : y : z = x'y'^2 : y'^3 : y'^2z' - x'^3.$$

In each plane, all the F-points are adjacent at (0, 0, 1) and the homaloids are cuspidal cubics with eight fixed intersections at the cusp, where they touch y or y', but do not osculate any linear branch. We cannot take O_1 and two simple F-points as F-trio of a T_2, and cannot resolve this T_3 into two T_2's; we can however resolve it into three.

Apply the T_2

$$x'' : y'' : z'' = x^2 : xy : yz,$$
$$x : y : z = x''y'' : y''^2 : x''z'',$$

whose homaloids in S touch (ϕ_3) at O, the third F-point of T_2 being the general point (0, 1, 0). This transforms

$$(\phi_3) \equiv c_1xy^2 + c_2y^3 + c_3(y^2z + x^3)$$

into

$$(\phi_3'') \equiv c_1x''y''^2 + c_2y''^3 + c_3(x''y''z'' + x''^3),$$

a net of nodal cubics with two fixed tangents x'', y'' at (0, 0, 1), one branch having four-point contact with the fixed linear branch $y''z'' + x''^2$.

A second T_2, whose homaloids osculate this branch:

$$x''' : y''' : z''' = x''y'' : y''^2 : y''z'' + x''^2,$$
$$x'' : y'' : z'' = x'''y''' : y'''^2 : y'''z''' - x'''^2,$$

converts (ϕ_3'') into the net of conics

$$(\phi_2''') \equiv c_1x'''y''' + c_2y'''^2 + c_3x'''z''',$$

and our T_3 is compounded of these two T_2's and a third:

$$x' : y' : z' = x'''y''' : y'''^2 : x'''z''',$$
$$x''' : y'' : z'' = x'^2 : x'y' : y'z'.$$

16. Metrical examples in one plane.

Draw PX, PY perpendicular to fixed rectangular axes OX, OY, and let P' be the reflexion in XY of a fixed point O' (**202**).

In Cartesian coordinates, if O' is (a, b), these conditions are expressed by

$$\frac{x'-a}{y'-b} = \frac{y}{x}, \qquad \frac{x'+a}{x} + \frac{y'+b}{y} = 2,$$

and the transformation is cubic:

$$x' : y' : 1 = a(x^2 - y^2) + 2xy(y - b) : -b(x^2 - y^2) + 2xy(x - a) : x^2 + y^2,$$
$$x : y : 1 = (y' - b)(x'^2 + y'^2 - a^2 - b^2) : (x' - a)(x'^2 + y'^2 - a^2 - b^2) : 2(x' - a)(y' - b).$$

The double F-points in S, S' are O, O', and the corresponding P-conics are the circle centre O through O', and a rectangular hyperbola centre O' through O. The simple F-points in S are the points A_2, A_3 at infinity on the axes and the imaginary points $(a \pm ib, b \mp ia)$ on the circular lines; in S', they are the circular points at infinity and the reflexions of O' in the axes.

In homogeneous coordinates, the invariant points lie on

$$z\phi_1' - x\phi_3' \equiv z(y - bz)\{y^2 - b^2z^2 - (x - az)^2\}$$

and $$z\phi_2' - y\phi_3' \equiv z(x - az)\{x^2 - a^2z^2 - (y - bz)^2\}$$

(p. 78).

There is no invariant curve, and the class of T_3 is 3. We expect $n + 2 = 5$ isolated invariant points; setting aside the F-points and the extraneous points on ϕ_3', we only find two at ∞ and one finite:

$$D_1(1, 1, 0), \quad D_2(1, -1, 0), \quad D_3\{a(a^2 + b^2), \quad -b(a^2 + b^2), \quad a^2 - b^2\}.$$

To investigate D_1, write $x + y \equiv \xi$, $x - y \equiv \eta$; then $\eta = z = 0$ at D_1. The first set of equations give

$$\xi' : \eta' : z' = 2(a - b)\xi\eta z + (\xi^2 - \eta^2)\{\xi - (a + b)z\}$$
$$: 2(a + b)\xi\eta z - (\xi^2 - \eta^2)\{\eta - (a - b)z\} : (\xi^2 + \eta^2)z.$$

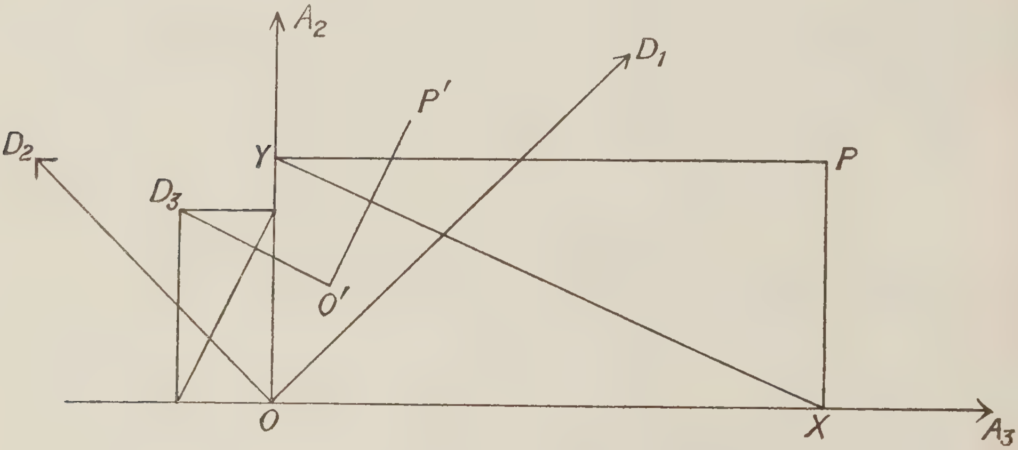

Fig. 4. A metrical T_3.

The homologue of the line $\eta' + cz'$, through D_1, touches

$$-\eta + (a - b)z + cz;$$

whence the invariant directions are z the line at ∞ and $\eta - \frac{1}{2}(a - b)z$, joining D_1 to the midpoint of OO'. To a first approximation,

$$\xi' : \eta' - \tfrac{1}{2}(a - b)z' : z' = \xi : -\{\eta - \tfrac{1}{2}(a - b)z\} : z,$$

and corresponding directions at D_1 form an involution.

In the first invariant direction, corresponding points are on opposite sides of D_1; in the second, the adjacent point is invariant.

Similarly at D_2 there are two adjacent isolated invariant points.

To investigate D_3, write $\xi = bx + ay$, $\eta = ax + by - (a^2 + b^2) z$, both of which vanish at D_3. We find

$$\xi' : \eta' : z' = 2xy\eta : 2\xi (xy - \xi z) : (x^2 + y^2) z.$$

This is a point of first kind with invariant directions $\xi \pm \eta$, that is, D_3D_2 and D_3D_1, and with no adjacent invariant point.

Thus the five isolated invariant points are two at D_1, two at D_2 and one at D_3.

If $O' \equiv O$, $a = b = 0$, then P' is the projection of P on the reflexion of OP in the bisector of XOY (53, 276, 415, 416).

We have
$$\begin{aligned} x' : y' : 1 &= 2xy^2 : 2x^2y : x^2 + y^2, \\ x : y : 1 &= y'(x'^2 + y'^2) : x'(x'^2 + y'^2) : 2x'y'. \end{aligned}$$

The imaginary simple F-points in S and the real ones in S' are adjacent to both double F-points at O; both P-conics break up. We have

$$P' \{OP, IJ\} = P \{OP', A_3A_2\} = -1,$$

and, except for questions of reality, the direct and reverse transformations have the same properties.

The rays of the pencil (O) are in involution, and the double rays $x \pm y$ are invariant lines, and absorb all the isolated invariant points. The class is reduced to 1. A general point of either invariant line is of first kind, and corresponding directions are in involution, being equally inclined to the invariant line.

Repeat the construction; in polar coordinates,

$$r^{(2n)} = r \sin^{2n} 2\theta, \qquad \theta^{(2n)} = \theta,$$

$$r^{(2n+1)} = r \sin^{2n+1} 2\theta, \quad \theta^{(2n+1)} = \frac{\pi}{2} - \theta,$$

and in Cartesian coordinates we have

$$\begin{aligned} &T_{4n+1} \quad x^{(2n)} : y^{(2n)} : 1 = x(2xy)^{2n} : y(2xy)^{2n} : (x^2 + y^2)^{2n}, \\ &T_{4n+3} \quad x^{(2n+1)} : y^{(2n+1)} : 1 = y(2xy)^{2n+1} : x(2xy)^{2n+1} : (x^2 + y^2)^{2n+1}. \end{aligned}$$

Each of these is a T_J of odd degree, with half the simple F-points adjacent to O_1 in the circular directions, and half at ∞ on the axes. If the successive reflexions of the radius vector are in different fixed lines, the F-points at ∞ are distinct (p. 105).

In a similar example, P' is the projection of X on OP; this gives a central T_3 (213, 331).

17. Space constructions (29, III·I).

The construction of p. 100 gives the general T_3; P, P' are the intersections with S, S' of rays meeting a fixed line l and a fixed conic k_2 which meets l once. Let l, k_2 meet S in O_1 and O_2, O_3 and meet S' in O_1' and O_2', O_3' respectively. The double F-point in S is O_1 and the simple F-points are O_2, O_3 and the projections from O_1' on S of O_2', O_3'.

In another construction, P, P' are the intersections with S, S' of the same variable conic passing through two fixed points, one on S, one on S' and meeting a fixed conic twice.

18. General involutory case; I_3 $1^2 4^1$.

If T_3 is involutory, the planes are superposed, the double F-points coincide and also the groups of simple F-points. The pencil of lines $(O_1) \sim$ itself, and in it I_3 sets up an involution which may be proper or identical.

If it is proper, the simple F-points are associated in pairs lying on conjugate rays, say $O_2 \equiv O_3'$, $O_3 \equiv O_2'$; $O_4 \equiv O_5'$, $O_5 \equiv O_4'$.

Since this I_3 is not central, the class $\nu \geqq 1$ and the invariant curve Δ is of degree $\mu \leqq 1$. Since $\mu + 2\nu = 3$, we have $\mu = \nu = 1$. Then Δ_1 passes through O_1 and is one of the double elements of the involution of rays; the other is a self-conjugate line, not invariant, bearing an involution of points with two d.p.'s D_1, D_2, isolated invariant points of I_3.

There are four other self-conjugate lines such as O_2O_4, joining F-points that are not associated. Each bears an involution whose d.p.'s are its intersection with Δ_1 and either D_1 or D_2. We can take as collinear

$$O_2O_4D_1, \quad O_3O_5D_1, \quad O_2O_5D_2, \quad O_3O_4D_2.$$

Since $O_2O_3 \sim O_4O_5$, these meet Δ_1 in the same point D_3 which with D_1, D_2 forms the diagonal triangle of the quadrangle $O_2O_3O_4O_5$.

The isologues all drop Δ_1, and are the net of cubics Ω_X through the seven points $O_1 \ldots O_5 D_1D_2$, of which five trios are collinear (p. 125). If X lies on any of the self-conjugate lines, this is a part of Ω_X; the residuals give five pencils of self-conjugate conics, $(O_2O_3O_4O_5)$ and four such as $(O_1O_3O_5D_2)$. Each of these conics meets its homologue in the four base points and two points of Δ_1, and therefore coincides with it.

If O_2, O_3 are adjacent, O_1O_2 is a double ray of the involution which is principal for I_3 and therefore not invariant. Hence it is OD_1D_2; the involution on it is improper, for every point $\sim O_2$; the d.p.'s D_1, D_2 coincide at O_2.

All the properties of I_3 can be studied by reducing it to I_1'' by a T_2 between S, S'' applied to $O_1O_2O_4$.

19. Central I_3; Arguesian transformation (90, 339, 340, 357, 376).

If the involution of rays of the pencil (O_1) is identity, each is self-conjugate for I_3, and bears an involution of conjugate pairs PP'. Since PP' always passes through O_1, a general line contains no conjugate pair, but meets its homologue in three invariant points; hence $\nu = 0$, $\mu = 3$; the invariant curve Δ_3 is of maximum degree, and there are no isolated invariant points.

Now Δ_3 meets each ray l through O_1 in the two d.p.'s D_1, D_2 of the involution on it, and in no other ordinary point; hence Δ_3 passes simply through O_1; and any pair P, P' divides D_1D_2 harmonically. Also $O_1 \sim$ the other intersection Q of l with the P-conic j which $\sim O_1$, and O_1, Q also divide D_1D_2 harmonically, and j the locus of Q is the first polar of O_1 with regard to Δ_3.

If l is the tangent to j at O_1, then $O_1 \equiv Q \equiv D_1$, and l also touches Δ_3. If l touches Δ_3 elsewhere, the involution is improper and D_1, D_2 coincide at the point of contact, which $\sim$ every point of l. These are the P-lines and simple F-points, each associated with itself, thus $O_2 \ldots O_5$ are the intersections, other than O_1, of j, Δ_3.

Any conic through $O_2 \ldots O_5 \sim$ another conic through $O_2 \ldots O_5$, also meeting the former in its two other intersections with Δ_3, and therefore coinciding with it. This pencil of self-conjugate conics with the pencil of self-conjugate lines (O_1) generate the I_3.

We can construct P' as the conjugate of P in the involution determined on OP by any two fixed conics of the pencil; the invariant points on OP are its points of contact with the two conics of the pencil that touch it. In this form, the I_3 is known as the Arguesian transformation.

If the F-points are not in general position, the degree of the involution may be reduced. If $O_1O_2O_3$ are distinct and collinear, their join drops away from j, Δ and (ϕ), and the transformation, known as the triangular Arguesian, is quadratic. If $O_1O_4O_5$ are collinear also, it becomes linear (341, 357).

Since the central I_3, in the general case, has an invariant curve of genus 1, it cannot be reduced to I_2 or I_1, which can only have rational invariant curves. A T_2 applied to $O_1O_2O_3$ or to O_1O_2 and a point of Δ_3, leads to another I_3 of the same type.

Reduction is possible if Δ_3 has a d.p. D. First suppose this distinct from O_1; then two of the intersections of Δ_3, j are at D, and two F-points coincide there, say $O_2 \equiv O_3 \equiv D$, being adjacent on a common tangent of j, (ϕ). Now apply a T_2, between S, S'', to $O_1O_2O_4$; a line in S'' becomes in succession a conic in S through O_1, O_2, O_4, a conic in S through O_1, $O_3 \equiv O_2$, O_5, and a conic in S'' through O_1'', O_2'', O_5''. Thus I_3 is reduced to I_2'', and we know that this can always be reduced to I_1.

If Δ_3 has two d.p.'s, at $O_2 \equiv O_3$ and $O_4 \equiv O_5$, it breaks up into O_2O_4 and a conic; the reduction to I_1 can then be effected by one T_2.

If the d.p. of Δ_3 is at O_1, each branch touches j, which also has a d.p., and breaks up into the two tangents to Δ_3. All the intersections of j, Δ_3 are at O_1, and all the simple F-points are adjacent to O_1, two on each of the two branches of (ϕ), which touch j and osculate fixed conics.

This I_3 is reduced to I_2 by a T_2 applied to O_1, an adjacent point of j and any point of Δ_3.

20. Description of $\mathbf{T}_4\ 3^2 3^1$ (78, 80, 340, 344).

Of the two quartic characteristics, one is $T_{J4}\ 1^3 6^1$, and the other is the semi-symmetric $T_4\ 3^2 3^1$.

Let $H \equiv (O_1O_2O_3)^2\, O_4O_5O_6$. By inspection, the P-curves are:

(i) the three lines j_4, j_5, j_6, joining pairs of $O_1O_2O_3$. These partition the group of double F-points, and each selects the point through which it does not pass; the group is of negative determination. These P-lines $\sim$ three simple F-points O_4', O_5', O_6' of H', associated with O_1, O_2, O_3 respectively.

(ii) the three conics j_1, j_2, j_3 through $O_1O_2O_3$ and two of $O_4O_5O_6$; these partition the group of simple F-points, which is also of negative determination, and $\sim$ three double F-points O_1', O_2', O_3' of H', associated with O_4, O_5, O_6 respectively.

This exhausts the total degree $3\,(4-1) = 9$ of J, and there are no more P-elements, $H' \equiv (O_1'O_2'O_3')^2\, O_4'O_5'O_6'$, and the characteristic is self-conjugate.

The ϕ_4 which $\sim O_2'O_3'$ breaks up into j_2, j_3, which are both conics through $O_1O_2O_3$. If the double F-points in each plane are taken as triangles of reference, the equations of T_4 can be put in the form

$$\frac{1}{x'} : \frac{1}{y'} : \frac{1}{z'} = \frac{a_1}{x} + \frac{b_1}{y} + \frac{c_1}{z} : \frac{a_2}{x} + \frac{b_2}{y} + \frac{c_2}{z} : \frac{a_3}{x} + \frac{b_3}{y} + \frac{c_3}{z},$$

equivalent to $x' : y' : z' = \quad j_2 j_3 \quad : \quad j_3 j_1 \quad : \quad j_1 j_2,$

where $j_1 \equiv a_1 yz + b_1 zx + c_1 xy,$ etc.,

and the simple F-points are $\left(\frac{1}{A_{23}}, \frac{1}{B_{23}}, \frac{1}{C_{23}}\right)$ etc., where A_{23} is the co-factor of a_1 in the determinant of the coefficients.

This shows how T_4 can be resolved into two T_2's in standard form between S, S'' and S', S''':

$$x'' : y'' : z'' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}, \qquad x''' : y''' : z''' = \frac{1}{x'} : \frac{1}{y'} : \frac{1}{z'},$$

together with a linear transformation between S'', S'''.

If the six F-points in S, and O_1', O_2', O_3' in S', are given, T_4 has two degrees of freedom, for j_1, j_2, j_3 each contains an arbitrary multiplier. It is shown in 344 that it is determined by any set of ten F-points.

From the homologue of the line O_1O_4, a P-conic and P-line are dropped, leaving as proper homologue the line $O_1'O_4'$ joining the associated F-points. In the same way, $O_5O_6 \sim$ the conic $O_2'O_3'O_4'O_5'O_6'$. The net of conics through $O_1O_2O_3 \sim$ the net of conics through $O_1'O_2'O_3'$; each net contains the three P-conics, and also three pencils of degenerate conics, the fixed part being a P-line, and the corresponding pencil of the other net having an F-point as additional base point.

The pencil of conics through $O_2O_3O_5O_6 \sim$ the pencil of conics through the four associated F-points. The two nets of cubics through all six F-points also correspond.

The normal resolution of T_4 consists of one step only; a T_2 applied to $O_1O_2O_3$ transforms (ϕ_4) into a family of conics through the homologues of O_4, O_5, O_6; and T_4 is resolved into two T_2's (p. 56).

21. Constructions.

A simple space construction is adapted from 383 (see also 114). It can also be regarded as a generalization of the construction for T_2 of p. 41, § 14.

Let l_1, l_2, l_1', l_2' be four skew lines on a cubic surface f_3; then points P, P' of S, S' correspond if the transversals λ, λ' drawn from P, P' to meet l_1, l_2 and l_1', l_2' respectively meet in a point X of f_3.

The degree of T_n is the number of pairs P, P' of which P lies on a given line l of S, and P' on a given line l' of S'. Now λ meets l, l_1, l_2 and lies on a quadric meeting f_3 in l_1, l_2 and a curve k_4 meeting each of l_1', l_2' twice; and λ' lies on a quadric through l_1', l_2', meeting k_4 twice on each of l_1', l_2' and in four other points, which are the possible positions of X; hence $n=4$.

Let O_1, O_2 be the traces on S of l_1, l_2, and O_1', O_2' the traces on S' of l_1', l_2'. The double F-points of S are O_1, O_2 and the trace on S of the one line of f_3 that meets l_1, l_2 but not l_1' or l_2'; the simple F-points are the homologue of $O_1'O_2'$ and the traces of the two lines of f_3 that meet l_1, l_2 and one but not both of l_1', l_2'.

If the planes are superposed, the transformation is not involutory. Now $P \equiv P'$ if P lies on f_3, and there is an invariant cubic Δ_3, which breaks up into a conic and line or three lines for special positions of S with regard to f_3. Since $\mu=3$, the class is 1, and the transformation is of the maximum degree for its class (p. 82).

A space construction of a special case is given in 78. Let S, S' meet in l, and let three points O_1, O_2, O_3; O_1', O_2', O_3' be given in each. Then $P \sim P'$ if the two pencils of conics $(O_1O_2O_3P)$, $(O_1'O_2'O_3'P')$ cut out the same involution on l. The double F-points in S are O_1, O_2, O_3, and the simple F-points are the intersections of l with $O_2'O_3'$, $O_3'O_1'$, $O_1'O_2'$. Thus all six simple F-points are collinear. See also 111·1.

A special case in one plane of some interest arises if each F-point coincides with its associate. Then $O_1O_4 \sim$ itself, and the intersections of O_1O_4, O_2O_5, O_3O_6 are invariant points. Also $O_2O_6 \sim O_5O_3$ in T_4 and in T_4^{-1}, and their intersection and the two similar points are also invariant. Since six is the maximum number of invariant points, there is no invariant curve and the class is 4 (p. 78).

22. Involutory case; $\mathrm{I}_4\, 3^2 3^1$ (50).

If this T_4 is involutory, the double F-points of H, H' coincide and also the simple F-points; and each of O_1, O_2, O_3 is associated with one of O_4, O_5, O_6. We can therefore take $O_\alpha \equiv O_\alpha'$, $\alpha = 1 \ldots 6$.

Apply to $O_1O_2O_3$ a T_2 between S, S''; this reduces I_4 to a linear involution I_1'' in S''. Let D'', Δ_1'' be its invariant point and line. By T_2, these $\sim$ the invariant system of I_4, consisting of an isolated invariant point D of second kind and an invariant conic $\Delta_2 \equiv O_1O_2O_3$. Hence the class of I_4 is 1.

By T_2, the lines and homaloids of $I_4 \sim$ conics through $O_1''O_2''O_3''$ and conics through $O_4''O_5''O_6''$ respectively; hence these trios correspond in I_1'', and the six points lie on a conic, which $\sim$ a line through $O_4O_5O_6$. Hence the three simple F-points of I_4 are collinear.

Also the lines joining the pairs $O_1''O_4''$, $O_2''O_5''$, $O_3''O_6''$, homologues in I_1'', meet in D''; hence O_1O_4, O_2O_5, O_3O_6 meet in D, and the three joins of associated F-points of I_4 are concurrent.

By T_2, the point of j_1 adjacent to $O_1 \sim$ the intersection of $O_5''O_6''$ with $O_2''O_3''$; these are homologues in I_1'', and meet on Δ_1''; hence Δ_2 touches j_1 at O_1 (p. 90).

The pencil of lines (D''), self-conjugate for I_1'', gives rise to a pencil of conics $(O_1O_2O_3D)$, self-conjugate for I_4. In the same way, the pencils of conics $(O_2''O_3''O_5''O_6'')$ and so on, self-conjugate for I_1'', give for I_4 three other pencils of self-conjugate conics such as $(O_2O_3O_5O_6)$. These are the families that arise from the F-points $O_1O_2O_3$, which lie on their corresponding P-curves.

If we apply T_2 to $O_1O_2O_4$ instead of $O_1O_2O_3$, we reduce I_4 to a general quadratic inversion, with an invariant conic, and the properties of I_4 can also be deduced from this.

Since $\mu = 2$, $\nu = 1$, every line l meets its homologue ϕ_4 in two invariant and two involutory points. The locus of the pair, as l turns about a centre X, is the cubic isologue

$$\Omega_X \equiv O_1O_2O_3O_4O_5O_6DXX'.$$

From Ω_{O_1}, which has an additional branch through O_1, the P-conic $j_1 \equiv O_1O_2O_3O_5O_6$ is dropped, leaving the self-conjugate line O_1O_4D. The d.p.'s of the involution of points on this are D and its intersection, other than O_1, with Δ_2.

From Ω_{O_4}, the P-line $j_4 \equiv O_2O_3$ is dropped, leaving a conic with a d.p. at O_4, breaking up into O_1O_4D, $O_4O_5O_6$.

If X lies on the self-conjugate line O_1O_4D, this is part of the isologue, and the residual is a conic of the pencil $O_2O_3O_5O_6$; and Ω_D consists of the three lines O_1O_4D, O_2O_5D, O_3O_6D.

23. Symmetric T_5 6^2 (73, 109, 278, 380).

Of the four symmetric types, the simplest is T_2, and next comes T_5 6^2. This can be defined as the relation between points P, P' which subtend projective pencils at two sets of five given points $O_1 \ldots O_5$ and $O_1' \ldots O_5'$ of S, S' respectively.

If P is given, P' is the one intersection, other than O_1', O_2', O_3', of the two conics defined by the cross-ratios

$$P'\{O_1'O_2'O_3'O_4'\} = P\{O_1O_2O_3O_4\}, \quad P'\{O_1'O_2'O_3'O_5'\} = P\{O_1O_2O_3O_5\}.$$

One or both of these are indeterminate if P coincides with one of the given points, say O_1; then the locus of P' is the conic j_1' given by

$$P'\{O_2'O_3'O_4'O_5'\} = O_1\{O_2O_3O_4O_5\}.$$

The only other case of failure is when the two conics on which P' lies coincide in $j_6' \equiv O_1' \dots O_5'$; then P has a determinate position O_6. Hence $O_1 \dots O_6$ are double F-points, and form H; and T is of degree 5. Similarly

$$H' \equiv (O_1' \dots O_6')^2.$$

A line joining two F-points $O_\alpha O_\beta \sim$ the line joining the associated points $O_\alpha' O_\beta'$; there are 15 such lines. A conic through any four F-points $\sim$ a conic through the associated set; there are 15 such pencils of conics. The six pairs of F-points enter in the same way; any five pairs determine the T_5 and the sixth pair.

If three F-points, $O_1 O_2 O_3$ say, lie on a line l, the homaloid ϕ_5 breaks up into l and $\phi_4 \equiv (O_4 O_5 O_6)^2 (O_1 O_2 O_3)^1$ and T_5 degenerates into the T_4 of the last paragraph. If all six lie on a conic q, then ϕ_5 breaks up into q and a cubic through $O_1 \dots O_6$, which is not a homaloid; the transformation fails entirely.

In general, referred to $O_1 O_2 O_3$, $O_1' O_2' O_3'$, the equations of T_5 are

$$x' : y' : z' = x j_2 j_3 : y j_3 j_1 : z j_1 j_2.$$

The point P and the conic $P' O_1' O_2' O_3' O_4'$ are related in a quadratic correlation between S, S'; and T_5 is generated by two such, just as T_2 is generated by two linear correlations.

The T_5 is normally reduced to a T_4 $3^2 3^1$, and is therefore compounded of three T_2's.

24. Involutory case; I_5 6^2 (73, 260).

If this T_5 is involutory, the F-points coincide, and we have to determine their association in 0, 1, 2 or 3 pairs, the others being self-associated.

If all are self-associated, the transformation is identity; hence there is at least one associated pair, say $O_1 \equiv O_2'$, $O_2 \equiv O_1'$.

(i) Let there be one self-associated point O_3 and therefore another O_4 say. Apply a T_2 between S, S'' to $O_1 O_2 O_3$; then I_5 is reduced to I_2'' with F-points $O_4'' O_5'' O_6''$.

Now in I_5, $O_1 O_4 \sim O_2 O_4$; therefore in I_2'', $O_1'' O_4'' \sim O_2'' O_4''$, and O_4'' is self-associated for I_2'', which is not central. Hence O_5'', O_6'' are also self-associated for I_2'' and therefore O_5, O_6 for I_5, which has not a second pair of associated F-points.

Now the lines $O_3 O_4$, $O_3 O_5$, $O_3 O_6$ are self-conjugate for I_5, and therefore $O_3'' O_4''$, $O_3'' O_5''$, $O_3'' O_6''$ for I_2''; and O_3'' is one of the four isolated invariant points of second kind of I_2''. Each direction at O_3'' gives rise, by T_2, to a point of $O_1 O_2$ invariant for I_5, which thus has an invariant line $\Delta_1 \equiv O_1 O_2$, and also three isolated invariant points D_1, D_2, D_3 of second kind. Therefore this I_5 is of class 2.

Now the sides of the quadrangle $O_3 O_4 O_5 O_6$ are all self-conjugate for I_5; hence its diagonal points are the invariant points D_1, D_2, D_3. The other d.p.'s of the involution of points on the sides are the intersections with Δ_1.

(ii) We have seen that there cannot be just two associated pairs of F-points, but there may be three. Let $O_1 \equiv O_2'$, $O_3 \equiv O_4'$, $O_5 \equiv O_6'$.

A T_2 applied to $O_1O_2O_3$ reduces I_5 to an I_3'' with F-system

$$O_4''^2 O_1'' O_2'' O_5'' O_6''.$$

$$\begin{array}{llll} \text{In } I_5, & O_4O_1 \sim O_3O_2; & \text{hence in } I_3'', & O_4''O_1'' \sim O_1'', \\ & O_4O_2 \sim O_3O_1; & & O_4''O_2'' \sim O_2'', \end{array}$$

and I_3'' is central; its invariant system is $\Delta_3'' \equiv O_4''O_1''O_2''O_5''O_6''$.

Now O_1O_2 is self-conjugate for I_5; hence O_3'' is invariant for I_3'' and lies on Δ_3''. Thus I_5 has an invariant curve $\Delta_3 \equiv O_1O_2O_3O_4O_5O_6$, and is of class 1. The H-line at a general point of Δ'' passes through O_4''; at O_1'', O_2'', being F-points of I_3'', these H-lines coincide with the tangent to Δ'', and do not give rise to isolated invariant points of I_5; but O_3'' is not an F-point of I_3'', and the direction of the H-line gives one isolated point D of I_5, lying on O_1O_2, and, by a similar argument, on O_3O_4 and O_5O_6 also. The six F-points lie on three concurrent lines.

A metrical specialization of I_5 is given in **22**, **348**, **407**. Let O_4 be the orthocentre of the triangle $O_1O_2O_3$, and O_5O_6 the circular points; let O_1, O_2, O_3, O_4 be self-associated, and O_5O_6 an associated pair. The line O_1O_4 is self-conjugate; conjugate points on it are harmonic with respect to its intersections with O_2O_3 and O_5O_6 which is the line at ∞, and are equidistant from O_2O_3.

The circle $PO_2O_3O_5O_6 \sim$ the circle $P'O_2O_3O_5O_6$, and these meet O_1O_4, which is perpendicular to the common chord O_2O_3, in points equidistant from O_2O_3; and the circles are the reflexions of each other in that chord. Hence van den Berg's construction: reflect the three circles through P and two vertices of a triangle in the corresponding sides; the three reflexions have a common point P', conjugate of P in an I_5.

The homaloids are bicircular quintics; the fifth intersection of each with the line at ∞ is invariant, and the asymptote is parallel to the corresponding line.

If the whole figure is inverted with regard to O_4, the inverses of P, P' are conjugate in the metrical I_2 of p. 52.

25. Example of space constructions; $\mathbf{T}_6$ $1^4 4^2 3^1$; $3^3 1^2 4^1$ (383).

In the space constructions for T_2 and T_4 (pp. 41, 119), we have a surface f_μ and two configurations C, C' lying on it, such that C together with a general point P of S determines a line λ, and C', P' another line λ'; then P, P' are homologues if λ, λ' meet in a point X of f. In the constructions of T_2 or T_J by skew projection, we have $C \equiv C'$, $\lambda \equiv \lambda'$, and f need not be considered; but these constructions fail if the planes are superposed. In the second construction of T_3, conics replace λ, λ'.

Now C, C' may each consist of a single curve on f with one apparent d.p., or of a pair of curves with one apparent intersection, such that the sum of the multiplicities on f_μ of the two branches is $\mu - 1$. In the above examples, C, C' are alike and with symmetric relations of incidence; this leads to self-conjugate characteristics. The following is a construction for the first characteristic that is not self-conjugate.

Let f be cubic, C two skew lines l_1, l_2, and C' a twisted cubic k_3' meeting l_1 once and not l_2. We shall show that this leads to T_6 with $H \equiv 1^4 4^2 3^1$ and $H' \equiv 3^3 1^2 4^1$.

As P describes a line l in S, the locus of λ is the quadric $f_2 \equiv l l_1 l_2$, meeting f_3 in l_1, l_2 and a k_4 which is the locus of X; and f_2 meets k_3' in six points, one on l_1 and five on k_4. As P' describes a line l' in S', then λ', meeting l' once and k_3' twice, describes an $f_4' \equiv k'^2_3$, meeting k_4 in 16 points, of which 10 are on k_3'; and the 6 others are the positions of X for which P lies on l and P' on l': the degree of T is 6.

To find the F-systems, consider how the construction may fail. First, λ is indeterminate if P is the trace O_1 of l_1 on S. Then the locus of X is a conic meeting l_2 twice, k_3' thrice, and f_4' in two other points; the locus of P' is a conic and O_1 is a double F-point of H.

If $P \equiv O_2$ the trace of l_2, the conic locus of X meets l_1 twice, k_3' twice, and f_4' in four other points; O_2 is a 4-fold F-point.

Next, X is indeterminate if λ lies on f_3. Of the five lines of f_3 meeting l_1 and l_2, two are chords of k_3' and three meet k_3' once. If λ is one of the first pair, it $\equiv \lambda'$, and P' is determinate; P is not an F-point. But if λ meets k_3' once, λ' describes a quadric, and P' a conic; this gives three more double F-points of H.

Thirdly, P' is indeterminate if λ' lies on S', joining two of the intersections $O_1'O_2'O_3'$ of k_3', S', and giving three simple F-points of H.

Similarly, T^{-1} fails first if $P' \equiv O_1'$; then λ' lies on a quadric cone through k_3', meeting k_4 in 8 points, 5 on k_3' and 3 others; the locus of P is cubic. Thus O_1, O_2, O_3 are triple F-points of H'.

It fails next if λ' is one of the six chords of k_3' lying on f_3; two of these meet l_1 and l_2, and do not give F-points; three meet l_2 only, and give P lines in S; and one meets neither l_1 nor l_2 and gives a P-conic. We have three simple and one double F-point of H'.

The third case of failure of T^{-1}, and the last simple F-point of H', arise when P lies on O_1O_2.

Thus the characteristics of T and T^{-1} are

$$n = 6; \quad 1^4 4^2 3^1; \quad 3^3 1^2 4^1.$$

26. Symmetric $T_8\ 7^3$; Geiser's involution (30, 50, 68, 103, 132, 275, 346, 364).

The next symmetric transformation is of degree 8, and has seven triple F-points, $O_1 \ldots O_7$ in S, and $O_1' \ldots O_7'$ in S'. Then $O_1' \sim$ the P-cubic

$j_1 \equiv O_1^2 O_2 \dots O_7$; the line $l_{12}' \equiv O_1'O_2' \sim$ the conic $q_{12} \equiv O_3 \dots O_7$, and the equations of T_8 are

$$l_{23}' : l_{31}' : l_{12}' = q_{23} j_2 j_3 : q_{31} j_3 j_1 : q_{12} j_1 j_2.$$

If any three F-points lie on a line, or any six on a conic, T_8 degenerates. For example, if $O_5 O_6 O_7$ are the diagonal points of the quadrangle $O_1 O_2 O_3 O_4$, the six sides break away from (ϕ_8), and T_8 becomes T_2 (239, 375).

This T_8 can be involutory in only one way, with each F-point self-associated (30). It is then one of the four standard types to which all plane Cremona involutions can be reduced.

This I_8 first arose as the relation between the two free intersections of cubics of a net (Ω) with seven base points $O_1 \dots O_7$. The involution so determined can have no F-points but these. If P is adjacent to O_1, all the cubics of the pencil $(O_1 \dots O_7 PP')$ touch at O_1, and one member is the determinate cubic $j_1 \equiv O_1^2 O_2 \dots O_7$, which is the locus of P' as $P \to O_1$ in all directions. Hence each O is a triple F-point, and $n^2 - 1 = 7 \cdot 3^2$, $n = 8$; we have an $I_8\ 7^3$, each F-point associated with itself.

We can also determine the nature of I from the principle of correspondence. The degree is the number of points P of a given line l whose homologues P' lie on another given line l'. Now I is determined by the two pencils selected from the net by two arbitrary additional base points X, Y. Any point P of l determines one cubic of the pencil $(O_1 \dots O_7 X)$; this determines three points P' of l', three cubics of the pencil $(O_1 \dots O_7 Y)$ and nine points of l. This 9, 9 correspondence of points of l has 18 coincidences; these include the three intersections of l with the one cubic $(O_1 \dots O_7 XY)$ common to both pencils, each counted three times, and the point (l, l') once; leaving 8 for the degree of I. If l passes through O_1, the correspondence is 6, 6, and from the twelve coincidences we remove $3 \times 2 + 1$, leaving five intersections other than O_1 of l with its homologue ϕ_8; hence O_1 is triple on (ϕ) (p. 96 of 385).

The net (Ω) contains 21 degenerate members $l_{\alpha\beta} q_{\alpha\beta}$; the two components are conjugate, and their intersections are d.p.'s of an Ω and lie on the Jacobian of the net, $\Delta_6 \equiv (O_1 \dots O_7)^2$. This is the invariant curve of I_8, and there are no isolated invariant points; for the cubics of the pencil through an invariant point D touch one another there, and one of them has a d.p.; hence D lies on Δ.

Every line meets its homologue in six points of Δ and one pair of conjugates: the class of I_8 is 1.

The isologues are obtained by dropping Δ from a net of 9-ics through $(O_1 \dots O_7)^3$; they are therefore the original net of cubics (Ω). Each of these has a centre X on it, such that the involution of conjugate points on Ω is cut out by the pencil of lines (X). There are four d.p.'s of this involution, points of contact of the four tangents from X to Ω; these are the free intersections of Ω, Δ.

An I_8 of this type arises as the relation between the two intersections of a line and its homologue in a general T_2; for the pair are the free intersections of the cubics of the isologue net of T_2, whose base points are the three F-points and four invariant points of T_2. Conversely, any $I_8\ 7^3$ can be so regarded; the line l joining any pair PP', conjugate for I_8, and the conic $q \equiv O_1O_2O_3PP'$ can be shown to correspond in a T_2. For, since the class of I_8 is 1, any l contains only one pair PP', and determines one q; and any $q \equiv O_1O_2O_3$ meets its homologue $k_7 \equiv (O_1O_2O_3)^2 (O_4 \ldots O_7)^3$ in six points at $O_1O_2O_3$, six other points of Δ and one conjugate pair PP', and therefore determines l. Thus l, q are in a 1, 1 relation, for which (q) is a homaloidal net and $O_1O_2O_3$ an F-trio.

27. Degenerations of $\mathbf{I}_8\ 7^3$ (30, 50, 346, 364).

If three F-points lie on a line, this breaks away from (ϕ_8), leaving a homaloidal net $(\phi_7) \equiv 4^3 3^2$. If six lie on a conic, since this meets any line twice, it breaks away twice from (ϕ_8), leaving $(\phi_4) \equiv 1^3 6^1$. These and their combinations give, as degenerations of $I_8\ 7^3$, all the characteristics with seven or fewer F-points:

No.	Base points collinear	and on a conic	Degree	F-system
1	567		7	$(1234)^3\,(567)^2$
2	567, 347		6	$(12)^3\,(3456)^2\,7^1$
3	567, 347, 127		5	$(123456)^2$
4	567, 347, 246		5	$1^3\,(235)^2\,(467)^1$
5	567, 347, 127, 246		4	$(135)^2\,(246)^1$
6	567, 347, 246, 235		4	$1^3\,(234567)^1$
7	567, 347, 127, 246, 235		3	$1^2\,(3456)^1$
8	567, 347, 127, 246, 235, 136		2	$(145)^1$
9		234567	4	$1^3\,(234567)^1$
10	167	234567	3	$1^2\,(2345)^1$
11	167, 145	234567	2	$(123)^1$
12	167, 145, 123	234567	1	

In (1), the line $l \equiv O_5O_6O_7$ breaks away from j_5, j_6 and j_7. Any point P of l determines the same degenerate pencil of cubics, consisting of l and the pencil of conics $(O_1O_2O_3O_4)$. By Geiser's construction, $P \sim$ all the points of l, including P itself, and l is part of Δ_6. But for I_7, the line l is self-corresponding, not invariant, and $P \sim$ another definite point of l, its other intersection with the conic $O_1O_2O_3O_4P$; and I_7 is not identical with Geiser's construction for points of l, though the two agree for general points of S. The invariant curve of I_7 is $\Delta_5 \equiv (O_1O_2O_3O_4)^2 (O_5O_6O_7)^1$.

Since (Ω) contains a degenerate pencil, this I_7 can be generated by two pencils, of cubics and of conics, with four common base points.

Similarly, (2) is generated by two pencils of conics with two common base points O_1, O_2 (409). This is the simplest example where the two pencils necessarily have a singular curve, the line O_1O_2 (p. 94). The F-point O_7 of I_6 which is not a base point is the intersection of the other components O_3O_4, O_5O_6 of the two conics having O_1O_2 as part, and $O_7 \sim O_1O_2$.

If P is a base point of one pencil only, one conic is indeterminate, and P' is any point of the other; hence O_3, O_4, O_5, O_6 are double F-points. If P is a common base point, O_1 say, the two conics touch here; this condition sets up a homography between the two pencils, and they generate a quartic, from which O_1O_2 drops away, leaving as locus of the other intersection the cubic P-curve which $\sim O_1$. Hence O_1, O_2 are triple F-points, and the homaloids are

$$(\phi_6) \equiv (O_1O_2)^3 (O_3O_4O_5O_6)^2 O_7^1.$$

The isologues are cubic and Δ is of degree 4; the class is 1, as for the general Geiser involution.

In (3), the point O_7 ceases to be an F-point and becomes invariant. It is triple on one of the cubics, and the other cubics of the pencil through O_7 osculate each other at O_7, which absorbs both their free intersections and is an invariant point of I_5.

In (6), the six simple F-points are vertices of a complete quadrilateral; and P' is the intersection of the other tangents from P and O_1 to the conic touching O_1P and the four sides of the quadrilateral (p. 105). This and (7) are non-central I_J's (375, 376).

The four last cases, with five points on a conic, are all central I_J's, each of the surviving simple F-points lying on the corresponding P-line joining it to O_1.

28. Example of class 2; $\mathbf{I}_{11}\ 4^5 5^2$ (30, 275).

The standard example of class 2 is of the semi-symmetric type $R5$ (Table II) and of degree 11; it can be generated by two pencils, of conics $q \equiv (O_1 \dots O_4)$ and of cubics $k \equiv (O_1 \dots O_9)$ with four common base points. By the formulae of p. 94, we find the homaloids

$$(\phi_{11}) \equiv (O_1 \dots O_4)^5 (O_5 \dots O_9)^2.$$

To determine the class, consider the coincidences of points of the two series cut out by (q), (k) on any line l. A point P of l determines one q, and one other intersection P' of q, l; then P' determines one k and two other intersections Q of k, l. If one of these $\equiv P$, then P, P' are free intersections of

q, k, and are conjugate for I_{11}. Now a point Q determines k and two other intersections P' of k, l, each of which determines one q and one P. Hence there is a 2, 2 relation between P, Q, with four coincidences, falling into the two pairs of conjugates of I_{11} lying on l; the class is 2, and the degree of Δ is 7.

29. Bertini's $I_{17}\,8^6$ (26, 30, 75, 275, 366).

The last of the symmetric transformations is also, in its only involutory form, the last of the standard types to which all involutions can be reduced.

This $I_{17}\,8^6$ is determined by the web of sextics $(f_6) \equiv (O_1 \ldots O_8)^2$: for the net through any point P has another base point P', related to P in involution.

Let $ak_1 + bk_2$ be the pencil of cubics through $O_1 \ldots O_8$ and let k_P be the member through P. Then if f_P is any one sextic of the net through P, this net is

$$f_P + k_P(ak_1 + bk_2),$$

and all its members pass through the last intersection P' of f_P, k_P. Thus I_{17} is generated by (k) and any of the pencils $(f_6) \equiv (O_1 \ldots O_8)^2 (XX')^1$.

If P is adjacent to O_1, the net has contact of one branch here, one member has a triple point and is the determinate sextic $j_1 \equiv O_1^3 (O_2 \ldots O_8)^2$, which is the locus of P', and the P-curve corresponding to O_1. Hence $O_1 \ldots O_8$ are 6-fold F-points, and there are no others; we have

$$n^2 - 1 = 8 \times 6^2, \quad n = 17.$$

If P is the ninth base point D of (k), the net of sextics is $(k_1, k_2)^2$, and all the free intersections are at D, which is invariant for I_{17}. The cubics and sextics are self-conjugate; any additional d.p. of an f_6 is invariant, the locus is found to be $\Delta_9 \equiv (O_1 \ldots O_8 D)^3$; hence the class is 4.

This transformation can be compounded of three of Geiser's I_7's (74). It can also be obtained by projection from the space $I_{15-15}\,7^8$ (77 and p. 326).

CHAPTER VII

RESOLUTION OF SINGULARITIES OF PLANE CURVES

1. Some characteristics of a singularity (79).

Cremona transformations afford one of the standard ways of dealing with the higher singular points of plane curves. The problem of singularity arises in relation both to a single point of a curve and also to the curve as a whole; and closely related is the problem of a singular intersection of two curves, each of which may or may not be itself singular at the point.

The properties of a singular point, after its multiplicity, which concern us are four: the effects on genus and class, the postulation and the equivalence. The first two are intrinsic to the curve itself, the class also involving the intersection with another curve, the first polar, determined by the given curve; the last two involve a family, or pair, of curves not wholly determined by the given curve, having the *same* singularity, the last phrase requiring definition, see below, p. 139. The reduction of genus is therefore the simplest of these characteristic numbers.

2. Ordinary and extraordinary singularities.

A *singularity* of a curve is a multiple point P of any kind. The chief geometrical property is that P absorbs more than one intersection of the curve with a general line through P. The chief analytical property is that near P there is no unique expansion of the form

$$y=\sum_{a=0}^{\infty} c_a x^a.$$

The simplest of all singular points of a curve is a d.p. with distinct tangents, and the next simplest are points of higher multiplicity with distinct tangents. These are called *ordinary* singularities of the curve. An *extraordinary* or *higher* singularity is a multiple point at which at least two tangents coincide. All simple points are ordinary; we do not consider inflexions, or other singularities of the curve regarded as an envelope.

If we apply the method of Cremona transformations to resolve higher singularities, we must accept all ordinary multiple points as elements needing no further resolution. For if a transformation T is applied to a general curve of degree μ, not specially related to the F- and P-system of T, it introduces a new set of singularities on the homologue; each i-fold F-point of T in the second plane is a μi-fold point with distinct tangents, that is, an ordinary multiple point. Since these are liable to be introduced by T, they must be admitted as elementary, and our aim is to reduce all others to this type.

We can go further if we use Riemann instead of Cremona transformations, birationally transforming the single curve and not the whole plane. Then every curve can be transformed into one having ordinary d.p.'s only (296, 406).

Geometrical constructions have been given for such curves of every possible degree and genus (363).

3. Statement of results (289, 291, 295).

We shall prove that the following problems can be solved by applying a Cremona transformation of finite degree (11, 31).

(i) To transform any singular point of an algebraic curve into a set of simple points.

(ii) To resolve any singular point into an equivalent set of ordinary points.

(iii) To transform any algebraic curve into one having ordinary multiple points only.

(iv) To transform any linear family into one having ordinary base points only.

Except where otherwise stated, we shall confine ourselves to the simplest Cremona transformation T_2, in its simplest form with all three F-points distinct, one coinciding with the point P which is being resolved and the other two in general positions. Any other transformations that may be used can be regarded as compounded of ordinary T_2's.

The problems (i) and (iii) deal only with the multiple components of P, and this is sufficient in discussing the genus p, on which a simple point has no effect. For example, they do not distinguish between an ordinary d.p. and a cusp of the first species; each reduces p by 1, and both can be transformed into simple points by one T_2. The only multiple component of the cusp is an ordinary d.p.; the difference lies in the disposition of the simple points that $\sim P$.

As regards the single curve f, an ordinary point P, lying as it must on one linear branch, is not capable of further resolution; T_2's applied to it in succession produce an indefinite number of similar points. But as regards characteristics, such as class or equivalence, involving the intersection of f with another curve f_1, owing to the possibility of contact, the point P, simple on each curve alone, may be a multiple intersection, and still call for resolution. We must go on applying T_2's till all the intersections are simple.

Thus to define fully the equivalent set of P, and what we mean by f_1 having the same singularity as f, we must know which of the simple points which $\sim P$ are essential to its nature, and therefore the same for f_1 as for f, and which are not essential, being variable with any parameters that may occur in f_1.

The definition must not be such as to oblige f_1 to be identical with f.

This can always be avoided if the degree μ_1 of f_1 is high enough, but it may impose a lower limit on μ_1.

If P is simple on the original curve f, no question arises; f_1 is merely required to have P as a simple point also; it need not touch f, and its tangent at P contains an arbitrary parameter. A T_2 applied to f, f_1 transforms P into two different simple points, one on f, fixed, and one on f_1, variable with the parameter. Similarly for an ordinary point of any multiplicity.

4. Transformation of an ordinary singularity.

An ordinary multiple point disappears entirely on transformation, being changed into a set of simple points. Let P be i-fold on f; apply a T_2 to $O_3 \equiv P$ and two general points O_1, O_2 as F-trio. Regarded as a point of the plane, $P \sim$ the whole of the P-line $j' \equiv O_1'O_2'$; regarded as a point of f, it $\sim$ the intersections $\Sigma P'$ of f', j', other than the F-points. These are a set of i points associated with the i tangents to f at P. Since P is ordinary, these tangents are distinct from each other; and they are also distinct from the P-lines PO_1, PO_2, since O_1, O_2 are general. Hence $\Sigma P'$ are distinct from each other and from O_1', O_2', and are simple intersections of f', j', and simple points of f'. The singularity of P is entirely removed by the transformation.

5. Accidental singularities.

This removal of P has been at the cost of introducing new singularities at the three F-points in S'; we must show that these are ordinary.

Since O_1, O_2 are in general positions, no P-line in S has a multiple intersection with f_μ other than P, and f_μ meets PO_1, PO_2 in $\mu - i$ general points and meets O_1O_2 in μ general points. With these simple intersections are associated distinct tangents to f' at O_2', O_1', O_3', and these are ordinary points of f' of multiplicities $\mu - i$, $\mu - i$, μ respectively. They are not points which $\sim P$, but are *accidental* singularities, arising from the T_2, quite independently of the nature of the singularity at P, except that the multiplicities of O_1', O_2' involve i.

For example, if f is a pair of lines and P their intersection, it might seem impossible that T_2 removes the d.p. But it does so, for $P \sim$ two simple points, one on each component of f'; the intersection of these components is the accidental d.p. O_3', arising from T_2 and not a homologue of P.

Besides introducing accidental singularities, the transformation also raises the degree of the curve f. In certain cases, this may be avoided, by taking the other F-points of T_2 at other multiple points of f, without thereby introducing accidental higher singularities. But unless f has points of high enough multiplicity, and is therefore of low enough genus, any Cremona transformation raises the degree.

6. Neighbourhoods.

Whatever the nature of P on f, or P' on f', the set of points $\Sigma P'$ constitutes the first *neighbourhood* of P on f; and any point is also regarded as belonging to this neighbourhood which $\sim P'$ by a series of T_2's regular at P' and its homologues. Similarly P^α is a point of f in the αth neighbourhood of P, if it $\sim P$ by a series of T_2's, for α of which the homologue of P^α is an F-point, and for all the rest an ordinary point. Then α is called the *rank* of the neighbourhood.

Thus by applying T_2's to each of the different points P', or their homologues, in succession, we obtain a set of points in the first neighbourhood of each; the aggregate of all these sets constitutes the second neighbourhood of P on f; and so on.

When P is an ordinary i-fold point, we have seen that T_2 removes it entirely. When P is of higher nature, we still say that T_2 removes an ordinary i-fold point, leaving a set of simpler points $\Sigma P'$; and that the singularity P is *equivalent* to an ordinary i-fold point together with all the singularities of f in the first neighbourhood of P, and possibly certain simple points of that neighbourhood to be specified below, p. 139. Hence it is also equivalent to all the singularities of f in the αth neighbourhood of P, together with certain simple points, and also a set of ordinary points of the same multiplicities as those points of f in the first $\alpha - 1$ neighbourhoods of P to which the T_2's of the series are applied.

This defines the equivalent set of P as regards its multiple points, and therefore as regards the effect of P on the genus of f.

7. Effect of T_2 on numerical characteristics.

This definition can be justified by showing that the values for P of the reduction R_p of genus, of the postulation P and of the equivalence E, are greater than their values R_p', P', E' for the set $\Sigma P'$ by their values for an ordinary i-fold point, namely

$$\tfrac{1}{2}i(i-1), \quad \tfrac{1}{2}i(i+1), \quad i^2 \quad \text{respectively.}$$

There are many definitions of genus, and from any of them it follows that p is invariant under any birational transformation, and is therefore the same for f' as for f. When the degree μ of f is given, the maximum of p is $\frac{1}{2}(\mu-1)(\mu-2)$. This is reduced by R_p due to P; let R_0 be the reduction due to all the other singularities of f:

$$p' = p = \tfrac{1}{2}(\mu-1)(\mu-2) - R_p - R_0.$$

The degree of f' is $\mu' = 2\mu - i$; its genus p' is reduced, from the maximum $\frac{1}{2}(\mu'-1)(\mu'-2)$ for this degree, by the accidental singularities O_1', O_2', O_3', of multiplicities $\mu - i$, $\mu - i$, μ, by the set $\Sigma P'$ which reduces it by R_p', and by the homologues of the other singularities of f, which are unaltered in nature by T_2, and reduce p' by R_0:

$$p' = \tfrac{1}{2}(\mu'-1)(\mu'-2) - 2 \cdot \tfrac{1}{2}(\mu-i)(\mu-i-1) - \tfrac{1}{2}\mu(\mu-1) - R_p' - R_0.$$

The consideration of P and E implies a linear family of curves having the same singularity at P as f. This phrase is not yet fully defined, but for the present we only assume it to give definite values to P, E for general members of the family. Let P_0, E_0 be the postulation and equivalence of all the other base elements of the family; these are unaltered by T_2. Expressing that the numbers of degrees of freedom, and of free intersections, are unaltered by transformation, we have

$$\tfrac{1}{2}(\mu+1)(\mu+2) - P - P_0$$
$$= \tfrac{1}{2}(\mu'+1)(\mu'+2) - 2 \cdot \tfrac{1}{2}(\mu-i)(\mu-i+1) - \tfrac{1}{2}\mu(\mu+1) - P' - P_0,$$
$$\mu^2 - E - E_0 = \mu'^2 - 2(\mu-i)^2 - \mu^2 - E' - E_0.$$

Hence

$$R_p - R_p' = \tfrac{1}{2}i(i-1), \quad P - P' = \tfrac{1}{2}i(i+1), \quad E - E' = i^2.$$

Thus these numerical characteristics are all reduced, unless $i = 1$, when R_p is unchanged.

More generally, if E, E' are the equivalences of P, $\Sigma P'$ on any two curves for which P is of multiplicities i, i_1 respectively, a similar argument shows

$$E - E' = ii_1.$$

8. Transformation of a higher singularity.

Let $P(0, 0, 1)$ be i-fold on

$$f_\mu \equiv z^{\mu-i}u_i + z^{\mu-i-1}u_{i+1} + \ldots + u_\mu.$$

The tangents at O are the factors of u_i, say $\prod_{a=1}^{i}(a_a x + b_a y)$. Apply the T_2,

$$x : y : z = x'z' : y'z' : x'y',$$

which is slightly more convenient than the standard form. We drop z'^i and obtain the homologue $f'_{\mu'} \equiv (O_1'O_2')^{\mu-i}O_3'^{\mu}$, where $\mu' = 2\mu - i$, and

$$f' \equiv x'^{\mu-i}y'^{\mu-i}u_i' + x'^{\mu-i-1}y'^{\mu-i-1}z'u'_{i+1} + \ldots + z'^{\mu-i}u_\mu'.$$

Thus $P \sim$ the set of points $\Sigma P'$, $z' = u_i' = 0$. If P is ordinary, the factors of Π are all different, and there are i distinct points P'.

If P is not ordinary, the tangents to f are not all distinct; let s of them coincide with $l \equiv ax + by$. Then

$$u_i \equiv l^s u_{i-s},$$

and the point P', $(b, -a, 0)$, associated with l by T_2, absorbs s intersections of f', j'. It may or may not be multiple on f'. Let this multiplicity be i'; then

$$i \geqq s \geqq i' \geqq 1.$$

If $i' > 1$, other conditions must be satisfied; f' has the form

$$f' \equiv (z', l' \overset{i'}{)\!(} x', y', z')^{\mu'-i'},$$

and therefore

$$u_i, \quad u_{i+1}, \dots u_{i+i'-1}$$

have as factors l raised to the power $i', i'-1, \dots 1$ respectively. Of these, only the condition on u_i is satisfied by virtue of the coincidence of tangents at P.

If $i + i' > \mu$, then f breaks up, and has $l^{i+i'-\mu}$ as a factor, as it would if P, P' were distinct multiple points lying on l.

The total number of intersections of f', j' at all the points $\Sigma P'$ is $\Sigma s = i$; but $s \geqq i'$. Hence

$$i \geqq \Sigma i',$$

that is, $i \geqq$ the sum of the multiplicities of all the points of f in the first neighbourhood of P.

Apply the same theorem to all the points P' and their later homologues; by summation we have $i \geqq$ the sum of the multiplicities of f at all its points in any one neighbourhood of P.

It follows that each P^a which $\sim P$ is of multiplicity $i^{(a)} \leqq i$; and further, $i^{(a)} = i$ only if P^a is the only point of the neighbourhood of P to which it belongs, and each earlier neighbourhood also consists of a single i-fold point.

9. **Example: cusp of first species.**

The simplest example of a higher singularity is a cusp of first species; then $i = 2$, $u_2 \equiv l^2$,

$$f \equiv z^{\mu-2} l^2 + z^{\mu-3} u_3 + \dots + u_\mu .$$

In S', the fact reappears in that two intersections of f', j' coincide at P'; the cusp at P imposes conditions both of incidence and contact on f' at a point of the first neighbourhood of P.

Hence P' must be regarded as a point of the equivalent set, and a second T_2 applied to it. Then f'' passes simply through the point P'' of intersection of the P-line j'' which $\sim P'$, with the line $j_1'' \equiv P''O_3''$ which $\sim j'$ by the second T_2. The tangent to f'' at P'' is in a general direction, depending effectively on the coefficients of u_3, to which the cusp presents no condition.

The point P'' also belongs to the equivalent set, for when P' and the second T_2 are fixed, P'' is a fixed and not a general point of j''. The condition of contact has disappeared, but f'' still satisfies a condition of incidence here.

To resolve P completely, we apply a third T_2 to P'', then f''' meets j''' in a general point P''' and passes there in a general direction; the last trace of the singularity at P disappears.

Thus P is transformed into a simple point of f' by one T_2; but to resolve it completely, transforming it into a general point at which the homologue of f is not restricted, we need the T_6 compounded of three T_2's.

In the T_4 compounded of the first two T_2's, P is a double F-point of partial contact, and, regarded as a point of S and not merely of f, its curve homologue in S'' consists of the two P-lines j'', j_1'', meeting in P'', which however is not an F-point of T_4^{-1}; the curve homologue of P in S''' by T_6 consists of the three P-lines j''', j_2''', j_1''', which $\sim P''$, j'', j_1'' by the third T_2.

The fact that P is a cusp and not an ordinary d.p. of f in S, reappears in S', in that f' touches the P-line j' at the simple point P'; and in S'', in that f'' passes through the d.p. P'' of the P-system (j'', j_1''). Both P' and P'' are double intersections of the curve homologues of f, P; but P''' is a simple intersection.

Thus the components of the cusp P, essential to its nature, are an ordinary d.p. and a simple point of simple contact with the P-line in its first neighbourhood; or an ordinary d.p. and two simple points, one in its first neighbourhood and one in its second, being multiple intersections of the curve homologues of f, P. The relation of these points to the P-systems is essential to the nature of the singularity; but for this, the description would also apply if, instead of the cusp, f had an ordinary d.p. at P, one branch osculating a fixed curve; only then the fixed tangent in S' would not be j', and the fixed point in S'' would not lie on j_1''.

If f_1 is another curve having a cusp of first species at P, with a different tangent l_1, its equivalent set has the same description, but it is not obtained by the same transformation. The first T_2 can be the same for both; but f_1' touches j' at a point P_1' different from P', and the second T_2's of the resolving series must be different. They can only be the same if $P_1' \equiv P'$; then $l_1 \equiv l$, and f_1, f have the same repeated tangent. But now f_1' touches f' at P', and f_1'' meets f'' at the fixed point P''; and the third T_2's can also be the same. The fact that P' is common to f_1, f causes P'' to be common also; for this reason P'' is called a *satellite* of P' (Vol. II, p. 350 of 124). The same T_6 removes the same equivalent set from f_1, f and transforms P into two different simple points of f_1''', f'''.

Thus we distinguish between curves having at P a singularity of the *same nature* as f, that is, a cusp of first species with a general tangent, and curves having the *same* singularity, that is, a cusp with the same tangent. This distinction is important, for we shall now show that in the latter case the curves form a linear family, and in the former case they do not.

In either case, f_1 is a member of the linear family of curves having a d.p. at P:

$$f_1 \equiv z^{\mu_1-2}v_2 + z^{\mu_1-3}v_3 + \dots + v_{\mu_1},$$

for which v_2 is a perfect square, l_1^2 say. The condition for this is quadratic in the coefficients of f_1; the family of all curves having a cusp at P is not linear; conversely, there is not in general a cusp on any other member of the linear family $cf + c_1 f_1$ determined by f, f_1.

But if we require either that every member of this pencil shall have a cusp, or that all curves having the same singularity as f shall form a linear family, $cl^2+c_1l_1^2$ must be a perfect square for all values of c/c_1; then $l_1\equiv l$, and the cuspidal tangent is fixed.

For a cusp with a given tangent, $R_p=1$, $P=5$, $E=6$, and we verify that these are all the same as for the equivalent set $1^2 2^1$.

We have now analysed the cases of the d.p., $i=2$, first when P is ordinary, $i'=s=1$, and next when P is a simple cusp, $i'=1$, $s=2$. There only remains the case $i'=s=2$. The condition for $i'=2$ is that l is a factor of u_3 (p. 133); this makes P a tacnode or cusp of higher species instead of an ordinary cusp. If there are no further conditions, P is a tacnode, and the tangents at P' are distinct from each other and from j', depending on the coefficients of u_4; and P' is an ordinary d.p. of f', and is completely removed by another T_2. The tacnode is composed of two adjacent ordinary d.p.'s.

10. Nature of a higher singularity.

In the general case, the nature of P depends, in the first instance, not only on the multiplicity i, but also on the numbers s in the sets of its coincident tangents. If any of the points P' is multiple on f', the nature of P must also include the similar characteristics i', s' for these points, and for any other multiple points P^α that arise in later neighbourhoods, by T_2's applied to $\Sigma P'$ or their homologues. Besides these numerical characteristics, the nature of P depends on the grouping of the points P^α in the successive neighbourhoods of their earlier homologues.

We call P an *explicit*, and P^α an *implicit* multiple point of f in S; then P^α is explicit on a homologue $f^{(\alpha)}$ of f, in a plane S^α. Each $i^{(\alpha)}$ is the actual multiplicity of an explicit singularity P^α in some plane, and disappears in the next T_2 applied to P^α. Each $s^{(\alpha)}$ is a number of coincident tangents at some P^α; it reappears in the next plane as a multiplicity of intersection of $f^{(\alpha+1)}$, $j^{(\alpha+1)}$ at $P^{\alpha+1}$, and a further series of T_2's must be applied to $P^{\alpha+1}$, whether this be multiple or simple on $f^{(\alpha+1)}$.

Consider in detail the cases that can arise in the second plane S'. We have $i\geqq s\geqq i'\geqq 1$.

(i) If $s=i'=1$, then P' is a simple point on f', which does not touch j'. It does not belong to the equivalent set and needs no further resolution.

(ii) If $s=i'$ and $i'>1$, then P' is multiple on f', but no branch of f' touches j'. The coincidence of tangents at P is ensured by the multiplicity of P', and s is not a separate characteristic. P' belongs to the equivalent set, and must be further resolved; it may be an ordinary or a higher singularity of f'.

(iii) If $s>i'$ and $i'=1$, then P' is a simple point of f' at which the linear branch has s-point contact with j'. In the next plane, f'' passes through the intersection P'' of j'', j_1'', and f'' has $(s-1)$-point contact with j_1'' there. To P'' and its homologues we apply $s-2$ more T_2's. The homologue of j_1'', in each S^α thus reached, is the line $j_1^{(\alpha)}\equiv P^\alpha O^\alpha$; the order of contact is reduced at each stage, and in S^s we have $f^{(s)}$ passing simply through P^s the intersection of $j^{(s)}$, $j_1^{(s)}$, but not touching either, the tangent being in a general direction. The next T_2 removes all trace of the singularity. $P' \dots P^s$ are s simple points of the equivalent set; and the last $s-1$ are satellites of P', the position of each depending on that of P' and on the choice of the T_2's, and not on the other parameters contained in f.

(iv) If $s > i' > 1$, then P' is a multiple point of the equivalent set, at which one or more branches of f' touch j'. First let only one tangent coincide with j'; then there is $(s-i'+1)$-point contact. In S'' as before, f'' has one linear branch through P'', with $(s-i')$-point contact with j_1'' and we obtain $s-i'$ simple points of the equivalent set, all satellites of P'.

(v) If $s > i' > 1$, and j' is a repeated tangent at P', then P'' is a multiple intersection of f'', j'' lying on j_1'', which must be analysed in the same way as P'. It and all its satellites are satellites of P'.

The same cases may arise in later planes. In the T_2 between $S^{\beta-1}$, S^β, a new P-line $j_\beta^{(\beta)} \equiv j^{(\beta)}$ arises from $P^{\beta-1}$; let the homologue of this in any later plane S^α be $j_\beta^{(\alpha)}$. Then in S'', ... S^α, ..., besides the cases enumerated for S', we may have an additional case, which does not occur in S' because there is only one curve $j_\beta^{(\alpha)}$ in that plane, but which arises in S'' in case (iii). This is a simple point of f^α at a point of intersection of two different curves $j_\beta^{(\alpha)}$.

In the compound transformation T^α between S, S^α, we have P as a complicated F-point, and the corresponding aggregate of P-curves is $\sum\limits_{\beta=1}^{\alpha} j_\beta^{(\alpha)}$. On account of the general choice of the P-lines of each T_2, no $j_\beta^{(\alpha)}$ has a multiple point which is free, that is, not an F-point of T^α; and all the free intersections of different curves $j_\beta^{(\alpha)}$ are simple.

The points of $f^{(\alpha)}$ that $\sim P$ are its free intersections with $\Sigma j_\beta^{(\alpha)}$; those which are satellites fall at intersections of two P-curves, the rest lie on only one.

When a simple point P^α of the equivalent set first arises, in the first neighbourhood of a multiple point, it is a satellite or has satellites. In either case, if it absorbs s intersections of $f^{(\alpha)}$, $\Sigma j_\beta^{(\alpha)}$, there are $s-1$ satellites in the series of simple points beginning with P^α.

We carry on the resolution as long as there are any multiple intersections of $f^{(\alpha)}$, $\Sigma j_\beta^{(\alpha)}$, other than F-points, whether these are multiple points of either, or points of contact. Finally, when P is completely resolved, every intersection is simple, where one linear branch of $f^{(\alpha)}$ meets one of a $j_\beta^{(\alpha)}$ without touching it.

Thus the nature of the singularity of f at P, the equivalent set, and the set of multiple points implicit at P, are defined in terms of certain explicit multiple and simple points P^α on curves $f^{(\alpha)}$ in other planes, arising from P and f by transformation; P^α may itself be of higher nature, but that does not enter into the definition directly, which only uses the multiplicities of ΣP^{α}, the way in which they are grouped in neighbourhoods, and their multiplicities on the P-systems. For a full discussion and a graphical method of exhibiting the resolution, see Vol. II, pp. 374–387 of 124.

If f_1 has the same singularity as f, this requires that P is of the same nature not only on these two curves but also on all the pencil $cf + c_1 f_1$. In the first plane, the characteristic numbers i, s must be constant for the pencil, and as for a cusp (p. 134) it follows that all the repeated tangents are fixed, and therefore, if the same T_2 is applied to all the curves, the positions of the points of the equivalent sets in S' are fixed and the second T_2's can be the same also. Since T_2 transforms a pencil into a pencil, the same holds at each P', and at all the points of the equivalent set; and the singularity is resolved for all the curves of the pencil by the same transformation.

If, at any P^{α}, there is a simple tangent to $f^{(\beta)}$ not coinciding with any $j_{\beta}{}^{(\alpha)}$, this is variable in the pencil, and leads to different points in $S^{\alpha+1}$. Any simple points that $\sim P$ and do not belong to the equivalent set, are variable; when the common singularity is completely resolved, the two homologues $f^{(\alpha)}$, $f_1{}^{(\alpha)}$ have no free common points that $\sim P$.

11. **Equivalent set finite** (31).

Before giving the final definitions, we must justify them in two ways, showing first that the equivalent set is finite, and then that it is independent of the arbitrary elements of the resolving T_2's.

First, the number of multiple points of the set is finite. For the reduction of genus of f due to P is finite, and that due to ΣP^{α} on $f^{(\alpha)}$ can never be negative at any stage, and is reduced by each T_2 applied to a multiple point. Thus a transformation of finite degree resolves P into a set of simple points ΣP^{α} of $f^{(\alpha)}$. Since $f^{(\alpha)}$ is the proper homologue of f, it has no common part with the P-system $\Sigma j_{\beta}{}^{(\alpha)}$; each P^{α} absorbs a finite number $s^{(\alpha)}$ of intersections with one P-curve, and possibly one with a second P-curve; it is followed by $s^{(\alpha)} - 1$ simple points of the equivalent set.

Hence the whole set is finite, and P is completely resolved by a transformation of finite degree.

If f is not proper, it consists of at most μ components, and $p \geqq -(\mu - 1)$. The argument follows as before.

If f is proper, we have an upper limit to the number, α say, of T_2's required to reduce the multiplicity of P. For α is the number of i-fold points in the equivalent set, and $\alpha \leqq$ the maximum number of i-fold points that f_{μ} can have without breaking up; since

$$p \geqq 0,$$

$$\tfrac{1}{2}(\mu-1)(\mu-2) - \alpha \, . \, \tfrac{1}{2} i (i-1) \geqq 0,$$

$$\alpha \leqq \frac{(\mu-1)(\mu-2)}{i(i-1)}.$$

Also $\alpha - 1$ is the greatest number of implicit i-fold points that can be adjacent to P, necessarily one in each of the first $\alpha - 1$ neighbourhoods (p. 133).

12. Uniqueness of the definition.

In the series of T_2's there are two kinds of arbitrary choice: first, of the other F-points O_1, O_2 of each T_2, and secondly, of the order in which the points of each neighbourhood are resolved. We have to show that neither affects the equivalent set.

This is because a regular transformation does not affect multiplicity or contact.

Consider two series V, W of T_2's, each of which resolves P entirely. Regarded as a point of f, let $P \sim$ the set of points $\Sigma P'_V$ of f'_V, lying on the P-line j'_V, by V_1, the first T_2 of V. Similarly W gives rise to $\Sigma P'_W, f'_W, j'_W$; and these are the ordinary homologues of the other set of elements, in the compound transformation $W_1 . V_1^{-1}$. Hence the multiplicities of P'_W on f'_W and on j'_W, and in their intersection, and the numbers of coincident tangents to f'_W, are all the same as for P'_V; if one is a point of the equivalent set, so is the other.

In that case, there is a later T_2 of each series which resolves the point. Let V_2 be the set of T_2's in V, after V_1, up to and including the next to be applied to P'_V or its homologue; and similarly W_2 of W. If the order of resolution of implicit points is different in the two series, V_2, W_2 may contain different numbers of T_2's. Then P'_V, P'_W are ordinary F-points without contact for V_2, W_2 respectively. By considering $W_2 . W_1 . V_1^{-1} . V_2^{-1}$, we see that the two sets of points $\Sigma P''_V$, $\Sigma P''_W$, forming the first neighbourhoods of P'_V on f'_V and of P'_W on f'_W, and part of the second neighbourhood of P on f, fall into pairs of ordinary homologues for this transformation.

The argument can be carried on to all later planes. If $P^{\alpha}{}_V$ is any point of the equivalent set, by the resolution V, this has a series of earlier homologues $P, P'_V \dots P^{\beta}{}_V \dots P_V^{\alpha-1}$, and gives a division of the process

$$V \equiv V_1 \dots V_\beta \dots V_\alpha \dots .$$

We can then determine a corresponding division of W, and a series of transformations in one of which $P^{\alpha}{}_V$ is the ordinary homologue of a point $P^{\alpha}{}_W$ of the other equivalent set, which is therefore of the same multiplicity as $P^{\alpha}{}_V$, and grouped in the same way in the neighbourhood of the same rank.

It follows that the equivalent set is independent of the particular series of T_2's used to resolve P.

It also follows that we can replace T_2 by any other transformation T_n which has P as an ordinary F-point, and whose P-curves through P are in general directions; for the two sets $\Sigma P'$ of points of f adjacent to P, made explicit by T_2, T_n respectively, are ordinary homologues in $T_n . T_2^{-1}$, and their natures are the same.

13. Final definition of resolution.

A singular point P of a curve is *resolved* by a series of T_2's applied to P and to each multiple intersection P^{α} in any later plane S^{α}, of the homologue

$f^{(\alpha)}$ with the set of P-curves $\Sigma j^{(\alpha)}$ which $\sim P$ in the transformation set up between S and S^α, other than intersections which fall at F-points.

The *equivalent set* of P is a set of ordinary points of the same multiplicities as all the points P^α to which the T_2's of the resolving series are applied.

A point P^α which is multiple on $\Sigma j^{(\alpha)}$ is a *satellite* of the nearest of its earlier homologues which is simple on $\Sigma j^{(\alpha)}$.

The *nature* of the singularity involves the number and multiplicities of the points of the equivalent set, and the way in which they are grouped, in successive neighbourhoods and sets of satellites.

A singularity of the same nature as that of $f^{(\alpha)}$ at P^α is *implicit* (or *latent*) on f at P.

Another curve f_1 has the *same singularity* at P as f, if P is of the same nature on each, and can be resolved by the same transformation. Then the general curve of the pencil $cf + c_1 f_1$ also has the same singularity.

Since any series of T_2's can be compounded into a single transformation, we can now solve all the problems of p. 129.

For the first we need only resolve P and the implicit multiple points; for the second, only such of these as are higher singularities. For the third we must apply the second process to each of the explicit higher singularities of f in turn; their number is finite. The ordinary singularities of f, and the accidental singularities introduced, which are all ordinary, remain so under the subsequent transformations; and the final homologue of f has ordinary singularities only.

The fourth is solved by resolving the higher singularities of the degenerate curve formed of two general members of the family (f); this set of points includes all base points of contact. The transformation changes (f) into another linear family all of whose base points are ordinary; these include the accidental singularities.

14. Transformation of first polars.

From the preceding work we see that the postulation, equivalence and reduction of genus are the same for P as for the equivalent set. The reduction of class R_c requires different treatment; for the class of f is not invariant under Cremona transformation. It is defined as the number of lines that can be drawn from a general point X to touch f properly, the lines joining X to the multiple points of f being improper tangents. It is also the number of variable intersections of f and the first polar f_X of X with regard to f, the fixed intersections falling at the multiple points. Thus R_c is the equivalence of P in the intersection of f, f_X.

In a general T_2 applied to P, the number of variable intersections of the two curves is invariant, but the homologues f', $(f_X)'$ are no longer a curve and one of its first polars; lines through X do not $\sim$ lines through a point X' unless X, X' are associated F-points of the T_2.

Now f_X is the locus of the polar groups, with regard to X as pole, of the groups of intersections of f with the rays l through X. If X, X' are associated F-points, then $l \sim$ a ray l' through X', and T_2 sets up a homography between l, l', in which $X \sim$ the intersection Q' of l' with the P-line j' which $\sim X$ in T_2. The group of intersections of $l, f \sim$ those of l', f' other than X', and the polar group of the former with regard to $X \sim$ that of the latter with regard to Q'. It is only if $X' \equiv Q'$, and lies on j', that the polar group of X' coincides with the homologue of that of X; and in this case the addition of the set of intersections of l', f' coinciding with X' has no effect on the variable points of the polar group; the first polar curves $f_X, f'_{X'}$ are homologues by T_2.

Then X also lies on its associated P-line, and T_2 is special, two F-points being adjacent at each of X, X'.

If $P\ (0, 0, 1)$ is i-fold on f,

$$f \equiv \sum_{\alpha, \beta=0}^{n} a_{\alpha\beta} x^\alpha y^\beta z^\gamma, \text{ where } \alpha+\beta+\gamma=n,\ \alpha+\beta \geqq i.$$

The first polar of $X\ (\xi, \eta, \zeta)$ is

$$f_X \equiv \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z}\right) f \equiv \Sigma a_{\alpha\beta} \left(\frac{\alpha\xi}{x} + \frac{\beta\eta}{y} + \frac{\gamma\zeta}{z}\right) x^\alpha y^\beta z^\gamma,$$

and P is $(i-1)$-fold on f_X.

If the T_2 with P as F-point is general, take it in the form

$$x' : y' : z' = xz : yz : xy.$$

The homologues of the two curves are

$$f' \equiv \quad z'^{-i} f(x'z', y'z', x'y') \quad \equiv \Sigma a_{\alpha\beta}\, x'^{\alpha+\gamma} y'^{\beta+\gamma} z'^{\alpha+\beta-i},$$

$$(f_X)' \equiv z'^{-(i-1)} f_X(x'z', y'z', x'y') \equiv \Sigma a_{\alpha\beta} \left(\frac{\alpha\xi}{x'} + \frac{\beta\eta}{y'} + \frac{\gamma\zeta z'}{x'y'}\right) x'^{\alpha+\gamma} y'^{\beta+\gamma} z'^{\alpha+\beta-i}.$$

But the first polar of $X'\ (\xi', \eta', \zeta')$ with regard to f' is

$$f'_{X'} \equiv \left(\xi' \frac{\partial}{\partial x'} + \eta' \frac{\partial}{\partial y'} + \zeta' \frac{\partial}{\partial z'}\right) f'$$

$$\equiv \Sigma a_{\alpha\beta} \left\{\frac{(\alpha+\gamma)\,\xi'}{x'} + \frac{(\beta+\gamma)\,\eta'}{y'} + \frac{(\alpha+\beta-i)\,\zeta'}{z'}\right\} x'^{\alpha+\gamma} y'^{\beta+\gamma} z'^{\alpha+\beta-i},$$

and cannot be identified with $(f_X)'$ for any choice of the coordinates of X, X', not even if they are associated F-points, with say $\xi = \zeta = \eta' = \zeta' = 0$.

If the T_2 is special, with $X\ (1, 0, 0)$ as an F-point of contact with z as fixed tangent, we can take

$$x' : y' : z' = xz : yz : y^2,$$

and the associated F-point is $X'\ (1, 0, 0)$ with z' as fixed tangent. Now

$$f_X \equiv \frac{\partial f}{\partial x} \equiv \Sigma a_{\alpha\beta}\, \alpha x^{\alpha-1} y^\beta z^\gamma,$$

$$f' \equiv \quad z'^{-i} f(x'z', y'z', y'^2) \quad \equiv \Sigma a_{\alpha\beta}\, x'^{\alpha} y'^{\beta+2\gamma} z'^{\alpha+\beta-i},$$

$$(f_X)' \equiv z'^{-(i-1)} f_X(x'z', y'z', y'^2) \equiv \Sigma a_{\alpha\beta}\, \alpha x'^{\alpha-1} y'^{\beta+2\gamma} z'^{\alpha+\beta-i},$$

$$f'_{X'} \equiv \frac{\partial f'}{\partial x'} \equiv (f_X)'.$$

Now the accidental singularity X' of f' is not ordinary, all the $n-i$ tangents coinciding with z'. This does not affect the process of resolution of P, or the nature of the equivalent set, but it prevents the resolution of the singularity of f as a whole, with which we are not at present concerned.

15. Reduction in class.

By the last theorem of § 7, since the reduction in class R_c is the equivalence of P on f, f_X,

$$R_c - \Sigma E_{P'} = i(i-1),$$

where $E_{P'}$ is the equivalence of P' on f', $(f_X)' \equiv f'_{X'}$; this difference being the value of R_c for the ordinary i-fold point removed. But $E_{P'}$ may not be the reduction in class due to P', on account of the special position of X', lying on j'.

Now if Y' is a general point near X', then the reduction in class R_c' due to $\Sigma P'$ is the equivalence of $\Sigma P'$ on f', f_Y'; and $\Sigma E_{P'} - R_c'$ is the number of proper tangents, drawn from Y' to f', that $\rightarrow j'$ as $Y' \rightarrow X'$. Since f' meets $P'Y'$ in i' points at P'; and meets j' in s points, the difference is

$$\Sigma(s-i') = i - \Sigma i'.$$

Hence $$R_c = i(i-1) + (i - \Sigma i') + R_c'.$$

Apply this result to each of the points P'. If R_c'' is the total reduction in class due to all the points of the second neighbourhood of P,

$$\begin{aligned} R_c &= i(i-1) + (i - \Sigma i') + \Sigma i'(i'-1) + \Sigma(i' - \Sigma i'') + R_c'' \\ &= i(i-1) + \Sigma i'(i'-1) + i - \Sigma i'' + R_c'', \end{aligned}$$

the sums extending to all the points of f in the first and second neighbourhoods of P respectively.

We can go on from plane to plane. When P is completely resolved, in the last plane we have a certain number N say of simple points, where N is the number of distinct branches of f at P; and for each of these, $i^{(a)} = 1$, $R_c^a = 0$.

Hence $$R_c = \Sigma i(i-1) + i - N,$$

the sum extending to all the points of the equivalent set.

Thus as regards reduction of genus and class, the singularity P is equivalent to $\Sigma \frac{1}{2} i(i-1)$ d.p.'s, of which $i - N$ are cusps of first species, and the rest are ordinary.

16. Sum of two singularities.

In the space theory, we need to consider the singularity P that arises, by free coalescence, when an i_2-fold point $Q_2 \rightarrow$ an i_1-fold point Q_1, both being of any nature; as, for example, when Q_1, Q_2 are points of two multiple curves of a surface, and f is its section by a plane which moves so that $Q_1, Q_2 \rightarrow$ an intersection P of the curves, of the simplest possible type. We call P the *sum* of Q_1 and Q_2.

If Q_1, Q_2 are ordinary d.p.'s, P is a tacnode B_4, equivalent to two successive d.p.'s. If Q_1 is a cusp B_3, and Q_2 an ordinary d.p., then P is a B_5, equivalent to a cusp in the first neighbourhood of a d.p. In each of these cases, the equivalent set of P is the sum of the elements of the equivalent sets of Q_1, Q_2, properly rearranged. The total reduction of class and of genus are unaltered.

If both Q_1 and Q_2 are simple cusps, P is a B_6; its equivalent set consists of three successive d.p.'s, with an increase of multiplicity at one point and a decrease at others, compared with the sum of the component sets. The total reduction of class is unaltered, that of genus is increased.

Just before the limit, apply T_2's, first to Q_1, and then to the homologue Q_2' of Q_2. In S', we have the cusp Q_2' which $\rightarrow$ the one point Q_{11}' of f' in the first neighbourhood of Q_1, where f' touches the P-line j_1'. In the second T_2, $Q_2' \sim$ a point Q_{21}'' of contact of f'', j_2'', and $Q_{11}' \sim$ a point Q_{11}'' of contact of f'' with j_1'', which is a conic meeting j_2'' only at a finite distance from Q_{11}'' and Q_{21}'', at the F-points of the second T_2. As $Q_2 \rightarrow Q_1$, so j_1' tends to pass through Q_2', and j_1'' tends to break up into a pair of lines, one of which is j_2'', and Q_{11}'' lies on the part of the conic j_1'' near j_2''; and neither j_1'' nor j_2'' meets f'' again near Q_{11}''.

Before the limit, f'' has two nearly parallel tangents which do not meet each other or the curve again near the points of contact. We cannot compare it to any linear branch whose curvature does not become infinite. It behaves like the part of a hyperbola near the vertices, when the axis $\rightarrow 0$, which acquires a d.p. in the limit.

In general, let $i_1 \geqq i_2$. If there is no restriction on the manner of approach, P is i_1-fold, and at least i_2 of the tangents coincide in the direction in which $Q_2 \rightarrow Q_1$. This may cease to hold if at either Q_1 or Q_2 there is a fixed tangent, and the direction of approach is not allowed to coincide with it, as in the last case, when two points simple on f'' coalesce in a d.p., which is not their sum, for the manner of approach is not free, but restricted.

If Q_1 is ordinary, in S' the only condition on f' is the singularity Q_2'; the equivalent set of P is obtained by prefixing an ordinary i_1-fold point to that of Q_2, and P consists of Q_2 in the first neighbourhood of Q_1.

If Q_1 is extraordinary, and has in its first neighbourhood another i_1-fold point Q_{11}', this is the only point of f' that $\sim Q_1$, and it is not a point of contact with j_1'. Then the point P_1' in the first neighbourhood of P is i_1-fold, and is the sum of Q_{11}' and Q_2', for their coalescence is free. Thus if $i_1 = i_2$, P has at least as many i_1-fold components as Q_1 and Q_2 together. We shall now show that it may have one but not two more. When we have removed all the i_1-fold components from Q_1 and Q_2, we have a set of points ΣQ_{11} on j_1, each of multiplicity $< i_1$, and together absorbing just i_1 intersections of f, j_1; these coalesce with a set ΣQ_{21} on j_2, and j_1, j_2 have no other intersections with f, and none with each other, near any of these points, and coalesce in the limit without acquiring any.

Let Q_{11} on j_1 be the point of either set which absorbs the highest number α of intersections with j_1 or j_2. The approach of a point Q_{21} of the other set may raise the multiplicity, but not beyond α, and it only reaches α if Q_{21} also absorbs α intersections with j_2. The approach of a point of the same set cannot raise the multiplicity, for it approaches along j_1, which is the only fixed tangent at either point. If there arises an i_1-fold point P_1, then $\alpha = i_1$, and each of Q_{11}, Q_{21} absorbs all the i_1 intersections of f with j_1, j_2, and is the only point of f in this neighbourhood of Q_1, Q_2 respectively.

Now the increase of multiplicity at P_1 relieves the conditions on the components of P in any later plane, and none of these can be of higher multiplicity than that of Q_{11} or Q_{21}, which $< i_1$. Hence P has no other i_1-fold component after P_1.

Thus the number of i_1-fold components of P is either the sum of those for Q_1 and Q_2, or exceeds it by just 1; and this excess occurs only if Q_1 has adjacent to it, after a series of i_1-fold points, a single point of multiplicity $< i_1$, and similarly at Q_2.

A proof of this result by the methods of infinitesimal geometry is given in 66; it is also shown that free coalescence does not affect the total reduction of class, though it may increase that of genus.

17. Twisted curve (207, 308, 406).

The singularities of a twisted curve k are its multiple points of any kind. For the purpose of resolving any one singularity, we can take as element an ordinary d.p. as above; but for the purpose of reducing k as a whole, we need not admit any kind of singularity as elementary, for we can always transform k into a curve quite free from multiple points.

We project k into a plane curve, and then transform it by a plane Cremona transformation T into a curve having ordinary multiple points only. This can always be regarded as the projection of a twisted curve without singularity. Both the projections and also the plane T can be brought about by Cremona space transformations of the monoidal type (p. 313). Then the space transformation, compounded of these three, changes k into a twisted curve free from multiple points.

CHAPTER VIII

NOETHER'S THEOREM

1. Statement of the theorem (61, 72, 285, 335, 405).

The central point of the whole plane theory is Noether's theorem:

Every Cremona plane transformation can be resolved into quadratic transformations.

This is proved by showing that every homaloidal net can be transformed into a net of lines by a series of T_2's applied to suitable trios of F-points. The transformation V compounded of the series of T_2's may not be exactly identical with the given transformation T_n; but since both T_n and V transform (ϕ_n) into lines, $T_n . V^{-1}$ transforms lines into lines, and is therefore of degree 1, say

$$T_n . V^{-1} \equiv T_1, \quad T_n \equiv T_1 . V.$$

The linear transformation T_1 can be compounded with the last T_2 of the series forming V, merely changing it into another T_2, and the series thus modified is identical with T_n. In this chapter, all transformations of (ϕ_n) into a net of lines will be considered as the same.

2. Reducing trios (279, 286, 335, 342).

If $n > 1$, there are at least three F-points. If a T_2 can be applied to the trio O_1, O_2, O_3 of multiplicities i_1, i_2, i_3, the degree of the homologue of (ϕ_n) is

$$n' = 2n - i_1 - i_2 - i_3,$$

so that $n' < n$ provided

$$i_1 + i_2 + i_3 > n. \qquad \text{..............................(1)}$$

A trio of F-points which satisfies (1) will be called a *reducing trio* if it defines a proper T_2. Now three points can fail to define a T_2 in two ways. First, they may be collinear, when the T_2 degenerates into a T_1; this is excluded when (1) holds, for the line $O_1O_2O_3$ would meet ϕ_n in more than n points, which is impossible since (ϕ_n) is proper. Hence if O_1, O_2, O_3 are distinct, they form a proper triangle. If O_2 is adjacent to O_1, and O_3 is distinct, the directions O_1O_2, O_1O_3 are different, and the trio defines a proper T_2 with one F-point of contact and one simple F-point. If they are all three adjacent lying on a linear branch, this has finite curvature, and again T_2 is proper, with one F-point of osculation.

The other case of failure, which is the only one possible when (1) holds, is when O_2, O_3 are both adjacent to O_1, but do not lie on the same linear branch, being either in different directions, or in the same direction but on a cuspidal branch. Then a curve through the three points necessarily has a d.p., and cannot be a proper conic, and T_2 breaks down (290, 354).

Whenever a reducing trio exists, a T_2 applied to it reduces n. If no obstacle arises, we can repeat the process, and finally reduce the degree to 1.

Then (ϕ_n) is transformed into a net of lines by a series of T_2's, into which T_n is resolved.

By Noether's inequality, the top trio $O_1O_2O_3$ always satisfies (1), and we can replace O_2 or O_3 by any of the major F-points (p. 75). These cannot be collinear with O_1, as explained above. Thus if any major F-point is distinct from O_1, or if any two are adjacent to O_1 on one linear branch, we have a reducing trio.

Thus the exceptional case, when no reducing trio exists, occurs only when all the major points are adjacent to O_1, and any branch which contains more than one of them is of order > 1. For all the major points, $\sum\limits_{\beta} i > i_1$ (p. 76); for the set of F-points adjacent to O_1 in different directions, $\Sigma i \leqq i_1$ (p. 133). Hence the major points cannot all lie in different directions, and some or all of them lie together on a cuspidal branch. No major point O_α, after O_1, can have two major points O_β, O_γ adjacent to it in different directions; for we should have

$$i_\alpha \geqq i_\beta + i_\gamma > 2\tau, \quad i_1 + i_\alpha > n.$$

When this exceptional case occurs, to resolve the transformation we have to begin with a T_2 which does not lower n. For example, the degree of the homaloidal net

$$c_1\{y^{n-1}z + (x, y)^n\} + c_2xy^{n-1} + c_3y^n$$

cannot be lowered by any Cremona transformation of degree $< n$ (354 and p. 113).

We can avoid this, and always lower n at each stage, if, instead of T_2, we use as auxiliary transformation the more general type T_J, which includes T_2 as its simplest case. We must prove separately that every T_J can be resolved into T_2's; then we finally have T_n resolved into a series of T_2's, each of which may raise or lower n, or leave it unaltered; and these T_2's are compounded by groups into a series of T_J's, each of which lowers the degree.

We shall first give an algebraic and a geometric method for the resolution of T_J; then the proof of Noether's theorem based on it; and lastly an alternative method of resolving T_n directly into T_2's.

The treatment of T_J is easier than the general case, because all its F-points are simple except O_1, which is $(n-1)$-fold. We have $\tau = \frac{1}{2}$, and all the simple F-points are major. As in the general case, no two of these are collinear with O_1, whether the three are distinct or adjacent, and if any F-point of T_J is distinct from O_1, there is a reducing trio.

3. Algebraic resolution of T_J (58).

First we shall show that, by a proper choice of the frames of reference, the equations of any T_J can be put in the form

$$x' = x, \quad y' = y, \quad z' = \frac{zu_{n-1} + u_n}{zv_{n-2} + v_{n-1}},$$

where u, v are functions of x, y only.

Take (0, 0, 1) at the multiple F-point O_1; the general homaloid has the form

$$\phi \equiv z u_{n-1} + u_n.$$

Now T_J has a P-curve $j_{n-1} \equiv O_1^{n-2}$ (p. 98). Let it be

$$j \equiv z v_{n-2} + v_{n-1};$$

this, together with any line through O_1, forms a degenerate ϕ. We can therefore take as base homaloids

$$\phi_1 \equiv xj, \quad \phi_2 \equiv yj, \quad \phi_3 \equiv z u_{n-1} + u_n.$$

Then

$$x' : y' : z' = \phi_1 : \phi_2 : \phi_3,$$

which, with a particular choice of the factor of proportionality, leads at once to the equations in the form stated.

Let the functions u_{n-1}, v_{n-2}, v_{n-1}, and also the combination Λ of degree $2n-2$,

$$\Lambda \equiv u_n v_{n-2} - u_{n-1} v_{n-1},$$

be resolved into their linear factors, say

$$u_{n-1} \equiv u^{(1)}_{n-1} u^{(2)}_{n-1} \dots u^{(n-1)}_{n-1}, \text{ etc.}$$

Then the relation between z' and z can be built up in five stages:

$$\text{(i)} \quad z_1 = \frac{v^{(1)}_{n-2}}{v^{(1)}_{n-1}} \cdot \frac{v^{(2)}_{n-2}}{v^{(2)}_{n-1}} \dots \frac{v^{(n-2)}_{n-2}}{v^{(n-2)}_{n-1}} z, \qquad (n-2 \ T_2\text{'s})$$

$$\text{(ii)} \quad z_2 = \frac{\Lambda^{(2n-3)} \Lambda^{(2n-2)}}{v^{(n-1)}_{n-1} + z_1}, \qquad (1)$$

$$\text{(iii)} \quad z_3 = \frac{\Lambda^{(n-1)}}{v^{(1)}_{n-1}} \dots \frac{\Lambda^{(2n-4)}}{v^{(n-2)}_{n-1}} z_2, \qquad (n-2)$$

$$\text{(iv)} \quad z_4 = \frac{\Lambda^{(1)}}{u^{(1)}_{n-1}} \dots \frac{\Lambda^{(n-2)}}{u^{(n-2)}_{n-1}} z_3, \qquad (n-2)$$

$$\text{(v)} \quad z' = \frac{u^{(1)}_{n-1}}{v^{(1)}_{n-2}} \dots \frac{u^{(n-2)}_{n-1}}{v^{(n-2)}_{n-2}} (u^{(n-1)}_{n-1} + z_4). \qquad (n-2)$$

4. Number of T_2's required.

This gives the resolution of T_J into a series of T_2's, in number $4n-7$.

The method must be modified if any of the functions u_{n-1}, v_{n-2}, v_{n-1} vanish identically; the number of transformations can then be reduced. For example, if all the tangents to (ϕ) at O_1 are fixed, $v_{n-2} \equiv 0$; then we can take

$$z_1 = \frac{u^{(1)}_{n-1}}{u^{(1)}_{n}} \dots \frac{u^{(n-2)}_{n-1}}{u^{(n-2)}_{n}} z,$$

$$z_2 = \frac{u^{(n-1)}_{n-1} z_1 + u^{(n-1)}_{n} u^{(n)}_{n}}{v^{(n-1)}_{n-1}},$$

$$z' = \frac{u^{(1)}_{n}}{v^{(1)}_{n-1}} \dots \frac{u^{(n-2)}_{n}}{v^{(n-2)}_{n-1}} z_2,$$

requiring $2n-3$ steps; and $2n-3 < 4n-7$ if $n \geqq 3$.

The full number $4n-7$ cannot be necessary in any case. If there exists a reducing trio, one T_2 reduces the degree, and then by the proof just given, at most $4(n-1)-7$ more are required, making $4n-10$ in all. And in the only case when there is no reducing trio, O_2 and O_3 are adjacent to O_1 on two distinct branches or one cuspidal branch, there are two fixed tangents at O_1, distinct or coincident; then u_{n-1}, v_{n-2} have two common factors, say $u^{(1)} \equiv v^{(1)}$, $u^{(2)} \equiv v^{(2)}$, and if $n \geqq 4$, the last two T_2's of stage (v) reduce to T_1's, leaving $4n-9$ T_2's. This is therefore an upper limit for the number of T_2's required.

If $n=3$, we have $2n-3=4n-9$, and the number required by either method may be as high as 3. In the only case in which there is no reducing trio, O_1 is a cusp on (ϕ_3), and all the simple F-points are adjacent to it; j consists of the cuspidal tangent taken twice, and

$$v_1 \equiv 0, \quad v_2 \equiv u_2.$$

In the cubic case of the example on pp. 113, 145:

$$x' = x, \quad y' = y, \quad z' = \frac{y^2 z + x^3}{y^2}.$$

If we put as above

$$z = \frac{xz_1}{y}, \qquad z_1 = \frac{yz_2 - x^2}{y}, \quad z_2 = \frac{yz'}{x},$$

then

$$\frac{y^2 z + x^3}{y^2} = \frac{x(yz_1 + x^2)}{y^2} = \frac{xz_2}{y} = z'.$$

Thus the best limits we can give for the least number of T_2's required are

$$1 = 4n-7, \quad 4n-9,$$

for

$$n=2, \qquad n \geqq 3.$$

5. Geometrical interpretation.

The geometrical meaning of the various steps in the above method is as follows.

Stage (i). The first T_2 has as F-trio, O, an adjacent point A_1 of j, and a distinct point A_2 also lying on j. If the T_J is general, neither A_1 nor A_2 is an F-point of (ϕ), and T_2 transforms (ϕ) into a homaloidal net of $(n+1)$-ic monoids, whose P-curve is of degree n. But T_2 lowers by 1 the degree of the proper homologue of j, which is part of the new P-curve (p. 17), which now consists of two rays through O_1 and a proper component of degree $n-2$. This indicates two fixed tangents at the vertex for the reverse transformation. Also (ϕ) acquires, as a simple F-point distinct from O_1, the F-point A_2' of T_2 in the new plane, associated with A_2, and also a new F-point A_1' adjacent to O_1.

The process is repeated, the F-points A_1, A_2 of T_2 always being taken on the proper part of j; this is possible as long as it passes through O_1, that is, as long as its degree $\geqq 2$. We continue till the proper part of j is reduced to a line λ, not passing through O_1, when A_1 no longer exists.

Stage (ii). Now λ meets (ϕ) in a set of two or more F-points of (ϕ) distinct from O_1. We use O_1 and two of these as F-trio of the next T_2; this step reduces n by 1, and λ disappears from j, which now consists wholly of rays. All the tangents to (ϕ) at O_1 are now fixed, since they $\sim$ the fixed intersections of λ, (ϕ), and there is a set of F-points O_2 adjacent to O_1.

Stage (iii). The net has also a set of $n-2$ F-points distinct from O_1, arising from the points A_2' of stage (i). The F-trio of the next T_2 consists of one of these, O_1 and an F-point O_2 adjacent to O_1. This reduces by 1 the degree of (ϕ) and the number of points A_2'. All the tangents at O_1 are still fixed. This is repeated as long as any points A_2' remain.

Stage (iv). We now choose a definite homaloid ϕ_3, not having j as a part, and break it up, just as j was broken up by the first stage, by a series of T_2's, the F-trio consisting of O_1, an adjacent F-point O_2 and a distinct point P, both O_2 and P belonging to the proper part of ϕ_3. This does not alter the degree of (ϕ), and its tangents at O_1 remain fixed; and ϕ_3 breaks up into a set of rays and a proper part of lower degree, which can be reduced to a line l not passing through O_1. The nature and degree of j are unaltered.

Stage (v). Now l meets j in a set of F-points distinct from O_1; we use one of these with O_1, A_1' as F-trio of the next T_2. This reduces by 1 the degree of (ϕ); its tangents at O_1 are still fixed, and ϕ_3 still contains a line not passing through O_1, the homologue of l in T_2.

We repeat the process till the degree of (ϕ) is reduced to 1, when the resolution of T_J is complete.

In all the stages except possibly the second, the auxiliary transformations are not general, but have two F-points adjacent.

We can now see why the series is shortened, some of the T_2's becoming T_1's, when certain of the functions have common factors.

Stage (i) is shortened if v_{n-2}, v_{n-1} have a common factor, which must be a factor of Λ also; for j is partly broken up at the outset. This also shortens (iii), which serves to eliminate the extraneous F-points A_2' introduced by (i).

Stage (iii) is again shortened if v_{n-1}, Λ have a common factor. If this is not a factor of v_{n-2}, it is one of u_n; then (ϕ) has a distinct F-point, given by $z = v_{n-1} = u_n = 0$, which serves as A_2 for some step of (i).

Stage (iv) is shortened if u_{n-1}, Λ have a common factor, which must be a factor of u_n or of v_{n-2}. In the former case, ϕ_3 is partly broken up at the outset. In the latter, (ϕ), j have a common fixed tangent, and one of the points O_2 serves as A_1 for some step of (i). Then (v) is also shortened, which serves to eliminate the A_1'.

In the case $v_{n-2} \equiv 0$, considered above, j is a set of rays at the outset, and the whole of stages (i), (ii), (iii) drop out.

6. Geometrical resolution of T_J (355).

The geometrical method consists in finding a set of four T_2's, which together lower the degree. If any F-point is distinct from O_1, then these and any third F-point form a reducing trio (p. 144); we need therefore only consider the case when all the simple F-points are adjacent to O_1. We shall show that the first T_2 raises the degree from n to $n+1$, and scatters some of the F-points; the second scatters them still further, leaving the degree unaltered; the third brings the degree back to n and the fourth reduces it to $n-1$.

Apply a T_2 to O_1 and two general points A_1, A_2; let O_1', A_1', A_2' be the associated F-points of T_2 in the next plane. Then $(\phi_n) \sim (\phi'_{n+1})$, of degree $n+1$, on which O_1' is n-fold; A_1', A_2' are distinct simple F-points of (ϕ') in general positions, which will be used at the fourth stage to reduce n. The $2n-2$ simple F-points $O_2 \dots O_{2n-1}$ of (ϕ_n), adjacent to O_1, $\sim$ simple F-points $O_2' \dots O'_{2n-1}$ of (ϕ'), at or adjacent to one or more distinct points of $A_1'A_2'$, and all distinct from A_1', A_2' and O_1'.

Since $n > 2$, there are at least four of these. The sum of their multiplicities is $2n-2 > (n+1)-2$, so they cannot all be intersections of $A_1'A_2'$ with ϕ'; at least one, O_3' say, is not a point of $A_1'A_2'$, but is adjacent to a point O_2' of that line, the direction $O_2'O_3'$ being different from $O_2'A_1'A_2'$, and also different from $O_2'O_1'$.

Apply a second T_2 to O_1', O_2' and a general point A_3'. The transformed net (ϕ'') is still of degree $n+1$; on it O_1'' is n-fold and O_2'' simple; A_3'' does not lie on it. $O_3' \sim$ a simple F-point O_3'' on $O_1''A_3''$, distinct from O_1''. Now A_1', $A_2' \sim$ two simple F-points A_1'', A_2'', collinear with O_2'', which do not lie on the sides of the proper triangle $O_1''O_2''O_3''$.

A third T_2 applied to $O_1''O_2''O_3''$ transforms (ϕ''_{n+1}) into (ϕ_n''') of degree n, on which O_1''' is $(n-1)$-fold and A_1''', A_2''' are distinct simple F-points. A fourth T_2 applied to $O_1'''A_1'''A_2'''$ reduces the degree of the net to $n-1$.

Thus a set of four T_2's reduces n by 1, and $4(n-2)$ T_2's reduce it to 2. All the auxiliary T_2's are general, with all three F-points distinct.

If $n=2$, the given transformation is a single T_2, but it may not be of general type. If two of its F-points are adjacent and one distinct, it is compounded of 2 general T_2's, or if all three F-points are adjacent, of 4; it never requires more than 4 (p. 37).

If $n \geqq 3$, then $4(n-2)$ T_2's reduce n to 2; at the end of this process, O_1 is distinct from any other F-point, but O_2, O_3 may be adjacent; the T_2 to which T_J is reduced is compounded of two general T_2's. Hence any T_J of degree $n \geqq 3$ can be resolved into at most $4n-6$ general T_2's, or $4n-4$ if the limit is to hold for $n=2$ also.

7. Proof of Noether's theorem (58, 128).

Let (ϕ_n) be a homaloidal net of any kind, of degree n; we have to find a T_J that reduces its degree.

Let μ be an integer such that $2 \leqq \mu \leqq \frac{1}{2}\beta + 1$, where β is the number of major F-points (p. 75). Since $\beta \geqq 2$, this is always possible. We shall prove that the F-system $O_1^{\mu-1} O_2 \ldots O_{2\mu-1}$ defines a proper T_J of degree μ. If this exists, it transforms (ϕ_n) into a net of degree n', equal to the number of free intersections of ϕ_n, ϕ_μ:

$$\begin{aligned} n' &= n\mu - i_1(\mu - 1) - \sum_{(2\mu-2)} i_\alpha, \text{ the sum extending to the } 2\mu - 2 \text{ points,} \\ &= n\mu - (n - 2\tau)(\mu - 1) - \sum_{(2\mu-2)} (i_\alpha - \tau) - (2\mu - 2)\tau \\ &= n - \sum_{(2\mu-2)} (i_\alpha - \tau). \end{aligned}$$

This sum has at least two terms, each $\geqq \frac{1}{2}$; hence

$$n' \leqq n - (\mu - 1) < n.$$

Let μ have its maximum value, $\frac{1}{2}\beta + 1$ or $\frac{1}{2}(\beta + 1)$, according as β is even or odd. Then $\sum\limits_{(2\mu-2)}$ is either the same as, or just one term short of $\sum\limits_{\beta}$, which extends to all the major points. For this value of μ,

$$\sum_{(2\mu-2)} (i_\alpha - \tau) = \sum_{\beta} (i_\alpha - \tau) - (0 \text{ or } i_\alpha - \tau).$$

But $\qquad i_\alpha \leqq 2\tau, \quad \sum\limits_{\beta} (i_\alpha - \tau) > n - 3\tau$ (p. 76).

Hence
$$\sum_{2\mu-2} (i_\alpha - \tau) > n - 4\tau,$$
$$n' < 4\tau.$$

Now the F-system proposed for T_J defines a net of monoids (ϕ_μ), however the points may coincide; but it might fail to determine a transformation if every member of (ϕ_μ) is forced to break up. If this is so, let ϕ_ν be the variable part of ϕ_μ. The fixed part $\phi_{\mu-\nu}$ cannot pass more than $\mu - \nu$ times through O_1; hence ϕ_ν passes at least $\nu - 1$ times, and is either a monoid or a set of rays through O_1.

If ϕ_ν is proper, $\nu < \mu$; it cannot be a single ray, for then (ϕ_μ) would be a pencil, and not a net; hence it is a monoid, variable within a net, which has for base $O_1^{\nu-1} \sum\limits_{(\gamma)} O_\alpha$, where the γ points O_α are some but not all of the base points of (ϕ_μ), in number $\gamma = 2\nu - 2 < 2\mu - 2$.

Then the number of free intersections of ϕ_ν, ϕ_n is

$$\begin{aligned} n'' &= n\nu - i_1(\nu - 1) - \sum_{(\gamma)} i_\alpha \\ &= n - \sum_{(\gamma)} (i_\alpha - \tau) \\ &> n - \sum_{(2\mu-2)} (i_\alpha - \tau), \end{aligned}$$

since the latter sum has additional terms > 0; hence $n'' > n'$.

But the free intersections of ϕ_ν, ϕ_n are some or all of those of ϕ_μ, ϕ_n; hence

$$n'' \leqq n',$$

giving a contradiction: ϕ_ν cannot be proper.

If ϕ_ν breaks up, since no part of it is fixed, it consists of two or more variable members of a fixed pencil (28) and can only be a set of $\nu \geqq 2$ rays, each meeting ϕ_n in i_1 points at O_1 and $n - i_1 = 2\tau$ variable points. In this case, the number of free intersections of ϕ_ν, ϕ_n is $n'' \geqq 4\tau$. But we have seen that we can choose μ so that $n' < 4\tau$; then

$$n'' > n',$$

giving the same contradiction as before: ϕ_ν cannot break up.

Hence ϕ_ν does not exist; (ϕ_μ) does not break up, but effectively defines a T_J which reduces the degree of (ϕ_n). Noether's theorem follows in its full generality.

Now T_J can be resolved into at most $4\mu - 4$ general T_2's. Let T_n require α T_J's. Then

$$\sum_{(\alpha)} (\mu - 1) \leqq n - 1;$$

and the number of general T_2's required to resolve T_n is at most

$$\sum_{(\alpha)} (4\mu - 4) \leqq 4n - 4$$

as for T_J.

8. **Alternative proof** (2, 67, 279, 290).

A T_2 applied to O_1 and two general points A_1, A_2 converts ϕ_n into

$$\phi'_{2n-i_1} \equiv O_1'^{n} (A_1'A_2')^{n-i_1},$$

and converts O_1 into a P-line $j' \equiv A_1'A_2'$. All the F-points of (ϕ) in the first neighbourhood of $O_1 \sim F$-points of (ϕ') of the same multiplicities at points Q' of j' other than A_1', A_2'. An F-point O_α of a higher neighbourhood $\sim$ a point O_α' adjacent to Q', which is an F-point of (ϕ') of multiplicity $\geqq i_\alpha$. If O_α lies on a linear branch through O_1, the direction $Q'O_\alpha'$ is different from $Q'A_1'A_2'$; if O_α lies on a cuspidal branch, the directions are the same, so that O_α' is among the intersections of ϕ', j'.

Let $\sum_{(\gamma)}$ apply to all the major F-points of (ϕ) in any neighbourhood of O_1, whose homologues lie on j'. Since j' meets ϕ' in $2n - i_1$ points, of which $2(n - i_1)$ lie at A_1', A_2',

$$\sum_{(\gamma)} i_\alpha \leqq i_1.$$

Hence $\sum_{(\gamma)}$, which by definition is part of $\sum_{(\beta)}$, cannot be the whole of it (p. 76), and there exists a set $\sum_{(\beta-\gamma)}$ of major F-points not belonging to $\sum_{(\gamma)}$, and therefore either distinct from O_1, or adjacent to it in neighbourhoods higher than the first, each lying on a linear branch through the two major F-points which it immediately succeeds.

Thus if $\underset{(\beta-\gamma)}{\Sigma}$ has no point on a cuspidal branch through O_1, it has at least one point either distinct from O_1, or in its second neighbourhood on a linear branch, which contains another major point in the first neighbourhood of O_1; in either case there is a reducing trio.

If all the points of $\underset{(\beta-\gamma)}{\Sigma}$ lie on cuspidal branches through O_1, we shall show how to transform this into the last case by a first series of T_2's, which raises the degree, and then show that a second series of T_2's reduces the degree to less than its original value.

Stage (i). Apply a special T_2 with two adjacent F-points, O_1 and any point O_a of $\underset{(\gamma)}{\Sigma}$ in the first neighbourhood of O_1, and the third A in general position. Then

$$\phi_n \sim \phi'_{n'} \equiv O'^{\,n'-2\tau}_1\, O'^{\,2\tau}_a\, A'^{\,2\tau-i_a},$$

where $$n' = n + 2\tau - i_a < n + \tau, \text{ since } i_a > \tau,$$

and A' is not a major point, since its multiplicity $< \tau$.

Now $\underset{(\gamma)}{\Sigma}$ may consist of points adjacent to O_1 in directions other than O_1O_a, of O_a itself, and of points adjacent to O_a. The first kind $\sim$ points of the same nature for (ϕ'); for the first neighbourhood of $O_1 \sim$ that of O_1', except for points near O_1O_a, which $\sim$ points near $O_1'A'$. But O_a itself is replaced by the 2τ-fold point O_a', adjacent to O_1' in a general direction, and with no other F-points of (ϕ') successive to it. Points of $\underset{(\gamma)}{\Sigma}$ of the last kind reappear with their multiplicities unaltered, each adjacent to O_1' in a neighbourhood of rank one lower than before. This can be proved from the definition of $\underset{(\gamma)}{\Sigma}$ by resolving the special T_2 into two with distinct F-points. Thus $\underset{(\gamma)}{\Sigma}$ is unaltered by the special T_2 except for the nature of O_a.

Apply γ T_2's in this way, at each step removing one of the γ original points from $\underset{(\gamma)}{\Sigma}$, replacing it by a 2τ-fold F-point, adjacent to $O_1^{(\gamma)}$ in a general direction, with no F-point successive to it. Then $(\phi_n) \sim (\phi^{(\gamma)}_{n^{(\gamma)}})$, where

$$n^{(\gamma)} = n + 2\gamma\tau - \underset{(\gamma)}{\Sigma} i_a,$$

on which $O_1^{(\gamma)}$ is $(n^{(\gamma)} - 2\tau)$-fold.

Stage (ii). Let $O_\delta^{(\gamma)}$ be a point of $\underset{(\beta-\gamma)}{\Sigma}$ for this net. Since it is not a point $O_a^{(\gamma)}$ of $\underset{(\gamma)}{\Sigma}$, nor successive to any $O_a^{(\gamma)}$, it is distinct from $O_1^{(\gamma)}$. We can apply a T_2 to $O_1^{(\gamma)}O_a^{(\gamma)}O_\delta^{(\gamma)}$; this gives a net of degree

$$n^{(\gamma+1)} = n^{(\gamma)} - i_\delta,$$

on which $O_1^{(\gamma+1)}$ is of multiplicity $n^{(\gamma+1)} - 2\tau$; the adjacent point $O_a^{(\gamma+1)}$ is of multiplicity $2\tau - i_\delta < \tau$, and does not belong to $\underset{(\beta)}{\Sigma}$; and $O_\delta^{(\gamma+1)}$ is not an F-point of $(\phi^{(\gamma+1)})$.

Thus $\underset{\gamma}{\Sigma}$ has lost one point completely, and is otherwise of the same nature as at the last step. Again $\underset{(\beta-\gamma)}{\Sigma}$ has a point distinct from $O_1^{(\gamma+1)}$; we can continue as long as $\underset{(\gamma)}{\Sigma}$ contains any point. After γ T_2's of this second series, the degree of the net is

$$\begin{aligned} n^{(2\gamma)} &= n^{(\gamma)} - \Sigma i_\delta = n + 2\gamma\tau - \Sigma i_\alpha - \Sigma i_\delta \\ &\leqq n - \gamma, \end{aligned}$$

for each of the 2γ terms i_α, $i_\delta > \tau \geqq \tau + \frac{1}{2}$, since 2τ is integral.

The combined series of 2γ T_2's reduces n by at least γ; Noether's theorem follows.

It also follows that n is reduced to 2 by at most $2(n-2)$ T_2's and to 1 by at most $2n-3$. But all the T_2's of the series are special, having two but not three adjacent F-points. If we require a resolution into general T_2's, each of the 2γ must be replaced by two, and the last T_2 to which T_n is reduced may have to be replaced by four; the total number is then $4n-4$ as before.

9. Rational resolution.

The question of the rationality of the resolution of a transformation is discussed in 193—195. A transformation is said to be *arithmetically* birational, if all the coefficients in the standard form of the equations are integral. In the equivalent series of T_2's, certain irrationalities may appear. If the components are all to be arithmetically birational, we must admit 16 classes of transformation as elementary, instead of T_2 only; of these, two are varieties of T_J, one of $R5$, these three being of general degree; the degrees of the others vary from 2 to 61.

Part II

CREMONA SPACE TRANSFORMATIONS

CHAPTER IX

OUTLINE OF THE GENERAL SPACE THEORY

1. Comparison with the plane (61, 76, 85–87, 117, 287, 385).

The theory of Cremona plane transformations is simple and complete. Any particular transformation is determined by one of its two homaloidal families of curves, whose degrees are equal, and this is determined by its *F*-system, which in turn is determined in all its main features by its characteristic, which merely enumerates a set of points and their multiplicities. If contact occurs, it can for most purposes be regarded as a particular case of the more general configuration without contact. The characteristics afford a satisfactory classification of the whole group of plane transformations, whose study is reduced, by Noether's theorem, to that of the simplest single transformation T_2.

In space, there is nothing simple to answer to a characteristic, except for very special cases (p. 318). A transformation is still determined by a homaloidal family, or by an *F*-system; but the two degrees are different, and the *F*-elements, instead of being all points, may be points or curves, and may be incident with each other in any way. The chief complication is that contact cannot in general be regarded as a limit, and for each *F*-element we must know not only the multiplicity, but also the order of contact, and the fixed element with which contact occurs (p. 173). The two plane equations of condition are replaced by formulae, ponderous in the simplest cases, which grow the less useful as they are the more general (p. 225). The incidence relations of the *F*- and *P*-elements are of great variety (p. 276); Clebsch's theorem has several partial extensions to space. It is still true that the two *F*-systems have the same total number of elements, properly reckoned.

It follows that there is no simple classification, beyond that given by the two degrees. Such other criteria as have been proposed do not throw much light upon the structure of the group of all Cremona space transformations.

A greater difficulty is the lack of any analogue of Noether's theorem. There must exist some base set of space transformations of which all others can be compounded; but we know practically nothing about it, except that it is an infinite set. More study of particular examples is needed before the underlying principles of a general theory can emerge; and at this stage we must develop methods for the simpler cases.

By means of auxiliary transformations, we could avoid the more complicated F-elements, and eliminate contact; but only at the expense of raising the degrees; and the theory of resolution is itself very complicated. In order to study the transformations of lower degrees, which we naturally attack first, we must be able to deal with contact directly (Chapter XII).

2. Homaloidal webs.

The definition of a *Cremona space transformation* is exactly parallel to that in the plane. It is a 1, 1 relation between the points P, P' of two spaces S, S' supposed to exist quite independently unless otherwise stated.

The immediate expression and consequences of the definition are also parallel to those of Chapter I. Let x, y, z, w be the homogeneous coordinates of P in S, and x', y', z', w' of P' in S'; then we must have a set of equations

$$x' : y' : z' : w' = \phi_1 : \phi_2 : \phi_3 : \phi_4, \qquad \text{..................(1)}$$

equivalent to the reverse set

$$x : y : z : w = \phi_1' : \phi_2' : \phi_3' : \phi_4', \qquad \text{..................(2)}$$

where the ϕ are homogeneous functions of x, y, z, w of degree n say, which is the degree of the direct transformation, and the ϕ' are homogeneous functions of x', y', z', w' of degree n' say, which is the degree of the reverse transformation; and in general, $n' \neq n$. This is the first great divergence of the space theory from the plane; a space transformation $T_{n-n'}$ has two degrees, which we show by the double suffix.

If the ϕ were unrestricted, equations (1) would define a Riemann transformation, which is 1, 1 between the points of a pair of homologous surfaces, but n^3, 1 between all the points of S, S'.

In order that (1) may be rationally reversible, and have the unique solution (2), we must specialize the web (ϕ), that is, the ∞^3 family of surfaces whose general member is $c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4$, corresponding to a general plane $c_1x' + c_2y' + c_3z' + c_4w'$ in S'. The family must be *homaloidal*, that is, it possesses the three characteristic properties of postulation, equivalence and genus:

(i) it is linear and ∞^3;

(ii) three general members have one and only one free intersection, that is, variable with the parameters $c_1 \dots c_4$;

(iii) all its members are rational.

These three properties are essential, since $(\phi) \sim$ the web of planes of S'; they are all invariant under Cremona transformation.

Cayley and some other writers call all rational surfaces homaloids (p. 341 of 315), but we do not use the term so widely (p. 3). For example, the family of rational quartic surfaces having a given double twisted cubic is ∞^5; but its members are not homaloids, for they have no free point of intersection, and we cannot select a homaloidal web from among them.

The web (ϕ) acquires its homaloidal properties by virtue of containing the base points and curves common to all its members; these are called *fundamental* elements (F-elements), and together form the *first F-system H* of $T_{n-n'}$; the base of (ϕ') is the *second F-system H'*. The multiple F-elements of H reduce the genus of ϕ to 0, and together with the simple elements, reduce (ϕ) to a web, and absorb all but one of the free intersections of three general members.

The particular surfaces $\phi_1, \ldots, \phi_4$, which $\sim$ the planes of reference in S', are called *base* homaloids in S. The transformation is determined in all essentials when one homaloidal web is given, defined by its F-system: we arbitrarily select four linearly independent members as base homaloids, and write down equations (1). This selection is equivalent to an arbitrary choice of the frame of reference in S'; any other base homaloids that might be chosen are linear functions of the first set, and their substitutions would effect a substitution of linear functions for x', y', z', w', equivalent to a change in this frame of reference.

3. Homaloidal twisted curves (192).

The total intersection, of two general members ϕ, ψ of the web (ϕ), breaks up into a set of F-curves ω and a residual c which varies with the parameters of ϕ, ψ. It is the latter only which $\sim$ the intersection in S' of the two planes which $\sim$ ϕ, ψ, that is, a general line l'. The intersection of l' with a third general plane $\sim$ the variable intersection of c with a third general homaloid χ; hence there is only one free intersection of c, χ, all the rest are F-points, either fixed or varying on ω.

Let m be the total degree of the F-curves ω, any multiple curve or curve of contact being reckoned as often as it counts in the intersection of ϕ, ψ. Then the degree of c is $n^2 - m$. This is the number of intersections of c with a general plane p; they are all variable with the parameters of ϕ, ψ, p, and they $\sim$ the intersections of l' with ϕ' the homologue of p. Since l' is a general line, these are all variable, and their number is n' the degree of the second homaloidal web:

$$n' = n^2 - m, \qquad m = n^2 - n';$$

and similarly $$n = n'^2 - m', \qquad m' = n'^2 - n,$$

giving the degree of the reverse transformation in terms of the first web, and the total degree of the F-curves in terms of the two degrees of $T_{n-n'}$. These relations take the place of the plane equality $n' = n$, which does not hold in general for space.

Since $m, m' \geqq 0$, we have limits for the second degree of $T_{n-n'}$ when the first is given:

$$\sqrt{n} \leqq n' \leqq n^2.$$

Thus if $n = 1$, then $n' = 1$; if $n = 2$, then $n' = 2$, 3 or 4; if $n = 3$, then n' can have any value from 2 to 9, and so on. Examples are given in Chapter XIV.

If ω consists of several multiple curves without contact, such as $\omega_m{}^i$, we replace m by $\Sigma i^2 m$; then

$$\Sigma i^2 m = n^2 - n', \text{ and can vary from } 0 \text{ to } n^2 - \sqrt{n}.$$

The same conditions that restrict (ϕ) to be a homaloidal web also restrict the family of twisted curves (c) to be ∞^4 and rational, for they $\sim$ the family of all lines of S'. This is called a *homaloidal family of twisted curves*; it is such that all the intersections of two general members are fixed or lie on fixed curves; but if the two lie together on a general ϕ, they have one and only one free intersection.

The homaloidal family of curves (c) can be taken instead of the web of surfaces (ϕ) to determine $T_{n-n'}$.

The net of curves (c) lying on a general ϕ is analogous to a homaloidal net of plane curves, being also a transformation of the lines of a plane. For a special homaloid ϕ_0, the net may have no free intersection. For example, if ϕ_0 has a d.p. at a simple F-point O, this is double on every c lying on ϕ_0, but simple on a general c; it absorbs two intersections of ϕ_0, ψ, χ but only one of three general homaloids. That is, O absorbs the one free intersection of ϕ_0, ψ, χ, and of any two curves c lying on ϕ_0.

Then ϕ_0 has no free intersection with a general c, and its homologue does not meet a general line, and is a curve or point, not a plane; ϕ_0 is an exceptional surface for T, of the kind called *principal* (p. 161).

4. Homologues.

A general element of S is transformed by T into an element of the same nature: a point into a point, a surface into a surface, represented by one equation between the coordinates, and a curve into a curve, represented by two or more equations.

By (2), a general surface in S, say, $f_\mu(x, y, z, w)$, of degree μ, $\sim$ the surface $f_\mu(\phi_1', \phi_2', \phi_3', \phi_4')$ or say $f'_{\mu'}(x', y', z', w')$, of degree $\mu' \equiv n'\mu$. It contains the whole of H' taken μ times; that is, an F-element i-fold of (ϕ') is μi-fold on f'. If f contains certain F-elements, we shall see that the result of substitution breaks up into principal surfaces, corresponding to these F-elements, and a residual surface f', of degree $< n'\mu$, which is the proper homologue of f, and $\sim$ all its general points. For example, if f is ϕ, then f' reduces to a plane.

A general curve k of degree $m \sim$ a curve k' of degree nm; for k meets ϕ in nm points, all variable with the parameters of ϕ; hence the homologue meets a general plane in nm points, and this is the degree of k'. If k passes through any F-points, we shall see that the transformed curve breaks up into a set of exceptional curves called *principal*, and a residual k' of degree $< nm$, which is the proper homologue of k. For example, if k is c, then k' reduces to a line.

The degree of k' can be simply verified when k is the total intersection of two surfaces f_μ, f_ν, where $\mu\nu = m$. These $\sim f'_{n'\mu}$, $f'_{n'\nu}$, containing μ, ν times respectively the F-curves ω' of total degree $n'^2 - n$. In the intersection of $f'_{n'\mu}$, $f'_{n'\nu}$ therefore, ω' counts as a curve of degree $\mu\nu(n'^2 - n)$, and the degree of the residual k' is $n'\mu . n'\nu - \mu\nu(n'^2 - n) \equiv nm$.

A given surface or curve can be studied by means of a homologue of lower degree, if we can find a Cremona transformation with such of its F-elements lying on the given variety, and of such multiplicities, that enough principal elements drop away from the homologue. Most of the space transformations in the earlier literature occur for this purpose (p. 393).

5. Postulation and equivalence.

As in the plane, each F-element has three numbers, answering to the three characteristic properties of the homaloidal web: the postulation P, equivalence E and reduction R in the genus of ϕ.

For a simple isolated F-point, $P = E = 1$, $R = 0$; for multiple points and points of contact, these numbers are higher, and $E > P$ (Chapter XI). For F-curves, P and E depend not only on the nature of the element, but also on the degree of the family.

These numbers refer to the element as imposed on the surface of degree n free from other restrictions. If (ϕ) already has F-elements with which the new element is incident, both P and E are reduced, and they may be so, even if there is no incidence with previous elements.

In the plane, if $i_1 + i_2 \geqq n + 1$, the homaloid breaks up, and the degree is lowered. In space, the only consequence is that (ϕ) acquires O_1O_2 as an F-line, as a necessary consequence of the two multiple F-points. If $i_1 + i_2 > n + 1$, not only does (ϕ) acquire O_1O_2 as a multiple line, but also the conditions presented to (ϕ) by O_1, O_2 cease to be independent, and the postulation of the pair becomes less than the sum of the postulations of the two points separately. Also the equivalence is entirely altered by the appearance of the F-line.

For example, let a quartic family (ϕ_4), which already has a triple F-point O_1, acquire a second distinct triple point O_2. The postulation of the first is found in Chapter XI to be 10, but of the two it is 19. For (ϕ_4) necessarily acquires yet another F-element, the double line $\omega_1 \equiv O_1O_2$. Now for a double line on a quartic family $P = 13$; but for the system O_1, O_2, ω_1, the sum of the separate postulations, $10 + 10 + 13 = 33$, is reduced by 7 for each incidence of ω_1 at a triple point, and the total is $33 - 2.7 = 19$.

In calculating P and E for a single element, we assume that n is high enough, compared with the multiplicity and degree of the F-element, to present the most general case. Otherwise ϕ_n may either break up, or contain other F-elements as a necessary consequence of its passage through those considered (327 and pp. 216, 220).

Thus the total numbers for the whole of H are not the sums of those for its elements taken separately but are reduced by the incidences of the elements.

The number of linearly independent terms in the general n-ic in four variables is

$$\binom{n+3}{3} \equiv \tfrac{1}{6}(n+1)(n+2)(n+3),$$

and three such surfaces have n^3 free intersections. The general homaloidal web has four linearly independent terms and one free intersection. Hence, for the whole of H,

$$P \leqq \binom{n+3}{3} - 4, \quad E = n^3 - 1.$$

We can show, just as on p. 5, that P has exactly this value, so that a homaloidal web is exactly determined by its F-system, and the conditions of postulation and genus follow from the condition of equivalence (61). A different proof is given in 146, by considering the net (c) of curves of intersection of one surface ϕ of the family with the remaining members. It is first shown that ϕ is rational; then any 1, 1 plane representation of ϕ converts (c) into a plane family satisfying the condition of equivalence, whose known properties (145) lead to the required result.

Further properties are given in 147—149. For example, the reduction in the genus of c due to any F-point is $E-P+R$; and if $\phi_1, \dots \phi_4$ are four general homaloids, the genus of a general plane section of $\phi_1\phi_2+\phi_3\phi_4$ exceeds twice the genus of a general plane section of ϕ by $n'-1$.

6. Types of F-elements (13).

The F-elements which compose H and H' may be either points or curves, either simple or of given multiplicity; and ϕ may be required either to contain the element, or to have contact of a given order with a given surface at the point or along the curve, or to have several sheets passing through the element, with separate conditions of contact for the different sheets.

Even then, H is not fully defined until we are given the incidences of the F-elements with each other; and at a point of contact we must know the relation of each branch of an F-curve, which passes there, to the fixed surfaces touched by ϕ.

The highest multiplicity possible for an F-element is $n-1$. For if O (0, 0, 0, 1) were an n-fold F-point, (ϕ) would all be cones with O as fixed vertex, the coordinate w would disappear from equations (1), which could not be reversed. If three of the cones had one free intersection P, they would have all the points of OP in common.

If there is an $(n-1)$-fold F-point, T is called *monoidal*. This type of space transformation is the most natural extension to space of the plane T_J, and has many important properties (p. 306).

It has been shown (28) that the homaloids cannot have any points of multiplicity > 2 except at F-points. But it does not follow that they all have exactly the same multiple points. An i-fold point may vary, for different members of the web, always lying on an $(i-1)$-fold F-curve of contact (97, 324, 325). For example (97, 158 Z_1), the cubic web

$$c_1 z(xw + z^2) + c_2 w(yz + w^2) + zw(c_3 z + c_4 w)$$

consists of cones with a common nodal edge $z = w = 0$, along which they touch the two fixed planes z, w, the vertex being the variable intersection of this edge with $c_1 x + c_2 y$. A cubic surface with a double line does not in general touch any surface of lower degree along it; if it is constrained to touch a fixed pair of planes, it can be shown that a triple point must occur at some point of the double line, and the cubic must be a cone as a necessary consequence of the conditions of contact, but the vertex is not thereby fixed (p. 249).

In general, the $(i-1)$-fold curve ω, with contact of all the sheets of ϕ, ψ, counts $(i-1)^2 + (i-1) \equiv i(i-1)$ times in the total intersection c_{n^2} of ϕ, ψ. Now at O, which is i-fold on ϕ, and $(i-1)$-fold on ψ, the tangent cone of ψ consists of $i-1$ planes, through t the tangent line to ω; that of ϕ is a proper cone of degree i, with t as an $(i-1)$-fold edge. The two cones meet in t counted $i(i-1)$ times, and have no other common generator. Hence c_{n^2} has $i(i-1)$ branches through O, all coinciding with ω, and the variable intersection $c_{n'}$ of ϕ, ψ does not pass through O, and has no condition presented to it by O, which is not an isolated F-point. The i-fold point, which every ψ has at some point of ω, is incidental to the contact conditions, in the same way that a certain number of pinch-points are incidental to a double curve (p. 254).

If H consisted of ordinary simple points only, the total P and E would both = the number of F-points. But if $n > 1$, then $\binom{n+3}{3} - 4 < n^3 - 1$, $P < E$. For a quadric family, $P = 6$, $E = 7$; but six simple F-points would leave two free intersections; and seven would not leave a web, and any three quadrics through the seven fixed points would also pass through an eighth, and would have no free intersection.

For any other type of F-point, $P < E$. For a simple point of simple contact, $P = 3$, $E = 4$, and accordingly there is a quadric transformation T_{2-4} (p. 292), for which H consists of a point of contact and three simple points.

Since $$m + m' = n(n-1) + n'(n'-1),$$
we cannot have $m = m' = 0$, and every transformation has an F-curve in one space or the other (p. 261).

The simplest transformation with no isolated F-point in one space is T_{2-3} (p. 287), whose second F-system consists of one double and three simple lines. The simplest with no isolated F-point in either space is T_{3-3} (p. 294) for which each of H, H' consists of a simple sextic curve.

7. The principal system.

The transformation breaks down for an F-point O, where x, y, z, w are such as to cause $\phi_1, \ldots \phi_4$ all to vanish at once. Equations (1) cease to be significant, and (2) cease to be independent, but reduce to two or one independent equation, instead of three, and represent a curve or surface corresponding to O, instead of a point. The elements of S' that ~ F-elements of S are called *principal*, and together form the *second P-system*; similarly there is a first P-system in S, corresponding to H'. These are discussed in Chapter XIII; for other terminologies see p. 14.

As in the plane, the break-down of the uniqueness of T for the F-points answers to the algebraic fact that the reverse equations (2), giving P' uniquely in terms of P, are not in all respects equivalent to (1) but only after the rejection of the common factor G' of degree $nn'-1$, that appears when we substitute in $\phi_1, \ldots \phi_4$ from (2) and obtain an identity. At points where $G'=0$, the reversal is not unique. We can write down G', without investigating the correspondence of its separate components with elements of H, most symmetrically as the factor which separates itself from the result of substitution in a general ϕ, leaving a general plane:

$$G' \equiv \sum_{\alpha=1}^{4} c_\alpha \phi_\alpha (\phi_1', \phi_2', \phi_3', \phi_4') / (c_1 x' + c_2 y' + c_3 z' + c_4 w'),$$

which is an integral function of x', y', z', w', of degree $nn'-1$, independent of $c_1, \ldots c_4$. Similarly,

$$G \equiv \sum_{\alpha=1}^{4} c_\alpha \phi_\alpha' (\phi_1, \phi_2, \phi_3, \phi_4) / (c_1 x + c_2 y + c_3 z + c_4 w).$$

As in the plane, we can get the equation of the first set of P-surfaces as the Jacobian

$$J \equiv \begin{vmatrix} \dfrac{\partial \phi_1}{\partial x}, & \ldots, & \dfrac{\partial \phi_4}{\partial x} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial \phi_1}{\partial w}, & \ldots, & \dfrac{\partial \phi_4}{\partial w} \end{vmatrix}.$$

This is of degree $4n-4$, different from that of G, and represents the same sheets but with different multiplicities. It will be shown on p. 281, that the multiplicity of any P-surface j in J is equal to the number of conditions presented to c' by each incidence of one of its branches with the F-element which ~ j; and its multiplicity in G is equal to the number of intersections of c', χ' absorbed by the incidence.

If $J \equiv G$, then

$$4(n-1) = nn' - 1, \quad n(4-n') = 3,$$

only satisfied if

$$n = n' = 1 \text{ or } 3.$$

We can show that an isolated i-fold F-point is $(4i-2)$-fold on J, and an i-fold F-curve is $(4i-1)$-fold on J.

Let O (0, 0, 0, 1) be an i-fold F-point; then

$$\phi \equiv w^{n-i} u_i + w^{n-i-1} u_{i+1} + \dots.$$

The leading terms in any column of J are

$$w^{n-i} \frac{\partial u_i}{\partial x}, \quad w^{n-i} \frac{\partial u_i}{\partial y}, \quad w^{n-i} \frac{\partial u_i}{\partial z}, \quad (n-i)\, w^{n-i-1} u_i;$$

but the determinant formed by these $\equiv 0$, since u_i is homogeneous in x, y, z, and $iu_i \equiv x \frac{\partial u_i}{\partial x} + y \frac{\partial u_i}{\partial y} + z \frac{\partial u_i}{\partial z}$. Hence the lowest terms in x, y, z in J are of degree $4i-2$ at least, and $J \equiv O^{4i-2}$.

If O is a point of an i-fold curve ω, and $x=y=0$ is the tangent to ω at O, then $u_i \equiv (x, y)^i$, being a set of planes through the tangent line, and $\frac{\partial u_i}{\partial z} \equiv 0$. The terms of J of degree $4i-2$ in x, y, z vanish identically, and $J \equiv \omega^{4i-1}$.

In the same way, if O is a simple F-point of contact, with z as fixed tangent plane,

$$\phi \equiv w^{n-1} z + w^{n-2} u_2 + \dots;$$

we find that $J \equiv O^4$, one sheet of the tangent cone to J at O being the fixed tangent plane to (ϕ).

8. Correspondence between J′ and H.

If j_α' is a factor of J' or G', of degree α, it gives a surface meeting a general line in α variable points. The homologue is therefore an F-element incident α times with a general c, where $\alpha > 0$.

If this is an F-point O, it is an α-fold base point of (c). The total intersection of ϕ, ψ in the neighbourhood of O consists of α branches of c, with or without some branches of F-curves ω. If O is wholly isolated, not lying on ω, since it lies on ϕ, ψ and on their total intersection, it lies on every c, and $\alpha > 0$; or O may be a point of ω, of such higher multiplicity or order of contact than the general points of the branches of ω through it, that the total intersection of ϕ, ψ at O is not exhausted by the F-curves.

Any point O which is a base point of (c) is called an *isolated* F-point, whether it lies on ω or not, in the sense of not being adjacent to ∞ points of the same nature. A general point O_1 of ω is not a base point of (c), though it may lie on particular positions of c; and it is one of an ∞ continuous series of points of ω all of the same nature. If an isolated F-point lies on ω, it is a special and not a general point of the curve.

Since a general point O_1 of ω lies on every ϕ but not on every c, the homologue meets every plane but not every line and is a P-curve κ', not a surface. The degree of κ' is the number of its intersections with a plane, equal to the number of times a general ϕ is incident at O_1. In general, this number is the multiplicity of ω on (ϕ); but for certain exceptional F-curves this does not hold (p. 270).

Thus F-points are of two kinds:

(i) isolated F-points, base points of (c), which $\sim$ P-surfaces of J';

(ii) points of F-curves, which are not base points of (c), and which $\sim$ P-curves.

For example, if O is an ordinary d.p. of a simple F-curve ω, it is a point of contact of (ϕ), and through it pass two branches of the total intersection of ϕ, ψ. Both of these are branches of ω, a general c does not pass through O which is of the second kind.

But if O is a triple point of ω, the three branches not all touching one plane, it is necessarily a d.p. of (ϕ); four branches of the total intersection of ϕ, ψ pass here, three being branches of ω and one of c. Hence O is a base point of c, and an isolated F-point, and $O \sim$ a P-surface, which is a plane since $\alpha = 1$.

There is a similar distinction between F-curves. Let ω have α intersections with a general c at points O_1 variable on ω, and β intersections at fixed points O, which are therefore isolated F-points lying on ω. The homologue of the whole set of points O, O_1 meets every line in $\alpha + \beta$ points, and is a surface or set of surfaces of degree $\alpha + \beta$. Of these, P-surfaces of total degree $\beta \sim$ the isolated points O alone, and the residual P-surface j_α' of degree $\alpha \sim$ all the general points O_1 of ω. As we have seen, any one point $O_1 \sim$ a P-curve κ', and j_α' is the locus of κ' as O_1 describes ω.

This assumes $\alpha > 0$; then it follows that κ' does vary with O_1. If $\alpha = 0$, the surface j' does not exist, or rather it reduces to a single curve κ', which may be regarded as a surface of degree 0, which remains fixed as O_1 describes ω. Then every point O_1' of $\kappa' \sim$ every point O_1 of ω, and O_1' is an F-point of H', and κ' is an F-curve of the same nature as ω; both ω and κ' are at the same time F-curves and P-curves; for an example see p. 273.

Thus we have two *species* of F-curves:

(i) F-curves which meet (c) in variable points, and which $\sim$ P-surfaces of J';

(ii) F-curves which meet (c) in isolated F-points only, and which $\sim$ F-curves of second species of H'.

For example, if a quartic family has an F-line ω_1 and a triple point O_1 lying on it, the residual intersection of ϕ, ψ has eight branches through O_1 and two meeting ω_1 in variable points. If none of these are other F-curves, they are all branches of c, and $O_1 \sim$ a P-surface j_8' and $\omega_1 \sim j_2'$. But if (ϕ_4) has a double F-point O_2 on ω_1, the residual has three branches through O_2 and no variable intersection with ω_1, which is now of second species and $\sim$ a P-line.

Thus we have F-points of two kinds and F-curves of two species. An F-point which is not isolated is usually not to be regarded as a separate F-element, but only as part of an F-curve. Thus F-elements are of three kinds, isolated F-points, and F-curves of first and second species.

9. Types of P-elements.

Conversely, we have three kinds of P-elements, homologues of the three kinds of F-elements. All the P-surfaces j' of S' are given by the different factors of J' or G', and are of two kinds according as they $\sim$ isolated F-points O or whole F-curves ω of first species. Since a general plane p does not contain O, and meets a curve ω_m of degree m in m points, if j' is of first kind, and $\sim O$, it meets (ϕ') in F-curves only; if j' is of second kind, and $\sim \omega_m$, it meets each ϕ' in m variable curves κ', which $\sim$ the m points of intersection of p, ω. Since it $\sim$ an F-point, each κ' is by itself a P-element, part of the P-surface j'.

But the sheets of J' may not exhaust the P-system, which may also include F-curves of second species, which $\sim$ F-curves of H, also of second species. These are discussed on p. 270.

These P-surfaces and curves drop away from the homologue of any surface f_μ or curve k_m in S, which contains the corresponding F-curve or point. Thus if f contains an F-curve ω of first species, and k contains a point O of ω, then a P-surface j' is dropped from f', and a P-curve κ', lying on j', is dropped from k'. One of the μm intersections of f, k is the point O, which $\sim \kappa'$, belonging to both the dropped parts, and O does not $\sim$ any intersection of f', k'. Thus, in equating the numbers of any two homologous sets of elements, we must first set aside any F-elements that belong to either set.

If a plane p contains an isolated F-point, or an F-curve of first species, the corresponding homaloid ϕ' breaks up, and one sheet is a P-surface. Unless p is a P-plane, ϕ' has one and only one sheet which is not a P-surface, but is the proper homologue of p. Conversely, if a ϕ' breaks up, all the sheets, except possibly one, are P-surfaces; and all P-surfaces which $\sim$ F-points or plane F-curves occur as sheets of degenerate homaloids.

The effect of T upon an F-point O is to spread it out into ∞^2 or ∞^1 P-points. Let an ordinary point $P \rightarrow O$ by a definite path; as long as P is distinct from O, it $\sim$ a definite point P', which $\rightarrow$ a definite point Q' as $P \rightarrow O$. There are ∞ paths of approach of P to O, and ∞ positions of Q', all P-points which $\sim O$.

In the simplest cases, the position of Q' depends only on the tangent at O to the path of P, its coordinates are rational functions of the two parameters defining this direction, and its locus is a rational P-surface j'. But it may be that sets of ∞^1 directions round O give the same Q', whose coordinates are rational functions of a single parameter, and whose locus is a rational P-curve κ'. If there is contact of (ϕ) at O, it may be that every line through $O \sim$ a curve approaching one of a finite set of points in S', and it is only certain higher curves through O, satisfying some of the same contact conditions as c, whose homologues approach the general points of the P-element. For an example, see p. 292.

If $O \sim$ a surface j', a surface f through O has a proper homologue f' from which j' has been dropped. The patch of f near $O \sim$ a part of f' near j', that is, the strip of f' along its curve of intersection with j'; and this curve $\sim$ the set of points of f adjacent to O, lying on the tangent plane or cone to f at O.

If $O \sim$ a curve κ', then f' contains κ', and the patch of f near $O \sim$ the strip of f' along κ'. If κ' is multiple on f' this strip may consist of strips of all the sheets of f' or of some only; then the strips of the other sheets of f' along $\kappa' \sim$ patches of f near other points such as O which also $\sim \kappa'$ and lie on f, and O is a point of an F-line of second species.

10. Transformation of adjoints (18).

The *adjoint family of order* α, of a given linear family with ordinary base points P and curves k,

$$(f_\mu) = \Sigma P^\sigma \, \Sigma k^\tau,$$

is the family, if it exists, of degree $\nu = \mu - 4\alpha$,

$$(g_\nu) \equiv \Sigma P^{\sigma-2\alpha} \, \Sigma k^{\tau-\alpha},$$

where the sums extend to the base points of (f_μ) of multiplicity $\sigma > 2\alpha$ and base curves of multiplicity $\tau > \alpha$. Simple and double points and simple curves of the base present no conditions to any adjoint, and the base elements of next lowest multiplicities drop out in succession as we form the higher adjoints.

We shall prove that adjunction is invariant under a Cremona transformation $T_{n-n'}$, in the sense that the proper homologue of (g_ν), supplemented if necessary by a set of P-surfaces of T, is adjoint of order α to the proper homologue of (f_μ).

We assume that there are no base elements of contact of (f_μ) and no F-elements of contact of T; for such, the definition of adjunction must be so extended that the transformation property holds. For example, if (f_μ) has a tacnode P, in a T_{2-2} with P as isolated F-point, $f \sim$ a surface f' having a double line κ' answering to the double tangent plane t to f at P (p. 362). If (g) is adjoint of first order to f, then $(g) \sim (g')$, which is adjoint to f', and therefore passes through κ', and g passes once through P and touches t there. The multiplicity of P on g is governed not by its explicit multiplicity on f, but by that of the implicit multiple line. In the same way, if f has a σ-fold point P of higher singularity, and adjacent to it an implicit infinitesimal τ-fold curve k, where $\sigma \geqq \tau$, then on the adjoint of order α the multiplicity of k is $\tau - \alpha$ and that of P is the greater of $\sigma - 2\alpha$, $\tau - \alpha$, that is, $\sigma - 2\alpha$ if $\alpha \leqq \sigma - \tau$ and $\tau - \alpha$ if $\alpha \geqq \sigma - \tau$. For an example, see p. 294.

Let the base elements of (g_ν) consist of a set O_1, ω_1 of F-elements of T and another set P_1, k_1. These are all base elements of (f_μ); let it also have other base elements O_2, ω_2, P_2, k_2, where O_2, ω_2 include all the other F-elements of T as base elements of (f_μ) of multiplicity $\geqq 0$.

Then $\quad \sigma_1 > 2\alpha, \; \tau_1 > \alpha, \quad 0 \leqq \sigma_2 \leqq 2\alpha, \quad 0 \leqq \tau_2 \leqq \alpha.$

Let $f'_{\mu'}, g'_{\nu'}$ be the proper homologues; then the results of substitution from the equations of T in f_μ, g_ν are

$$f'_{\mu n'} \equiv \Sigma O'^{i'\mu}\, \omega'^{i'\mu}\, P'^{\sigma} k'^{\tau} \qquad \equiv f'_{\mu'}\, \Pi j'^{\sigma}_{O} j'^{\tau}_{\omega},$$

$$g'_{\nu n'} \equiv \Sigma O'^{i'(\mu-4\alpha)}\, \omega'^{i'(\mu-4\alpha)}\, P_1'^{\sigma_1-2\alpha} k_1'^{\tau_1-\alpha} \equiv g'_{\nu'}\, \Pi j'^{\sigma_1-2\alpha}_{O_1}\, j'^{\tau_1-\alpha}_{\omega_1}.$$

Now the second Jacobian of T is

$$J'_{4(n'-1)} \equiv \Sigma O'^{4i'-2}\, \omega'^{4i'-1} \qquad \equiv \Pi j'^{2}_{O}\, j'_{\omega} \text{ (pp. 161, 281).}$$

Thus we can form a family of degree $4\alpha(n'-1) + \nu n' = \mu n' - 4\alpha$,

$$J'^{\alpha} g'_{\nu n'} \equiv \Sigma O'^{i'\mu-2\alpha}\, \omega'^{i'\mu-\alpha}\, P_1'^{\sigma_1-2\alpha} k_1'^{\tau_1-\alpha} \qquad \equiv g'_{\nu'}\, \Pi j'^{\sigma_1}_{O_1} j'^{\tau_1}_{\omega_1} j'^{2\alpha}_{O_2} j'^{\alpha}_{\omega_2},$$

which is adjoint to $(f'_{\mu n'})$.

But adjunction is not altered by dropping a common factor from the family and its adjoints; since $2\alpha \geqq \sigma_2$, $\alpha \geqq \tau_2$, we can drop $\Pi j'^{\sigma}_{O} j'^{\tau}_{\omega}$, and find that the family $g'_{\nu'}\, \Pi j'^{2\alpha-\sigma_2}_{O_2}\, j'^{\alpha-\tau_2}_{\omega_2}$ is adjoint of order α to $(f'_{\mu'})$.

If $\mu = 4\alpha$, and there is no $\sigma > 2\alpha$ and no $\tau > \alpha$, it is convenient to say that the adjoint of order α exists and consists of ∞^0 surfaces of degree 0 subject to no conditions. The zero system exists in the sense of offering no contradiction; and in this case the transformed family has an adjoint of order α composed of P-surfaces of T. Thus (f_4) with no base elements has a zero adjoint system of order 1. If we transform by a T_{2-2} (p. 187), we obtain the family $f_8' \equiv O'^4 \omega_2'^4$, whose first adjoint system $J_4' \equiv O'^2 \omega'^3$ exists, while the second would be $(f_0') \equiv \omega'^2$, which is self-contradictory and does not exist.

11. F-elements of highest multiplicity (18).

A homaloidal web (ϕ_n) has no adjoint systems of any order, for they would still be adjoint when (ϕ_n) is transformed into a family of planes, which has no adjoints at all. Hence for every value of $\alpha \leqq \frac{1}{4}n$, the F-elements that would survive on the adjoint present impossible conditions to a general surface of degree $n - 4\alpha$.

Let α be the quotient and β the remainder in $n \div 4$. The adjoint of lowest degree is f_β of order α and degree β. This is prevented from existing by the F-curves of multiplicity $i \geqq \alpha + 1$ and F-points of multiplicity $i \geqq 2\alpha + 1$. These are therefore at least either

(i) an aggregate ω_m of curves, of total degree m, of multiplicities $\geqq 1$ on f_β, and therefore of multiplicities $\geqq \alpha + 1$ on (ϕ); or

(ii) a point of multiplicity $\geqq \beta + 1$ on f_β and $\geqq 2\alpha + \beta + 1$ on (ϕ); or

(iii) a set of points of multiplicities $\beta, \beta - 1, \ldots 1$ on f_β, in suitable numbers and positions to prevent f_β existing, and therefore a suitable set of points of multiplicities $2\alpha + \beta, \ldots 2\alpha + 1$ on (ϕ).

None of these must be such as to prevent (ϕ) existing. In case (i), if ω lies on a surface of degree $\leqq \beta \leqq 3$, the condition $i \geqq \alpha + 1$ is necessary but not sufficient, and either $i \geqq \alpha + 2$, or there exist other F-elements that also present conditions to f_β.

Now $\omega_m^{\alpha+1}$ cannot exhaust the intersection of ϕ, ψ, for there is a free intersection of degree $n' \geqq \sqrt{n}$; hence

$$(\alpha+1)^2 m \leqq n^2 - \sqrt{n}$$

$$m < \left(\frac{4\alpha+\beta}{\alpha+1}\right)^2 \leqq 15, \text{ since } \beta < 4.$$

Hence there are for m at most the 16 values 0, 1, ... 15, and this gives a possible basis for the classification of transformations. For the lower values of n, the limit for m is much lower; for example,

if $\qquad n=4, \quad \alpha=1, \quad 4m \leqq 16-2, \qquad m \leqq 3,$

if $\qquad n=8, \quad \alpha=2, \quad 9m \leqq 64-\sqrt{8}, \quad m \leqq 6.$

Again, every quadrisecant of ω_m meets ϕ in at least $4(\alpha+1) > n$ points, and lies on it, and is an F-line. Hence ω_m has only a finite number of quadrisecants: if ω_m is plane, $m \leqq 3$; if ω_m lies on a quadric, $m \leqq 6$, for ω_m meets every tangent plane in at most three points of each generator of the quadric. If ω_m lies on a cubic f_3, the total intersection of f_3 with ϕ is of degree $3(4\alpha+\beta) \geqq (\alpha+1)m$; hence in this case, $m \leqq 11$.

By such methods, limits have been found for the F-elements that must exist in the various cases (18).

12. Cremona's theorem (85, 87).

Cremona has shown how to obtain all the transformations $T_{n-n'}$ whose general homaloid is any given rational surface ϕ_n of known degree. The first degree n of T is given by ϕ, but the second is unknown between the limits $\sqrt{n}$ and n^2 (p. 156). Since ϕ is rational, we assume that we have some 1, 1 representation V, not necessarily a Cremona transformation, of ϕ upon a plane p. Then Cremona's theorem is that the set of space transformations $T_{n-n'}$, having ϕ as a general member of the first homaloidal family, answers to a set of plane Cremona transformations of p, whose F-systems satisfy certain conditions, and whose degree $\leqq$ a certain limit m, these conditions depending only on ϕ and on V.

Thus when ϕ, V are given, m is a known number. We can obtain all the plane transformations of degree $\leqq m$ (p. 64), and examine which of them satisfy the other conditions; from the set thus selected we can construct the set of space transformations required.

The first homaloidal family (ψ) of T must have the same multiple elements as ϕ, with the possible exception that if ϕ has an i-fold point O lying on an $(i-1)$-fold curve ω, then (ψ), instead of having O as an i-fold F-point and ω as an ordinary multiple curve, may have ω as an $(i-1)$-fold curve of contact, with a variable i-fold point on it in place of O, which in this case is not an isolated F-point (p. 160). Then we may have to examine the two possibilities separately.

The multiple F-elements of (ψ) being thus fixed, it is the choice of the simple F-elements that determines each different transformation T. Unless there are enough multiple elements to reduce this choice within reasonable limits, and to keep m low, the method becomes unwieldy.

The total intersection c_{n^2} of ϕ, ψ consists of

(i) the known multiple F-curves of T,

(ii) the unknown simple F-curves of T,

(iii) the variable residual $c_{n'}$, also unknown.

The last is subject to certain conditions of incidence, which reduce it to a net. Since ϕ is a general homaloid, and not a P-surface, the net $(c_{n'})$ on ϕ has one and only one free intersection.

The unknown parts of (c_{n^2}), consisting of (ii) and (iii), form a net of degree $\leqq n^2$, with one free intersection, which is subject to certain known conditions of incidence with multiple elements of ϕ, as well as to certain unknown conditions. By V, the image in p of this net is a plane net (k_m) of definite degree m say, and each k_m consists of a fixed part $k_{m-\mu}$, image of (ii), and a variable part k_μ, image of (iii); and (k_μ) is a net with one free intersection, that is, a homaloidal net, determining a plane Cremona transformation W of p, of degree $\mu \leqq m$. Thus W is one of the finite set of plane transformations of degree $\leqq m$; and it can be any one of them which is such that its homaloidal family (k_μ), supplemented, if $\mu < m$, by a suitable fixed curve or set of curves of degree $m-\mu$, gives a (k_m) satisfying the known conditions of incidence, at the images of the multiple points of ϕ, and at the base points of the plane representation V. Thus we determine all the possible forms of W.

For each of these, (k_m) is a known net, consisting of a known homaloidal net (k_μ) and a fixed part $k_{m-\mu}$. Its image on ϕ, by V^{-1}, consists of a net $(c_{n'})$, determined by a known base, and a fixed set of curves ω, either images of $k_{m-\mu}$, or arising from additional incidences of (k_m) at base points of V, beyond those required by the known conditions. Thus we determine (ii), and the unknown base points of (iii). These are all simple elements of ϕ, and if any of them is reckoned more than once, it is an element of contact of (ψ). These simple elements, together with the known multiple F-elements, form an F-system H; and the family of surfaces (ψ) containing H is homaloidal: it is a web, for it cuts out a net on ϕ, which is a general member; and it has one free intersection, for ϕ has one free intersection with any two other general members. Thus every W determines one T; different W's may determine T's of the same type, with their F-systems differently related to the base of V.

13. Quadric transformations.

As the simplest example, let ϕ be a quadric, and V projection from an F-point O of ϕ. The base points in p are A_1, A_2, the traces of the two generators l_1, l_2 of ϕ through O. The total intersection of ϕ, ψ is $c_4 \equiv O$, and its image in p is $k_3 \equiv A_1 A_2$. All the plane homaloidal nets (k_μ) with $\mu \leqq 3$ can serve. If X_α is a general fixed point of p and O_α its image on ϕ, the possibilities are:

In P		On ϕ	
Homaloid	Fixed curve	Homaloidal curve	F-curve
k_1	$k_2 \equiv A_1 A_2$	$c_2 \equiv O$	ω_2
$k_2 \equiv A_1 A_2 X_1$	k_1	$c_2 \equiv O_1$	$\omega_2 \equiv O$
$k_2 \equiv A_2 X_1 X_2$	$k_1 \equiv A_1$	$c_3 \equiv O O_1 O_2$	ω_1
$k_2 \equiv X_1 X_2 X_3$	$k_1 \equiv A_1 A_2$	$c_4 \equiv O^2 O_1 O_2 O_3$	
$k_3 \equiv A_1^2 A_2 X_1 X_2 X_3$		$c_3 \equiv O_1 O_2 O_3$	$l_1 \equiv O$
$k_3 \equiv A_1 A_2 X_1^2 X_2 X_3$		$c_4 \equiv O O_1^2 O_2 O_3$	

Now a fixed simple point on c is an ordinary F-point of T, and a d.p. is an F-point of contact. Hence there are three types of quadric homaloidal webs, the F-systems being

(i) a conic and a simple point, T_{2-2} (p. 182),

(ii) a line and three simple points, T_{2-3} (p. 287),

(iii) a point of contact and three simple points, T_{2-4} (p. 292),

the degree of the reverse transformation being $4-m$, where m is the degree of the F-curve.

14. General cubic transformations.

Next, let ϕ be a general cubic surface. Take three skew lines $\lambda_1, \lambda_2, \lambda_3$ on it, and three points A_1, A_2, A_3 in p. We can map ϕ on p so that if a point X of p is the image of a point O on ϕ, then the lines XA_1, XA_2, XA_3 of the three pencils $(A_1), (A_2), (A_3)$ are related to the planes $O\lambda_1, O\lambda_2, O\lambda_3$ of the pencils $(\lambda_1), (\lambda_2), (\lambda_3)$ by three homographies.

Then A_1 is the image of a base line l_1 on ϕ, intersection of the planes related to A_2A_1, A_3A_1 in the second and third homographies; it meets λ_2, λ_3, but not λ_1. Similarly for A_2, A_3. The other base lines l_4, l_5, l_6 on ϕ are the three that meet all of $\lambda_1, \lambda_2, \lambda_3$; every point of l_4 has for image the same point A_4 of p, lying on the three lines related to $\lambda_1 l_4, \lambda_2 l_4, \lambda_3 l_4$. Thus there are six base points A in p, in general positions, and six base lines l on ϕ, skew to each other, and five meeting each axis. The axes themselves are not base lines; if $O \rightarrow$ a point Q of λ_1, it approaches it along the tangent plane to ϕ at Q, which contains λ_1, and the plane of the first pencil is determinate.

The image of a curve c_m in ϕ meeting l_α in γ_α points is a curve k_μ in p with γ_α branches through O_α, where $3\mu = m + \Sigma\gamma_\alpha$. The intersection of ϕ, ψ is a c_9 meeting each l three times; its image is a $k_9 \equiv \Sigma A^3$.

We examine all the plane homaloidal nets

$$(k_\mu) \equiv \Sigma A_\alpha^{i_\alpha} \Sigma X_\beta^{s_\beta};$$

if there exists a proper or degenerate $k_{9-\mu} \equiv \Sigma A_a^{3-i_a}$, the index being 0 if $i_a > 3$, then $k_\mu k_{9-\mu}$ is a $k_9 \equiv \Sigma A^3$, and (k_μ) determines a homaloidal web containing ϕ. There are 40 such cases, and the same space $T_{3-n'}$ occurs repeatedly, with its F-system in different relations to the base lines.

If k_μ passes $3+s$ times through A_a, then l_a counted s times is part of c_9, and is an F-line of s-point contact. A base point X, of multiplicity s on k_μ, represents a fixed point O of multiplicity s on c_9, which is an isolated F-point of s-point contact.

We find seven distinct space transformations, of which details are given in 85, 87, 217, 384, 385. The following table gives examples of how they can arise.

In P		On ϕ		No. in Table VI
Homaloid	Fixed curve	Homaloidal curve	F-curve	
k_1	$k_8 \equiv (A_1A_2A_3A_4A_5A_6)^3$	c_3	ω_6, genus 3	2
$k_2 \equiv A_1A_2X$	$k_7 \equiv (A_1A_2)^2(A_3A_4A_5A_6)^3$	$c_4 \equiv O$	ω_5, „ 1	6
$k_3 \equiv A_1A_2A_3A_4X^2$	$k_6 \equiv (A_1A_2A_3A_4)^2(A_5A_6)^3$	$c_5 \equiv O^2$	ω_4, „ 0	23
$k_4 \equiv A_1A_2A_3A_4A_5A_6X^3$	$k_5 \equiv (A_1A_2A_3A_4A_5A_6)^2$	$c_6 \equiv O^3$	ω_3, „ 0	51
$k_6 \equiv A_1^4(A_2A_3A_4A_5)^2A_6X_1X_2$	$k_3 \equiv A_2A_3A_4A_5A_6^2$	$c_5 \equiv O_1O_2$	ω_3, l_1	12
$k_9 \equiv A_1^5A_2^4(A_3A_4A_5A_6)^3X_1X_2X_3$		$c_6 \equiv O_1O_2O_3$	l_1 (contact), l_2	28
$k_9 \equiv (A_1A_2A_3)^4(A_4A_5A_6)^3X_1^2X_2$		$c_6 \equiv O_1^2O_2$	l_1, l_2, l_3	41

15. Singular cubic transformations (85, 159).

So far it has been assumed that ϕ is not singular. If (ϕ_3) has a double F-point O, we can use the simpler plane representation by projection from O on to p. The image of the total intersection of ϕ, ψ, with four branches through O, is a plane quintic through the traces on p of the six lines of ϕ through O.

We have to distinguish the cases of conic node, binode, and nodal line, and the numbers of each. The total number of distinct cubic transformations is 75, of which 42 can be derived from the other 33 by composition; they are tabulated on p. 447. The simplest example is the Arguesian transformation (p. 322), a particular involutory case of no. 3.

Cremona's method has also been applied to certain quartic and quintic surfaces, and a few types of general degree, including monoids (21, 85, 283, 316 and p. 306).

16. Extension of the method.

The same method may be applied to determine those transformations $T_{n-n'}$ for which ϕ is a special and not a general member of the homaloidal web. Now (ψ) is a rational family, from which ϕ can be obtained by imposing three conditions. Since ψ is rational, and ϕ is proper and of the same degree, ϕ can have no additional singularity that would affect the genus; but it may have additional d.p.'s.

If a d.p. O of ϕ_n is not an F-point of (ψ), it lies on the Jacobian of (ψ) and is a P-point. It presents three conditions, and there can be only one such point on ϕ. A d.p. at an ordinary simple F-point again presents three conditions; at a point of a simple F-curve, two; and at an F-point of contact, only one. Thus there may be as many as three of the last kind on ϕ.

Thus it may be that a rational surface ϕ_n, which has too many singular elements to be a general member of a homaloidal family, may yet be a special homaloid of $T_{n-n'}$. For example, the quartic

$$\phi_4 \equiv w^2u_1^2 + x^2yz, \quad u_1 \equiv ax + by + cz,$$

has two ordinary double lines, l_1, $x = w = 0$; l_2, $x = u_1 = 0$; the tacnode A_4 and unodes A_2, A_3, with u_1, w, w as repeated tangent planes respectively; and the ordinary d.p. A_1. These singular elements define a net, not a web, and so cannot all belong to an F-system; but ϕ_4 is a member of the homaloidal family

$$(\psi_4) \equiv c_1w^2u_1^2 + c_2x^2yz + c_3xwu_1(x-y) + c_4xyz(x-y),$$

which has the three F-elements of contact l_1, l_2, A_1, which are double on ϕ_4.

By applying Cremona's method, with each assumption in turn, we obtain all the forms of $T_{n-n'}$ which have ϕ_n as a proper homaloid, whether general or special.

A space transformation $T_{n_1-n'}$, where $n_1 > n$, may have a special homaloid which breaks up into a P-surface j_{n_1-n} and a residual ϕ_n which T converts into a plane. Then both the degrees n_1, n' of T are unknown; with ϕ we must associate the unknown surface j, and it is the multiple curves of ϕj that are F-curves of T; the F-system need not all lie on ϕ, and the method does not apply.

If V, W are two transformations, of any degree, which convert ϕ into two planes p_1, p_2, then $T \equiv WV^{-1}$ is a space transformation which converts p_1 into p_2; conversely if V is known, and T is any transformation which converts p_1 into a plane, then $W \equiv TV$ also converts ϕ into a plane. The problem, unsolved at present, of finding all transformations W, of degree $\geqq n$, which convert ϕ into a plane, is equivalent to finding one of them, and also all space transformations which convert one plane into another; examples of these are given on p. 313.

17. Loria's method (217).

A more direct method than Cremona's does not use an auxiliary transformation, but it requires a closer study of the given surface. We find all the simple curves of low enough degree lying on ϕ_n, consider in turn every set of them which, with the multiple curves, gives an intersection of degree between $n - \sqrt{n}$ and 0, and find whether, with the multiple points and any set of simple points, ordinary or of contact, it can form an F-system. This has been done for a non-singular cubic surface, and for a type of singular quartic (6).

18. Genus of a transformation (216).

Since a homaloid and a plane in $S \sim$ a plane and a homaloid in S', the two plane sections are homologues and have the same genus p, which is called the *genus of* T. This is a common characteristic of T, T^{-1}, and of all the transformations for which a given homaloid ϕ is a general member of either family.

If $p=0$, then ϕ_n is a monoid, either ruled with an $(n-1)$-fold directrix, or a Steiner quartic, with three double lines meeting in a triple point (319). Transformations of genus 0 are of four types:

$$T_{2-2},\ T_{2-4},\ T_{4-4},\ T_{n-n'},$$

described on pp. 182, 292, 293, 316.

The general T_{2-3} (p. 287) is a particular case of $T_{n-n'}$, but T_{2-2} is not, unless the F-conic breaks up into two lines, one of which is regarded as the $(n-1)$-fold directrix. The three types with equal degrees, T_{2-2}, T_{4-4}, T_{n-n}, can be involutory (259).

There are 32 irreducible types of genus 1 (21, 283). There exist transformations of all genera; for example, in the three types of general degree given on p. 327 we have

$$\phi_n \equiv l^{n-2},\ p=n-2;\quad \phi_{2n+1} \equiv \omega_4{}^n,\ p=n;\quad \phi_n \equiv l_1{}^i l_2{}^{n-i-1},\ p=i(n-i-1).$$

19. Contact conditions as limits.

Contact in space cannot in general be treated as a limiting case, arising from the coalescence of distinct elements, as in the plane. The true analogy for a point of contact of surfaces is not with a point of contact of plane curves, but with a common double point, for an F-point of contact of (ϕ) is a common double point of (c).

If two distinct simple F-points → a third O in different directions, then in the limit O is a point of contact of (ϕ). Without any further condition, another intersection of ϕ, ψ, χ, variable before the limit, also → O and becomes fixed, thus increasing the equivalence, without increasing the postulation, of the point of contact compared with the set of three distinct points. This follows from the corresponding property of the family of plane curves, images of (c) in any 1, 1 plane representation of (ϕ) (p. 228).

With curves of contact, this difficulty does not arise, but there is another. In general, ϕ does not contain a continuous series of curves of the same nature as the curve of contact, and a second curve cannot → the first in the same way as one point can → another, passing through a continuous series of intermediate points of ϕ (224).

For example, if two quadrics touch along a generator, we can imagine them to intersect in two distinct generators of the same system, which then coalesce; because there are ∞ intermediate generators with which the second can coincide in turn as it → the first. But if two general cubic surfaces intersect in two distinct lines, which then coalesce, the surfaces are modified as the second line → the first, and in the limit are no longer general cubics, but each is specialized in that two of its 27 lines coincide. But two general cubics, whose 27 lines are all distinct, can be in contact along one of them; and this case cannot arise as a limit (p. 250).

It is not a definite condition to require that a family of n-ic surfaces shall have contact along a given curve. Such a definition of the family would be circular, since each surface would be defined as touching another as yet undefined member of the family. If ω is an F-curve of contact, we must be given a series of fixed tangent planes to (ϕ) at the points of ω, forming a developable f; and to avoid the vicious circle, f must be defined quite apart from (ϕ). Thus (ϕ) must be required to touch a given fixed surface f along ω; the degree of f need not be n, and (ϕ) may or may not prove to include f as a member. The postulation and equivalence of the contact condition depend on the degree and nature of f as well as of ω.

20. Spaces superposed.

Let the spaces S, S' be superposed. This introduces a new element of variety, for a transformation T of given type, arising from the possible relations of the F- and P-systems of S, S'.

Every point $P \equiv P_1'$ may now be regarded as belonging either to S or to S' and has two homologues, P' in T, and P_1 in T^{-1}. The three points are in general distinct.

All the properties dealt with so far continue to hold; the special matters of interest that arise in addition are

(i) invariant, involutory and cyclic points, curves and surfaces,

(ii) self-corresponding curves and surfaces,

(iii) the associated complex of lines joining homologous points,

(iv) constructions for special transformations.

No general theory exists; examples of the methods available are given in the detailed discussion of T_{2-2} on p. 201, and in connection with some other transformations on pp. 302, 311, 322.

21. Invariant system (122·1).

There are always certain invariant points D which coincide with their homologues, $D \equiv D' \equiv D_1$. There may be ∞ such points, forming an invariant locus, either a curve δ_ν, or a surface Δ_μ, or both. This reduces the number of isolated invariant points, which depends not only on the degrees of the invariant loci, but also on their incidences with H.

The homologue of δ is itself, meeting any plane in ν points; hence δ meets any ϕ in ν points other than F-points, and in $\nu n - \nu$ F-points; the sum of the multiplicities of the F-points on branches of δ is $\nu(n-1)$. Also the homologue of Δ is itself, meeting any line in μ points; hence Δ meets $c_{n'}$ in

μ points other than F-points, which are its free intersections with ϕ, ψ. Hence the equivalence on Δ, ϕ, ψ of the F-elements that lie on Δ is $\mu(n^2-1)$.

The coordinates of D satisfy

$$x : y : z : w = \phi_1 : \phi_2 : \phi_3 : \phi_4,$$

and D lies on the six surfaces such as

$$x\phi_4 - w\phi_1 \equiv \begin{vmatrix} 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ x, & y, & z, & w \\ \phi_1, & \phi_2, & \phi_3, & \phi_4 \end{vmatrix} = 0,$$

which expresses that P, P', A_2, A_3 are coplanar, and belongs to the family whose general member is

$$\begin{vmatrix} \xi_1, & \eta_1, & \zeta_1, & \theta_1 \\ \xi_2, & \eta_2, & \zeta_2, & \theta_2 \\ x, & y, & z, & w \\ \phi_1, & \phi_2, & \phi_3, & \phi_4 \end{vmatrix}, \equiv \Delta\Omega_{X_1X_2} \text{ say},$$

locus of points P coplanar with P' and two general points $X_1(\xi_1, \eta_1, \zeta_1, \theta_1)$ and $X_2(\xi_2, \eta_2, \zeta_2, \theta_2)$. This contains the invariant system, the F-system and the line $l \equiv X_1X_2$. The invariant surface Δ of degree μ can be separated; the residual Ω, of degree $n+1-\mu$, is the locus of a point P such that PP' meets l, and is called the *first isologic surface* of l. It contains l, δ and all of H that does not lie on Δ, any F-element which is i-fold on (ϕ) and i_Δ-fold on Δ being $(i-i_\Delta)$-fold on Ω. Similarly the locus of P' is the *second isologic surface* of l.

If p is any plane through l, it meets $\Delta\Omega$ in l and the locus of points P of p such that P' also lies on p and therefore P on the homaloid ϕ that $\sim p$. This locus is the section of ϕ by p, and breaks up into the section of Δ and a residual curve $k_{n-\mu}$, which with l makes up the section of Ω by p.

If X_3 is any third point, the intersection of $\Omega_{X_1X_2}$, $\Omega_{X_1X_3}$ is the locus of P when PP' meets both X_1X_2 and X_1X_3. If P, P' are distinct, PP' either lies in the plane $p \equiv X_1X_2X_3$, or passes through X_1. The intersection therefore consists of δ, $k_{n-\mu}$, the F-curves on (Ω) and a residual o, locus of P when PP' passes through X_1, called the *first isologic curve* of X_1. It passes through X_1, the isolated F-points on (Ω) and the isolated invariant points. Similarly the locus of P' is the *second isologic curve* o' of X_1.

Thus $\Delta\Omega$ is the product of the pencil of planes $(p) \equiv l$ and the projective pencil of their homologues (ϕ); and $\Delta\delta o$ is the product of the star of lines through X_1 and the projective net of their homologues (c). Hence o touches X_1X_1' at X_1.

The isologic surfaces of all the lines in the plane p form a net $(\Omega)_p$, whose base consists of

(i) the F-elements at which Δ has lower multiplicities than (ϕ);

(ii) the invariant curve δ;

(iii) a plane $k_{n-\mu}$ which, with the section of Δ, makes up the section by p of its homologue ϕ;

(iv) the isolated invariant points D.

The number of D's is therefore the difference between $(n+1-\mu)^3$ and the equivalence on $(\Omega)_p$ of the other base elements. This can be calculated by the methods of Chapter XI, when we know the incidences of the base curves with each other.

Now p meets $\Omega_{X_1X_2}$ in $k_{n-\mu}$ and the variable line X_1X_2; and $\Omega_{X_1X_2}$, $\Omega_{X_1X_3}$ have $k_{n-\mu}$ and X_1 in common, and no other point of p; and X_1 lies on o and not on any base curve. Hence δ meets k in the ν points in which it meets p, and any ω_m which is i_Ω-fold on Ω meets k in m points through each of which k has i_Ω coplanar branches. This demands a correction to the general equivalence formula (p. 226).

In the simplest case, all the invariant points are isolated, and there is no invariant locus Δ or δ. Then Ω is of degree $n+1$, and contains all the F-curves, whose total degree, in the intersection of two Ω's, is the same as in that of two ϕ's, namely n^2-n'. The degree of o is therefore

$$(n+1)^2-(n^2-n')-n=n+n'+1.$$

It meets p in X_1 and in $n+n'$ points P, which are such that PP' lies in p and contains X_1. Similarly for the second isologic curve. We can then find the number of D's by considering the correspondence between planes joining any fixed line l of p to corresponding points P, P' of o, o'. A plane lP meets o in $n+n'+1$ points P, which determine as many points P' and planes lP'; in the same way, one plane lP' determines $n+n'+1$ planes lP. Hence there are $2(n+n'+1)$ pairs PP' lying on the same plane through l. If P, P' coincide, we have an invariant point; if they are distinct, the plane contains $PP'X_1$, and coincides with $p\equiv lX_1$, which contains, as we have seen, $n+n'$ pairs. Hence the number of invariant points is $n+n'+2$, in the simplest case when they are all isolated (417).

22. Invariant directions (112).

In the star of rays (D), a homography is set up by T, such that if any curve k touches a ray, then k', which also passes through D, touches the associated ray. This relation is linear, for rays lying in a plane are associated with rays touching a ϕ' at D, and lying in its tangent plane, which is definite since ϕ' is not singular at D, which does not lie on J. The self-corresponding rays of this homography are in the *invariant directions* at D.

Thus D can be of three kinds, according as the invariant system of the homography consists of

(i) three distinct rays,

(ii) a plane and one distinct ray, or

(iii) the whole star;

in the last case, the homography is identity, and every curve through D touches its homologue there.

If the homography is improper, a general plane of the star (D) answers to a plane which either passes through a fixed line or is itself fixed; and the planes of $(D) \sim$ a net of homaloids ϕ' all touching a fixed line or plane; either one or ∞^1 of them have a d.p. at D, and D lies on J', and is a P-point of S', which coincides with its homologue, and is therefore an F-point of S.

Conversely, if an ordinary F-point O is invariant, it lies on the corresponding P-surface j'. Let O be a simple point of j', and not an F-point of S'. The points of j' are associated with the rays of the star (O) (p. 164) and in particular, O is associated with a definite ray, which is the only invariant direction at O. No sheet of Δ, and at most one branch of δ, can pass through O in general.

Next, let D be a general point of an ordinary F-curve ω, not an F-point of S', and a simple point of its corresponding P-curve κ'. Each point Q' of κ' is associated with a particular plane p through the tangent line to ω at D, such that $p \sim$ a ϕ' having a d.p. at Q', and the rays through D in p are associated with the generators of q' the quadric tangent cone to ϕ' at Q' (pp. 268, 280). If $Q' \equiv D$, then p, q' have definite positions, and their common rays are the only directions at D that can be invariant. Thus at most one sheet of Δ or two branches of δ can pass through D in general.

If $D\,(0, 0, 0, 1)$ is not an F-point of S or S', there are at least three invariant directions, not coplanar. Take these as lines of reference; then the homography has the form

$$x' : y' : z' = ax : by : cz,$$

and the base homaloids are

$$\phi_1 \equiv axw^{n-1} + \dots, \quad \phi_2 \equiv byw^{n-1} + \dots, \quad \phi_3 \equiv czw^{n-1} + \dots, \quad \phi_4 \equiv w^n + \dots.$$

The condition for a point of second kind is that two of a, b, c are equal, and for a point of third kind, $a = b = c$.

The tangent plane to Ω is

$$t \equiv (1-a)(\eta_1\zeta_2 - \eta_2\zeta_1)x + (1-b)(\zeta_1\xi_2 - \zeta_2\xi_1)y + (1-c)(\xi_1\eta_2 - \xi_2\eta_1)z,$$

and Ω has a d.p. at D provided X_1X_2 passes through D. If D is of second kind, two of a, b, c are equal, and t contains the intersection of the plane DX_1X_2 with the plane of invariant rays. If D is of third kind, then $t \equiv DX_1X_2$.

Also D is an invariant point of contact if two of $a, b, c = 1$; it is then of second kind, and t is the plane of invariant rays. It is a double invariant point if $a = b = c = 1$, and is then of third kind. In the same way we define a *multiple invariant point*, as a multiple base point of (Ω) which is not fundamental.

If D is a general point of an invariant surface Δ, we can write (p. 85)

$$\phi_1 \equiv x\phi_{n-1} + \Delta\psi_1, \ldots, \quad \phi_4 \equiv w\phi_{n-1} + \Delta\psi_4,$$

and the proper isologic surface is

$$\Omega \equiv \begin{vmatrix} \xi_1, & \eta_1, & \zeta_1, & \theta_1 \\ \xi_2, & \eta_2, & \zeta_2, & \theta_2 \\ x, & y, & z, & w \\ \psi_1, & \psi_2, & \psi_3, & \psi_4 \end{vmatrix}.$$

Every direction in the tangent plane x of Δ is invariant, and D is of second kind; let $y = z = 0$ be the isolated invariant ray, which is the H-line of D. We have

$$\phi_{n-1} \equiv w^{n-1} + \ldots, \quad \Delta \equiv xw^{\mu-1} + \ldots,$$

$$\psi_1 \equiv a_1 w^{n-\mu} + \ldots, \quad \psi_2, \psi_3 \equiv (x, y, z \between x, y, z, w)^{n-\mu-1}, \quad \psi_4 \equiv d_4 w^{n-\mu} + \ldots.$$

The homography is

$$x' : y' : z' = (1 + a_1)\, x : y : z,$$

and the leading term in Ω is

$$-\, a_1 (\eta_1 \zeta_2 - \eta_2 \zeta_1)\, w^{n-\mu+1}.$$

Thus Ω does not pass through D unless either $X_1 X_2$ meets the H-line, or else $a_1 = 0$ and D is of third kind, and is a base point of (Ω), equivalent to an adjacent isolated invariant point of first kind. If every point of Δ is of third kind, it is equivalent to two adjacent invariant surfaces, and another factor Δ is separated from Ω.

If D is a point of δ, in the simplest case δ is a line, $x = y = 0$ say; we have

$$\phi_1 : \phi_2 : \phi_3 : \phi_4 \equiv x\psi_1 + y\chi_1 : x\psi_2 + y\chi_2 : z\phi_{n-1} + x\psi_3 + y\chi_3 : w\phi_{n-1} + x\psi_4 + y\chi_4.$$

One of the invariant rays at D is δ; let the other two be the lines of reference; then

$$\phi_{n-1} \equiv w^{n-1} + \ldots, \quad \psi_1 \equiv a_1 w^{n-1} + \ldots, \quad \chi_2 \equiv b_1 w^{n-1} + \ldots,$$

$$\psi_2, \chi_1, \psi_3, \chi_3 \equiv (x, y, z \between x, y, z, w)^{n-2}.$$

The homography is

$$x' : y' : z' = a_1 x : b_1 y : z,$$

and
$$t \equiv (1 - a_1)(\eta_1 \zeta_2 - \eta_2 \zeta_1)\, x + (1 - b_1)(\zeta_1 \xi_2 - \zeta_2 \xi_1)\, y,$$

and contains δ; and Ω has a d.p. at D provided $X_1 X_2$ meets the two invariant rays other than δ.

Now D is of second kind if either $a_1 = 1 \neq b_1$, or $b_1 = 1 \neq a_1$, or $a_1 = b_1$. If a_1 or $b_1 = 1$, the plane of invariant rays contains δ and touches (Ω); thus D is an invariant point of contact on δ, equivalent to an adjacent isolated invariant point of the first kind. If every point of δ is of this nature, δ is an invariant line of contact, equivalent to two adjacent invariant lines.

If $a_1 = b_1$, the plane of invariant rays does not contain δ; it meets $X_1 X_2$ on t, which is not fixed.

If D is of third kind, $a_1 = b_1 = 1$, and D is a double invariant point. If every point of δ is of third kind, δ is a double invariant line.

23. Involutory system (186).

There is always a certain system of involutory pairs of conjugate points K, K', each of which is the homologue of the other in both T and T^{-1}; that is, $K' \equiv K_1$ without coinciding with K. A repetition of T carries K into itself again, and K is an invariant point of the compound transformation T^2, the nature of which gives a basis for the classification of the involutory pair. There may be ∞ involutory points, whose locus is a curve, or a surface or the whole of space. In the last case, every point is one of a pair of conjugates, T is the same as T^{-1}, and T^2 is identity: then T is a Cremona involution.

The involutory system of T is the set of points invariant for T^2 but not for T. In the simplest case, the degrees of T^2 are n^2, n'^2, it has no invariant locus, and $n^2 + n'^2 + 2$ isolated invariant points. Then T also has no invariant locus, and $n + n' + 2$ invariant points. The number of involutory pairs of T is therefore

$$\tfrac{1}{2}\{n^2 + n'^2 + 2 - (n + n' + 2)\} = \tfrac{1}{2}n(n-1) + \tfrac{1}{2}n'(n'-1).$$

The involutory system can be found directly from T without using T^2. Since the two homologues of K coincide, it satisfies

$$\phi_1(x, y, z, w) : \phi_2 : \phi_3 : \phi_4 = \phi_1'(x, y, z, w) : \phi_2' : \phi_3' : \phi_4'$$

and K lies on all the surfaces such as $\phi_1\phi_4' - \phi_4\phi_1'$, or more generally,

$$\begin{vmatrix} \xi_1, & \eta_1, & \zeta_1, & \theta_1 \\ \xi_2, & \eta_2, & \zeta_2, & \theta_2 \\ \phi_1, & \phi_2, & \phi_3, & \phi_4 \\ \phi_1', & \phi_2', & \phi_3', & \phi_4' \end{vmatrix},$$

the locus of P when $P'P_1X_1X_2$ are coplanar. This contains the invariant and involutory systems. The variable part, after setting aside any fixed surfaces, is the *M-surface* of X_1X_2, and is the locus of P when $P'P_1$ meets X_1X_2, and is the product of the two projective pencils (ϕ), (ϕ'), homologues in T and T^{-1} of the pencil of planes through X_1X_2. The involutory system can be obtained from the base of the net of M-surfaces of all the lines of a plane p, for example, if $p \equiv A_1A_2A_3$, from the base determined by

$$\phi_1\phi_4' - \phi_4\phi_1', \quad \phi_2\phi_4' - \phi_4\phi_2', \quad \phi_3\phi_4' - \phi_4\phi_3',$$

after setting aside the two F-systems, the invariant system and the curve

$$\phi_4 = \phi_4' = 0.$$

Even for the simplest transformation, the involutory system is capable of a great variety of forms. For an example, see p. 205.

The invariant points and involutory pairs are cycles of indices 1, 2 respectively (p. 42). A few cases of cycles of higher index have been considered. If the index is ∞, the cycle may become a self-corresponding curve or surface; other self-corresponding loci arise from ∞ cyclic groups of finite index, such as the invariant and involutory loci.

If l is a self-corresponding line, its homologue c_n' breaks up into l and a set of P-curves of S', and l contains F-points of H whose homologues are of total degree $n-1$, and F-points of H' whose homologues are of total degree $n'-1$. The general points of $l \sim$ each other, and T establishes a 1, 1 relation between them, that is, a homography. If this contains one involutory pair, it is an involution and l is an involutory line. Its two double points are invariant for T, and if l contains three invariant points, the homography is identity, and l is an invariant line.

24. The associated complex.

The ∞^3 set of lines l, joining each point P of space to its homologue P', is the *associated complex* of T. The same complex is associated with T^{-1}, the ray l arising from P when the complex is associated with T, and from P' when it is associated with T^{-1} (306).

Now P' is an intersection of l and its homologue c_n' in T, and P is an intersection of l and its homologue $c_{n'}$ in T^{-1}. If l contains more than n such pairs, an involutory pair counting as two, it meets c_n' in more than n points, and c_n' breaks up into l and a set of P-curves; l is a self-corresponding line.

The complex may reduce to a congruence; then each ray contains ∞^1 pairs of homologues, and is therefore a self-corresponding line. In the simplest case, it is a star (O); then T is called *central* (or *perspective*), and T is monoidal (p. 309).

If the rays all meet two given lines, the congruence is bilinear, and T is called *skew* (408).

Thus the two forms of linear involution are one central (harmonic homology), and the other skew.

Most of the special constructions for transformations in one space use surfaces or curves containing both P and P', and, in particular, use the associated complex. When this is given, we must be able to determine the pair of homologues on each ray, for example, by its intersection with the related members of two families of planes in 1, 1 relation to the complex of lines (260). Instead of planes, we may be given surfaces of higher degree, provided we can isolate one intersection of each with its associated line; or we may be given a correlation, and then P' is the intersection of the ray of the complex through P with the plane related to P (249).

If the given complex reduces to a congruence, we must be able to determine the homography on each ray, by assigning three pairs; any of these may consist of an F-point and one of its homologues, if all the rays meet a given F-element and we are given the corresponding P-surface; or a pair may coincide at an invariant point on the ray if we are given an invariant surface or curve met by all the rays.

25. Involutions (367).

An involution is the same as its reverse, and its two degrees are equal; we denote it by I_{n-n}. It relates all the points of S in conjugate pairs. The best known examples are reflexion in a plane and inversion with regard to a sphere, which are metrically special cases of I_{1-1} and I_{2-2} (p. 208).

The two F-systems coincide, and the two P-systems. If there is an invariant locus and an F-point O on it, then O is a common F-point of H, H', and its two corresponding P-elements coincide and contain O and any invariant point adjacent to O. Thus if O lies on Δ, either O is an isolated F-point whose P-surface j touches Δ, or O is a point of an F-curve ω, and Δ contains the whole of ω and touches the corresponding P-surface along ω; and Δ cannot meet ω in a general point. The two isologic surfaces of any line coincide, and (Ω) touches at O any sheets of j that do not touch Δ.

If three pencils, or a pencil and a net, or a web, of surfaces are such that three general members have just two free intersections, these are thereby related in a Cremona involution, for which the surfaces are self-corresponding. For example, Geiser's I_{7-7} is generated by quadrics through six fixed points (p. 323). In general, there may be a great variety of singular points, curves and surfaces of the generating families, and no concise treatment is possible (301, 347, 374).

In particular, a 2, 1 relation T between two spaces defines in the double space S a web of surfaces, answering to the planes of the simple space, with just two free intersections, and the relation between these is the Cremona involution *associated with* T (105, 361, 368, 370).

It is probably not true that every I_{n-n} can be obtained in this way; this would imply that we could always map the conjugate pairs of I_{n-n} on the points of another space, and I_{n-n} would be rational. In the plane, every involution, whether 1, 1 or not, is rational; this is not true in space, but so far the only irrational involutions that have been constructed are not 1, 1, and the rationality of all Cremona involutions remains an open question (p. 94).

If the associated complex of I_{n-n} is given, we must be able to determine the conjugate pair or pairs on each ray; for example, by its intersections with a related quadric.

Involutions have been investigated whose complexes have special forms:
- linear (99, 245, 248), special linear (247, 321);
- quadratic (253, 320), tetrahedral (250, 412), Hirstian (322), rays meet a conic (323);
- cubic (257);
- rays meet a twisted cubic (47);
- set of tangents (266, 326).

If the complex reduces to a congruence, each ray is self-conjugate (136). Since any point of intersection of two self-conjugate lines is either invariant or fundamental, the congruence is of first degree, and consists of either (104, 246)

(i) rays of a star; I is central, and monoidal; the simplest example is the Arguesian involution (229 and p. 322);

(ii) chords of a cubic curve; for examples, see pp. 304, 323, 325;

(iii) rays meeting a line l and a curve of degree m meeting l in $m-1$ points. If $m=1$, the congruence is bilinear, and I is skew; for example, the I_{5-5} of p. 322.

CHAPTER X

THE QUADRO-QUADRIC TRANSFORMATION

I. SPACES DISTINCT

1. Quadratic transformations in plane and space (8, 209, 282, 299, 343).

In the plane, the quadratic transformation T_2, whose reverse is also quadratic, is the simplest, the most used in applications, and is chiefly important as the one from which all other plane Cremona transformations can be compounded. In space, there also exists a transformation T_{2-2} which is quadratic both ways. It is, in some connections but not in all, the simplest, and the most often applied. It is one of the space transformations which we must regard as elementary, along with others, such as the bilinear and monoidal types, because they are used in standard methods of resolving higher singularities of surfaces. But there is as yet no general space theory of composition, and certainly all Cremona space transformations cannot be built up from T_{2-2}, or from any finite set of transformations (p. 385).

T_{2-2} is not the only space analogue of the plane T_2; in many ways, the bilinear T_{3-3} (p. 294), or one of its particular cases, is a more natural extension.

Consider a family of quadrics (ϕ) defined by the conditions of passing through a fixed conic ω and a fixed point O not lying in the plane of ω. Since a conic, which does not lie wholly on a quadric, meets it in four points only, if ϕ is subjected to the five conditions of passing through five points of ω, it contains the whole curve; ω therefore presents five linear conditions to (ϕ), and O presents one condition. Thus for the proposed F-system, the postulation $P=6$. Now the general homogeneous quadratic function in four variables has ten arbitrary coefficients and therefore nine degrees of freedom. After six conditions are satisfied, there remain three degrees of freedom; then (ϕ) is linear in the four surviving coefficients, and is ∞^3, that is, a web.

Now two quadrics ϕ, ψ of the web intersect in a twisted quartic k_4. Since they both pass through ω, this k_4 breaks up into ω and a residual c_2 of degree 2, which meets ω in the two points in which it meets the plane of ω. Since O is a common point of ϕ, ψ, not lying on ω, it lies on c. A third quadric χ of the family has in common with ϕ and ψ the whole of ω and the four points of intersection of c, χ; two of these are the points of intersection of c, ω. Now three general quadrics intersect in eight points; in the case of ϕ, ψ, χ, all but two are absorbed by ω, and its equivalence is $E=6$. Of the remaining points, one is O, and the other is the free intersection of ϕ, ψ, χ. Hence for the whole F-system, $E=7$, and any three members of (ϕ) have one and only one variable point of intersection.

These two properties show that (ϕ) is homaloidal, and O, ω form its F-system H.

Thus there exists a transformation which converts the web of quadrics (ϕ) into the planes of S', and the family of conics (c) into the lines of S'. Since c meets any plane in two variable points, the corresponding line meets any surface of the second homaloidal web (ϕ') in two points; the degree of ϕ' and of the reverse transformation is also 2.

2. Uniqueness of T_{2-2}.

We shall show next that this is the only possible form of H, if the direct and reverse transformations are both quadratic.

As we have just seen, since (ϕ) is quadric, H must have $P=6$, $E=7$. And since (ϕ') is quadric, the variable part c of k_4 is of degree 2, and k_4 breaks up into c and a fixed part ω also of degree 2, and ω is either a conic or a pair of skew lines.

If ω is a pair of skew lines l_1, l_2, then l_1 meets its total residual l_2, c in two points, as we shall prove in the next paragraph. Since l_2 does not meet it, c meets it twice, and also meets l_2 twice; the four intersections of c, ψ lie two on l_1 and two on l_2, and there is no free intersection. All the eight common points of ϕ, ψ, χ are absorbed by l_1, l_2, and E is too high.

Thus ω cannot be a pair of skew lines, and must be a conic, either proper or breaking up into a pair of intersecting lines. This has $P=5$, $E=6$, and H has another element for which $P=E=1$, which can only be a simple point O. Thus H consists of two elements not in general incident with one another, the conic ω and the point O.

3. Intersection of a line with its residual.

Let k_4 break up into l, k_3. An intersection of l, k_3 is a d.p. of k_4, and therefore a point of contact of ϕ, ψ; we have to show that there are two points of contact on l.

Let l be $z=w=0$; the general quadric through l is

$$\phi \equiv zu_{1,1}+wu_{1,2}+(z, w)^2,$$

where $u_{1,1}$, $u_{1,2}$ are linear functions of x, y only. The tangent plane to ϕ at the point $(x, y, 0, 0)$ of l is

$$t \equiv zu_{1,1}+wu_{1,2},$$

if in this x, y are treated as constants and z, w as current coordinates. Let the corresponding tangent plane to ψ be

$$zv_{1,1}+wv_{1,2}.$$

The points of contact of ϕ, ψ, where these tangent planes coincide, are given by

$$u_{1,1}/v_{1,1}=u_{1,2}/v_{1,2},$$

that is,

$$u_{1,1}v_{1,2}-u_{1,2}v_{1,1}=0,$$

which is quadratic in x/y, and gives the two points of contact of ϕ, ψ lying on l, which are the two intersections of l, k_3.

This holds whether k_3 breaks up or not.

The same method can be applied to surfaces of any degree (p. 213).

4. Equations of the transformation.

Take O for the vertex $A_4(0, 0, 0, 1)$ of the tetrahedron of reference, and take the plane of ω for w. Let q be the cone vertex O standing on ω; then q is a homogeneous quadratic function of x, y, z, and the equations of ω are $w = q = 0$. The conditions imposed on ϕ by ω reduce it to a sum of multiples of w, q, and the condition imposed by O removes the term in w^2. Hence the form of the general ϕ is

$$\phi \equiv w(c_1x + c_2y + c_3z) + c_4q.$$

We also obtain ϕ if we can find four linearly independent quadrics through H, multiply by arbitrary constants and add. The ∞^3 family to which this leads must be the same as (ϕ). Now the simplest quadrics through ω and O are those which break up into the plane of ω and any plane through O; this gives three independent quadrics, and a fourth simple quadric, linearly independent of these, is the cone q. Hence if

$$\phi_1 \equiv xw, \quad \phi_2 \equiv yw, \quad \phi_3 \equiv zw, \quad \phi_4 \equiv q,$$

we have the family in the equivalent form

$$\phi \equiv c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4.$$

To obtain the equations of T_{2-2}, we must set x', y', z', w' proportional to any four independent ϕ's as base homaloids. Taking the four just found, we have the equations in their simplest form,

$$x' : y' : z' : w' = xw : yw : zw : q \quad \text{......................(1).}$$

In order to reverse these, we have

$$x : y : z = x' : y' : z',$$

$$\frac{xw}{q} = \frac{x'}{w'},$$

$$\frac{w}{x} = \frac{x'}{w'}, \quad \frac{q}{x^2} = \frac{x'}{w'}, \quad \frac{q'}{x'^2} = \frac{q'}{x'w'},$$

where q' is the same function of x', y', z' that q is of x, y, z. Therefore the equations of the reverse transformation T_{2-2}^{-1} are

$$x : y : z : w = x'w' : y'w' : z'w' : q' \quad \text{......................(2).}$$

These are of exactly the same form as those of T_{2-2}; hence (ϕ') is of the same nature as (ϕ) and H' consists of a conic ω', $w' = q' = 0$ and a simple point $O'(0, 0, 0, 1)$.

5. F- and P-systems.

The equations (1) determine a unique point $P'(x', y', z', w')$ corresponding to $P(x, y, z, w)$, for all positions of P except those which make $\phi_1 : \phi_2 : \phi_3 : \phi_4$

indeterminate. This only happens when all four ϕ's vanish at once, and P is a point common to them all, that is, an F-point, either O or a point of ω.

If $P \equiv O$, then $w = 1$, $x = y = z = 0$. The equations (2) give

$$0 : 0 : 0 : 1 = x'w' : y'w' : z'w' : q',$$

which reduce to the single equation $w' = 0$, and are satisfied by any point Q' of this plane, and by no other point.

Now in general Q' is not an F-point, for the reverse transformation does not break down, and $Q' \sim$ the definite point O. But Q' is a P-point in S' because it $\sim$ an F-point in S. The whole plane w', locus of Q', is the P-element which $\sim O$.

Let

$$p \equiv ax + by + cz$$

be any plane through O. This $\sim$ the quadric

$$w' (ax' + by' + cz'),$$

which breaks up into two planes, one the P-plane w', and the other the plane p' through O' which has the same equation in S' that p has in S. Since every point of $w' \sim O$ only, it is p' which is the proper homologue of p, and is the locus of a point which $\sim$ a general point of p.

A general line l through O,

$$x : y : z = \xi : \eta : \zeta,$$

is the intersection of two planes through O, and the locus of (ξ, η, ζ, w); then l has for its proper homologue the line through O',

$$x' : y' : z' = \xi : \eta : \zeta,$$

locus of the homologue of a general point of l; but the single point O of l still $\sim$ the whole plane w'.

Next, let $P \equiv Q$, an F-point lying on ω. Then $w = 0$, and x, y, z satisfy $q = 0$. Now (1) are again indeterminate, and (2) reduce to the two equations

$$x' : y' : z' = x : y : z,$$

since $q' = 0$ for any value of w'. These represent a certain generator $\kappa' \equiv O'Q'$ of the cone q', where Q' is the point having the same coordinates in S' that Q has in S; and P' may be any point of κ'. The P-element which $\sim$ the F-point Q is in this case the line κ', and its locus, the cone q', is the P-surface which $\sim$ the whole F-curve ω.

Thus the whole P-system G' in S', which $\sim H$, consists of the P-plane w' which $\sim O$, and the P-cone q' which $\sim \omega$. Similarly in S we have G consisting of the P-surfaces w, q, which $\sim O'$, ω' respectively.

6. Examples of transformation of elementary properties.

In transforming the properties of any element into those of its homologue, we must remember that it is the number of variable elements in any set in S, depending effectively upon arbitrary parameters, that is the same as for the corresponding set in S'; either set can have F-elements in addition. For example:

A general plane in S does not pass through O, contains two points Q of ω, meets w in a line, meets every κ once.	A general ϕ' in S' meets w' in the F-curve ω' only, contains two generators κ' of q', has a simple sheet through O', passes once through each point Q' of ω'.
A general line does not pass through O, does not meet ω, meets w once, meets q twice, in points of two generators κ.	The variable intersection c' of ϕ', ψ' meets w' in F-points only, meets q' in F-points only, passes once through O', contains two points Q' of ω'.
A general quadric meets each κ twice, and meets ω four times.	The homologue of a quadric passes twice through ω', and its remaining intersection with q' consists of four generators κ'.

7. Transformation of the neighbourhood of an F-point.

Consider further the correspondence between the single point O of S and the whole plane w' of S'. An ordinary point P near $O \sim$ a definite point P' near some point of the plane w'. As $P \to O$ along some definite path, $P' \to$ a definite point Q' of w', the intersection of w' with one branch of the curve which $\sim$ the path of P. The different points of $w' \sim$ the same point O approached along different paths.

Let $P \to O$ along the line

$$l, \quad x : y : z = \xi : \eta : \zeta;$$

we can take P to be $(\xi\epsilon, \eta\epsilon, \zeta\epsilon, 1)$, where $\epsilon \to 0$. Then P' moves on the line

$$l', \quad x' : y' : z' = \xi : \eta : \zeta,$$

and $P' \to Q'\,(\xi, \eta, \zeta, 0)$, the intersection of l', w'. Hence the points (Q') are associated in a 1, 1 relation with the rays (l) of the star (O), and we may say that $Q' \sim$ the point adjacent to O on l.

In the same way, we can study the correspondence between the F-conic ω and the P-cone q'. Consider any definite point Q of ω. As triangle of reference in w take $A_1 \equiv Q$, another point A_2 of ω, and the pole A_3 of the chord A_1A_2 with respect to ω. Then for the time being we can assume $q \equiv xy - z^2$, and the tangent to ω at Q is $y = w = 0$.

Let $P \to Q$ along a fixed line; we may take the coordinates of P to be $(1, \eta\epsilon, \zeta\epsilon, \theta\epsilon)$ and proceed to the limit as $\epsilon \to 0$. The coordinates of P' are proportional to

$$\theta\epsilon, \quad \eta\theta\epsilon^2, \quad \zeta\theta\epsilon^2, \quad \eta\epsilon - \zeta^2\epsilon^2,$$

that is, as long as $\epsilon > 0$, to $\quad \theta, \quad \eta\theta\epsilon, \quad \zeta\theta\epsilon, \quad \eta - \zeta^2\epsilon.$

As $\epsilon \rightarrow 0$, so $P' \rightarrow$ the definite point Q' $(\theta, 0, 0, \eta)$ of the P-line $\kappa' \equiv O'A_1'$ which $\sim Q$.

Now Q' is independent of ζ, and is the same for all paths of approach to Q which touch a given plane $y/\eta = w/\theta$ through the tangent to ω at Q, without touching ω. The different points of κ' are associated with the different planes of the pencil whose axis is this tangent, or with the different rays of the pencil (Q) in the normal plane to ω.

8. Transformation of general elements.

A general surface $f_\mu(x, y, z, w) \sim$ the surface

$$f_\mu(x'w', y'w', z'w', q'), \equiv f'_{2\mu}(x', y', z', w') \text{ say,}$$

of degree 2μ with ω', O' as μ-fold elements.

If f passes through O, then $f = 0$ when $x = y = z = 0$, and every term has either x, y or z as a factor,

$$f \equiv (x, y, z \between x, y, z, w)^{\mu-1};$$

after substitution, every term has either $x'w'$, $y'w'$ or $z'w'$ as a factor, and f becomes

$$w'(x', y', z' \between x'w', y'w', z'w', q')^{\mu-1}, \equiv w'f'_{2\mu-1} \text{ say,}$$

which breaks up into two sheets; the P-plane w' which $\sim$ the single F-point O, and f' of degree $2\mu - 1$, the proper homologue of all the rest of f. This surface passes μ times through O', but only $\mu - 1$ times through ω'; for $w'f'$ has μ sheets through ω', one of which is w' and only $\mu - 1$ are sheets of f'.

Similarly, if O is i-fold on f,

$$f \equiv (x, y, z \overset{i}{\between} x, y, z, w)^{\mu-i}.$$

The result of substitution is $w'^i f'_{2\mu-i}$, where

$$f' \equiv (x', y', z' \overset{i}{\between} x'w', y'w', z'w', q')^{\mu-i};$$

the P-plane w' is dropped i times, the proper homologue f' is of degree $2\mu - i$, and on it O' is μ-fold and ω' is $(\mu - i)$-fold.

If f passes through ω it has the form

$$f \equiv wf_{\mu-1} + qf_{\mu-2}.$$

Now T_{2-2} replaces w by q' and q by w'^2q'. After substitution, the P-cone q' is a component, the whole of which $\sim$ the F-conic; on it, O' is double and ω' simple. The proper homologue f' is of degree $2\mu - 2$, and on it O' is $(\mu - 2)$-fold and ω' is $(\mu - 1)$-fold.

In the same way, if O is i-fold and ω is i_1-fold on f, the factor dropped is $w'^i q'^{i_1}$; the proper homologue f' is of degree $\mu' = 2\mu - i - 2i_1$; on it the multiplicity of O' is $i' = \mu - 2i_1$, and that of ω' is $i_1' = \mu - i - i_1$. We have the symmetrical relations

$$\mu - \mu' = i - i' = i_1 - i_1',$$
$$\mu + \mu' = i + i' + 2(i_1 + i_1').$$

In particular, for a homaloid, $\mu = 2$, $i = i_1 = 1$; after substitution, $w'q'$ is dropped, $\mu' = 1$, $i' = i_1' = 0$, and the homologue is a general plane, not passing through either O' or ω'. For $c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4$ becomes

$$G'(c_1x' + c_2y' + c_3z' + c_4w')$$

where $G' \equiv w'q'$, which is the whole of the second P-system.

If f has O as a simple point, with a general tangent plane, then

$$f \equiv w^{\mu-1}(ax + by + cz) + (x, y, z \overset{2}{\between} x, y, z, w)^{\mu-2},$$

and $$f' \equiv q'^{\mu-1}(ax' + by' + cz') + w'(x', y', z' \overset{2}{\between} x'w', y'w', z'w', q')^{\mu-2}.$$

Hence f' meets w' in ω' counted $\mu - 1$ times and a residual line

$$l', \quad w' = ax' + by' + cz' = 0.$$

The points of f adjacent to $O \sim$ the points of l', which is associated with the tangent plane to f at O.

If O is i-fold on f, with $u_i(x, y, z)$ as tangent cone,

$$f \equiv w^{\mu-i}u_i + (x, y, z \overset{i+1}{\between} x, y, z, w)^{\mu-i-1},$$

and $$f' \equiv q'^{\mu-i}u_i' + w'(x', y', z' \overset{i+1}{\between} x'w', y'w', z'w', q')^{\mu-i-1},$$

which meets w' in ω' counted $\mu - i$ times and a residual k_i', $w' = u_i' = 0$, projective with the section of the tangent cone at O; and the points of $k_i' \sim$ the points of f adjacent to O.

Next consider a general curve k_m of degree m. Its ∞^1 general points are neither fundamental nor principal, and therefore $\sim$ another ∞^1 points, that is, a curve $k'_{m'}$ of degree m' say. The m' intersections of k' with a plane $\sim$ the intersections of k with a ϕ; hence $m' = 2m$. Also k contains no F-points; but it contains m P-points of w and $2m$ of q; hence k' has m branches through O' and $2m$ intersections with ω', and these account for all its intersections with G', so that k' contains no P-points other than F-points.

If k passes through O, it meets ϕ in $2m - 1$ variable points only, and k' is of degree $m' = 2m - 1$. Now k meets w in m points and q in $2m - 2$ points other than O; hence k' has m branches through O and meets ω' in $2m - 2$ points. This accounts for all the intersections of k', q' and for all but one of k', w'. The remaining intersection of k', $w' \sim$ the point of k adjacent to O, and is associated with the tangent to k at O.

If, instead, k meets ω once, in Q say, it has $2m - 1$ other intersections with ϕ, and $m - 1$ with w, and $2m - 1$ with q. Then k' is of degree $m' = 2m - 1$, with $m - 1$ branches through O', and $2m - 1$ intersections with ω'. It does not meet w' again, but has one other intersection, Q' say, with q', lying on the generator $O'Q'$, the whole of which $\sim Q$, and Q' is associated with the plane containing the tangents at Q to ω and k.

In the same way, if k has α branches through O and β intersections with ω, then k' is of degree $m' = 2m - \alpha - \beta$, and has $\alpha' = m - \beta$ branches through

O' and $\beta' = 2m - 2\alpha - \beta$ intersections with ω'; we have the symmetrical relations

$$m - m' = \alpha - \alpha' = \tfrac{1}{2}(\beta - \beta'),$$
$$m + m' = \alpha + \alpha' + \beta + \beta'.$$

Then k' has α further intersections with w', at the points associated with the α tangents to k at O; and β further intersections with q', lying on the generators which $\sim$ the intersections of k, ω, and at the points of these generators associated with the tangent planes to ω containing the tangents to k.

Let k be a general curve, passing near O but not through it, and let it move so that one of its points $P \to O$. Before the limit, $k \sim$ a k'_{2m} which does not break up. Now k contains a point near O on each ray of a set lying near the plane p determined by the tangent line t at O to the final position of k, and the tangent to the path of approach of P to O.

Then k'_{2m} contains a point near to each point of the line l' of w' associated with p, which line is a definite member of the plane pencil whose vertex is the point associated with t. In the limit, it is l' that breaks away from k'_{2m}. If we are only given the final position of k, we cannot say which ray of this fixed pencil has broken away.

If k passes near a point Q of ω but not through it, k contains a point near Q in each plane through the tangent line to ω at Q. Hence k' contains a point near each point of $O'Q'$. In the limit, it is the line $O'Q'$ which breaks away, depending on Q only, and not on the path of approach.

In the more general case, we can suppose $k'_{2m-\alpha-\beta}$ to arise from a k'_{2m}, homologue of a k_m not incident with H, by the breaking away of $\alpha + \beta$ lines, of which α are unknown lines in w', one through each of the points associated with the α tangents to k at O, and β are the known generators of q' which $\sim$ the β intersections of k, ω.

9. The P-system determined directly.

We have determined G as the homologue of H'; but by consideration of the first space alone, without using the reverse transformation, we can also determine the various P-elements, from their property of meeting (ϕ) in F-points only.

If there is a P-line κ_1, it meets ϕ in two F-points, either two points of ω or O and one point of ω. Thus κ_1 either lies on w or is a generator of q. A P-curve κ_m of degree $m > 1$ meets ϕ in $2m$ F-points, of which at most $m-1$ lie at O, since κ is proper, and therefore at least $m+1$ lie on ω and in w. Hence κ is a plane curve lying wholly in w; and conversely any curve κ_m in w meets ϕ in $2m$ points of ω and no other point; the whole plane w is a P-element; and the only P-surfaces are w and q.

Having thus found G, we can find from it the nature of its homologue H', again without direct reference to (ϕ').

We have the following correspondences:

The whole plane w;	An isolated point O';
any plane meets w in a line;	any ϕ' has a tangent plane at O';
any line meets w once.	any c' passes once through O'.
The whole cone q;	A curve ω';
a generator κ of q;	a point Q' of ω';
any ϕ contains two κ's;	any plane contains two points Q' of ω', which is a conic;
any plane contains one point of each κ;	ϕ' passes once through each Q' and therefore simply through ω';
any line meets two κ's.	c' meets ω' twice.

Thus H' consists of the simple isolated F-point O' and simple F-conic ω'.

Finally, H' being determined, the second homaloidal web is thereby determined: (ϕ') is the family of quadrics through O', ω'; and all the properties of the reverse transformation follow from this.

10. The Jacobian.

Since a quadric with a d.p. is a cone, the Jacobian, or locus of d.p.'s of (ϕ), is the locus of vertices of cones belonging to the homaloidal web.

Taking as before,

$$\phi_1 \equiv xw, \quad \phi_2 \equiv yw, \quad \phi_3 \equiv zw, \quad \phi_4 \equiv q,$$

we have

$$J \equiv \begin{vmatrix} w, & 0, & 0, & x \\ 0, & w, & 0, & y \\ 0, & 0, & w, & z \\ \frac{\partial q}{\partial x}, & \frac{\partial q}{\partial y}, & \frac{\partial q}{\partial z}, & 0 \end{vmatrix} \equiv -2w^2 q,$$

and J consists of the P-plane counted twice and the P-cone counted once. In the same way, J' is a multiple of $w'^2 q'$.

If ϕ is a proper cone, vertex Q, it stands on ω, and one generator passes through O; hence OQ meets ω and Q lies on q. There is just one such cone vertex Q. But if the cone is a pair of planes, one is the plane w of ω, and Q lies on w; the other is any plane of the pencil through OQ. There are two linearly independent ϕ's with a d.p. at Q; hence w is squared in J. Since J is quartic, it can have no other sheet than $w^2 q$.

Consider the algebraic effect of requiring a homaloid,

$$\phi \equiv (c_1 x + c_2 y + c_3 z) w + c_4 q,$$

to pass through a general point of a P-surface. Let $Q\,(\xi, \eta, \zeta, 0)$ be any point of w not lying on ω; then $q\,(\xi, \eta, \zeta) \neq 0$, and the condition that ϕ passes through Q is $\phi\,(\xi, \eta, \zeta, 0) = 0$, which reduces to $c_4 = 0$. Now this is independent of ξ, η, ζ, and is the same for all points such as Q; any ϕ which contains one of them satisfies the condition and contains them all, and breaks up into w and the arbitrary plane $c_1 x + c_2 y + c_3 z$ through O.

If $Q(\xi, \eta, \zeta, \theta)$ is any point of q not lying at O or on ω, so that

$$q(\xi, \eta, \zeta) = 0, \ \theta \neq 0,$$

the condition that ϕ passes through Q becomes $c_1\xi + c_2\eta + c_3\zeta = 0$. This is independent of θ, and is the same for all positions of Q on the line

$$x : y : z = \xi : \eta : \zeta,$$

which is a generator κ of q. Any ϕ which contains one position of Q satisfies the condition and contains the whole of κ, but ϕ need not break up.

11. The homaloidal curves.

The homaloidal curves of T_{2-2} are the ∞^4 conics (c) in S which $\sim$ the lines (l') of S'. The family of all conics in S is ∞^8, hence (c) is subject to four conditions; these are all conditions of incidence with H. Each of the two incidences of c with ω imposes one condition; and the incidence with O imposes two, being equivalent to incidence with two lines through O. These four conditions exactly determine (c).

An incidence of c with an F-element corresponds to an incidence of l' with a P-element, that is, with a sheet of J'; and here we have to count once the sheet q', which $\sim \omega$, and twice the sheet w', which $\sim O$, and which is squared in J'. The four intersections of l' with J' answer to the four conditions presented to c by H.

Now J is the locus of d.p.'s of (c) as well as that of (ϕ). If c has a d.p. Q, it breaks up into a pair of lines through Q; only one can properly $\sim l'$, and the other must $\sim$ an F-point on l', and is a P-line through Q: hence Q lies on J.

Conversely, a d.p. of ϕ is a d.p. of the intersection c of ϕ with every ψ through that point; every point of J is a d.p. of ∞^1 c's.

12. Homography between the stars (O), (O').

A ray l of the star $(O) \sim$ a ray l' of (O'); the relation between the two stars is homographic, reducing to $x' : y' : z' = x : y : z$ if the equations of T_{2-2} are in their simplest form (1). The relation between homologous points on l, l' is also a homography, in which $O \sim$ the intersection of l', w', and $O' \sim$ the intersection of l, w. Hence we can construct T_{2-2}, if we are given the homography between the stars, and can find three pairs of homologues on each pair of rays; as, for example, if we know w, w' and the quadric ϕ' through O' which $\sim$ a given plane p, and which then determines ω' as its intersection with w'; or we may be given three planes and the corresponding quadrics, all passing through O' and having the common conic ω'.

Again, the homography on l, l' may be determined by a correlation between the points P of S and the planes p' of S', such that P' is the intersection of p', l' (8). Then if P describes a plane, p' describes a star of planes, homographic with (P) and therefore with the star of lines (l'). The locus of P' is therefore a quadric, and the transformation is T_{2-2}.

13. First special form: F-conic degenerate.

T_{2-2} can be specialized in two ways. In the first, the F-conic breaks up into a pair of intersecting lines. If these are l_1, $y = w = 0$ and l_2, $z = w = 0$, then $q \equiv yz$, and the equations of transformation are

$$x' : y' : z' : w' = xw : yw : zw : yz,$$

$$x : y : z : w = x'w' : y'w' : z'w' : y'z';$$

and the reverse transformation is specialized in the same way.

This has all the properties of the general T_{2-2}, and special features arise when the two components of ω play different parts.

Now $J' = -2w'^2 y'z'$; the P-cone in S' breaks up into two planes y', z', which $\sim l_1, l_2$ respectively. If any surface f_μ passes through l_1 and not through l_2, after transformation only one P-plane is dropped, and f' is of degree $\mu' = 2\mu - 1$.

We have $$f \equiv (y, w)\!(x, y, z, w)^{\mu-1},$$
the result of substitution is $y'f'$ where

$$f' \equiv (w', z')\!(x'w', y'w', z'w', y'z')^{\mu-1},$$

on which the multiplicities of O', l_1', l_2' are $\mu - 1$, $\mu - 1$, μ respectively.

More generally, if f passes i times through O, i_1 times through l_1 and i_2 times through l_2, after substitution $w'^{i} y'^{i_1} z'^{i_2}$ is dropped; f' is of degree

$$\mu' = 2\mu - i - i_1 - i_2,$$

and on it the multiplicities of O', l_1', l_2' are

$$i' = \mu - i_1 - i_2,$$

$$i_1' = \mu - i - i_1,$$

$$i_2' = \mu - i - i_2.$$

The point A $(1, 0, 0, 0)$, being a d.p. of ω, is a point of contact of (ϕ), with w as fixed tangent plane. This is not an independent condition, but a necessary consequence of the passage of (ϕ) through ω; and c does not pass through A, which is not an isolated F-point as defined on p. 162.

The points A, A' are centres of corresponding stars, in which

$$l,\quad y : z : w = \eta : \zeta : \theta \quad \text{and} \quad l',\quad y' : z' : w' = \eta\theta : \zeta\theta : \eta\zeta$$

are corresponding rays; and T_{2-2} sets up a homography on l, l', given by

$$x : y = x' : y',$$

in which A, A' are corresponding points.

14. Transformation of polars (67, 205).

From this there follows a property used in resolving a singular point of a surface, namely that if f_A is the βth polar of A, with regard to any surface f in a sufficiently general position with regard to A, then the homologue $(f_A)'$ of f_A coincides with $f'_{A'}$, the βth polar of A' with regard to f'. The proof is as follows.

If f is of degree μ, and on it O is i-fold and A is α-fold, a general ray l through A meets f in α points at A and $\mu-\alpha$ general points ΣP. These $\sim$ $\mu-\alpha$ general points $\Sigma P'$ of $f'_{\mu'}$, l', the remaining $\mu'-(\mu-\alpha)$ intersections falling at A'. Since A, A' correspond in the homography on l, l', the βth polar of the group ΣP, with respect to A as pole, $\sim$ the βth polar of $\Sigma P'$ with respect to A'; and the addition of any number of points coinciding with the poles does not affect the rest of the polar group. Hence the intersections of l, f_A other than A $\sim$ those of l', $f'_{A'}$ other than A'.

The argument fails if l or l' lies wholly on one of the surfaces. Hence $f'_{A'}$ coincides with $(f_A)'$ except possibly for a conical sheet vertex A or A'. If f itself has no such sheet, this could only arise as a P-surface of T_{2-2}. If A has a special position with regard to f, and a general point $X \rightarrow A$, then $(f_X)'$ breaks up only if the multiplicity of f_A at an F-element $>$ that of f_X, and this occurs when either OA is of multiplicity $> i-\beta$ on the tangent cone to f at O, or if l_1 or l_2 lies on f_A. We only consider $\beta \leqq i$.

If f is in sufficiently general position to avoid this, O is $(i-\beta)$-fold on f_A, which does not pass through l_1 or l_2, and we have

$$f \equiv f(x, y, z, w), \quad f_A = \frac{\partial^\beta}{\partial x^\beta} f(x, y, z, w),$$

$$f' \equiv w'^{-i} f(x'w', y'w', z'w', y'z'),$$

$$(f_A)' \equiv w'^{-(i-\beta)} \frac{\partial^\beta}{\partial (x'w')^\beta} f(x'w', y'w', z'w', y'z')$$

$$\equiv w'^{-i} \frac{\partial^\beta}{\partial x'^\beta} f(x'w', y'w', z'w', y'z')$$

$$\equiv \frac{\partial^\beta f'}{\partial x'^\beta} \equiv f'_{A'}.$$

In the application that we have in view (p. 368), a series of T_{2-2}'s are applied having as isolated F-points a singular point O of f, and certain singularities O' of the corresponding surfaces f', arising from O, f at each stage. In the first space, A, l_1 and l_2 are general. The F-conic of the second T_{2-2} consists of two lines passing through A' and otherwise general, not lying on f'. The later F-conics are similarly chosen.

Now A' is not in general position with regard to f', being an accidental singularity arising from the first T_{2-2}; but it is in sufficiently general position for the second T_{2-2} to preserve the polar relation between f', $f'_{A'}$; for we can show that $O'A'$ is of multiplicity $\leqq i'-\beta$ on the tangent cone to f' at O'. For if this were not so, and if OQ is the ray through O associated with the point O', then the plane OAQ would be a repeated tangent plane of the tangent cone to f at O, which is not so, since A is general. This can be verified from the equations.

The argument can be extended to later spaces; hence the polar relation of f, f_A is invariant for this series of T_{2-2}'s.

In a second application (p. 371), the multiple point of f is not O, but a general point of l_1. Then $i = 0$, $i - \beta < 0$, and we find $f'_{A'} \equiv w'^{\beta} (f_A)'$. The polar of A' consists partly of the homologue of the polar of A, the other part being the P-plane. In particular, $f'_{A'}$ coincides with $(f_A)'$ in the neighbourhood of any point of either not lying on w'.

This form of T_{2-2} can be further specialized by letting the two components of ω coincide. The equations can be taken as

$$x' : y' : z' : w' = xw \ : \ yw \ : \ zw \ : z^2,$$
$$x \ : y \ : z \ : w \ = x'w' : y'w' : z'w' : z'^2,$$

and (ϕ) touches the plane w all along the coincident line-pair l, $z^2 = w = 0$.

If A is any point of this line, the star $(A) \sim$ another star (A'), and $A \sim A'$ in the homography on each pair of rays. The polar property is therefore invariant for a series of transformations of this type also, provided that, at each stage, the line of contact passes through the pole.

Now the general homaloid

$$\phi \equiv c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_3\phi_4 \equiv (c_1x + c_2y + c_3z)\, w + c_4z^2$$

is a cone whose vertex $(c_2, -c_1, 0, 0)$ is an F-point variable on l with the parameters of ϕ (97 and p. 160).

15. Second special form: F-point on F-conic (84).

In the second specialization, the F-point lies upon the F-conic. We can obtain this as the limit of the general case, when $O \rightarrow$ a point A_1 of ω, and we shall show that then (ϕ) all touch a fixed plane at A_1.

We have assumed above that O does not lie on w. Since a proper quadric through ω can meet w in no other point, if O approaches w, its final position must lie on ω. As on p. 186, we take $q \equiv xy - z^2$; let $O\,(1, 0, 0, \epsilon) \rightarrow A_1\,(1, 0, 0, 0)$, moving along the line A_4A_1, $y = z = 0$, as $\epsilon \rightarrow 0$.

Of the four base homaloids of the general case, the last three, yw, zw, q, are still available; O is no longer the vertex of q, but lies on its generator A_4A_1 for all values of ϵ. But x no longer passes through O, and we must replace xw by $w\,(w - \epsilon x)$.

In the limit, $O \rightarrow A_1$ and $\epsilon \rightarrow 0$. The equations are more symmetrical if we change the order of ϕ_1, ϕ_2, ϕ_3, ϕ_4, and take

$$x' : y' : z' : w' = \ xy \ - z^2 \ : w^2 : \ zw \ : \ yw,$$
$$x \ : y \ : z \ : w \ = x'y' + z'^2 : w'^2 : z'w' : y'w',$$

showing that the second system is specialized in the same way as the first.

The general homaloid in S,

$$\phi \equiv c_1\,(xy - z^2) + w\,(c_2w + c_3z + c_4y),$$

touches at O the fixed plane y, which contains both $y = w = 0$ the tangent to ω, and $y = z = 0$ the path of approach of O. In S', the F-conic ω' is

$$w' = x'y' + z'^2 = 0,$$

and the F-point O' is $(1, 0, 0, 0)$, lying on ω', all the homaloids touching y'.

The first Jacobian is

$$J \equiv \begin{vmatrix} y, & x, & -2z, & 0 \\ 0, & 0, & 0, & 2w \\ 0, & 0, & w, & z \\ 0, & w, & 0, & y \end{vmatrix} \equiv -2yw^3;$$

the P-cone is replaced by a pair of planes, y the fixed tangent plane at O, and w the plane of ω over again. As $O \rightarrow A_1$, the cone $O\ (\omega)$ flattens down. If O approached a point of w not lying on ω, each sheet of the cone would ultimately coincide with w. But as $O \rightarrow$ the point A_1 of ω, the generators of the cone which join O to the nearer points of ω all lie near the plane y through O and the tangent to ω at A_1; and the generators which join O to the further points of ω all lie near w. In the limit, the cone breaks up into y, w.

Regarded as a P-plane, w' now plays several distinct parts: it corresponds to O; it partly corresponds to ω, for O belongs to ω; and it also corresponds to the contact condition.

If a general surface f_μ passes through O, each of its terms contains y, z or w as a factor, and after substitution the factor w' is dropped, just as in the general case. If f passes through ω and therefore through O, the quadratic factor dropped is $y'w'$, of which $w' \sim O$ regarded as a point of ω, and $y' \sim$ all the rest of ω.

Now this T_{2-2} has a contact condition as well as conditions of incidence. Let f satisfy the same contact condition as ϕ, touching y at O:

$$f \equiv x^{\mu-1}y + (y, z, w \overset{2}{\between} x, y, z, w)^{\mu-2},$$

which becomes w'^2f' where

$$f' \equiv (x'y' + z'^2)^{\mu-1} + (w', z', y' \overset{2}{\between} x'y' + z'^2, w'^2, z'w', y'w')^{\mu-2};$$

and the factor dropped is w'^2, one w' answering to the incidence, and the second w' to the contact, of f with (ϕ) at O.

Thus if f passes through O, the degree of f' is reduced by 1; if f also touches y there, it is further reduced by 1. If O is i-fold on f, the degree of f' is reduced by i; in this case also, let f touch y.

If this meant merely that the tangent cone $u_i\,(y, z, w)$ to f at O touched y, along $y = z = 0$ say,

$$u_i \equiv w^{i-1}y + (y, z \overset{2}{\between} y, z, w)^{i-2},$$

f could contain the terms $x^{\mu-i}z^2\,(z, w)^{i-2}$, which would become

$$(x'y' + z'^2)^{\mu-i}z'^2\,(z', y')^{i-2}w'^i,$$

and the power of w' to be dropped would still be only w'^i.

We therefore take the condition of touching y to mean that the tangent cone breaks up and has y as one sheet:

$$u_i \equiv yu_{i-1},$$

which becomes $$w'^{i+1}u_{i-1}(w', z', y');$$
then w'^{i+1} is dropped, and the degree of f' is reduced by $i+1$.

But if u_i has two sheets coinciding with y, it does not follow that another w' is dropped. Now

$$f \equiv x^{\mu-i}y^2u_{i-2} + (y, z, w \overset{i+1}{\between} x, y, z, w)^{\mu-i-1},$$

the second group containing the terms $x^{\mu-i-1}(z, w)^{i+1}$ from which, after substitution, only w'^{i+1} can be dropped. Now f has not two separate sheets touching y, but two connected half-sheets, forming a single cuspidal sheet. If there are two separate sheets, O is a tacnode,

$$f \equiv x^{\mu-i}y^2u_{i-2} + x^{\mu-i-1}y\,u_i + (y, z, w \overset{i+2}{\between} x, y, z, w)^{\mu-i-2},$$

and then w'^{i+2} can be dropped.

The case when more than two sheets of f touch y can be treated in the same way.

Of the two intersections of c with ω, one is fixed at O and the other variable. Now O presents three conditions to c, two of incidence at a fixed point and one of contact with a fixed plane. This answers to the fact that w' is cubed in J'.

Since c passes through O, this is an isolated F-point.

In this specialized T_{2-2}, the P-planes are still loci of lines meeting (ϕ) in F-points only. Just as before, $w \sim$ a single F-point O' in S'. In the P-plane y, every line κ through O touches ϕ there and does not meet it again. Every ϕ meets y in a conic having a d.p. at the point of contact O, that is, in two lines κ; hence $y \sim$ an F-element which meets every plane in S' twice, that is, the conic ω'. There is one κ which also lies in w, namely the intersection of the two P-surfaces, which in this case is not an F-curve; hence the two F-elements in S' are incident, and O' lies upon ω'.

16. Transformation of the neighbourhood of the F-point of contact.

Let $P(1, \eta\epsilon, \zeta\epsilon, \theta\epsilon)$ be a point near O, and let it $\rightarrow O$ along any path as $\epsilon \rightarrow 0$. Then P' is $(\eta\epsilon - \zeta^2\epsilon^2, \theta^2\epsilon^2, \zeta\theta\epsilon^2, \eta\theta\epsilon^2)$, and $\rightarrow O'$ $(1, 0, 0, 0)$ unless η becomes small of the same order as ϵ. If so, let $\eta \equiv \eta_1\epsilon$, where η_1 remains finite; then $P(1, \eta_1\epsilon^2, \zeta\epsilon, \theta\epsilon) \rightarrow O$ along a path which is approximately a conic touching y, along the line $y = \theta z - \zeta w = 0$, and having a definite curvature depending on η_1. Then P' $((\eta_1 - \zeta^2)\epsilon^2, \theta^2\epsilon^2, \zeta\theta\epsilon^2, \eta_1\theta\epsilon^3) \rightarrow Q'$ $(\eta_1 - \zeta^2, \theta^2, \zeta\theta, 0)$, a definite general point of the P-plane w', which is associated with this path of approach to O.

Thus all general points near $O \sim$ points near the single point O'; but points of the neighbourhood of O lying near the fixed tangent plane are spread out into the neighbourhood of the whole P-plane w'.

If $Q(1, \epsilon^2, \epsilon, 0)$ is a point of ω which $\rightarrow O$ along ω as $\epsilon \rightarrow 0$, then $Q \sim$ the P-line $y' = w' - \epsilon z' = 0$. When Q reaches O, this becomes $y' = w' = 0$, the common line of the two P-planes. Since it lies on w', it is dropped with the

P-plane when f passes through O, and does not lie on the proper homologue (p. 266).

17. Specializations combined.

Both special forms of T_{2-2} may occur together. If the components of ω are distinct, and O lies at a general point of one of them, the equations can be

$$x' : y' : z' : w' = xy : w^2 : zw : yw, \quad J \equiv -2yw^3,$$
$$x : y : z : w = x'y' : w'^2 : z'w' : y'w'.$$

Now (ϕ) has two points of contact, but of different natures. $A\,(0, 0, 1, 0)$ is a d.p. of ω, the contact there is incidental to the passage of (ϕ) through ω, and c does not pass through A. But $O\,(1, 0, 0, 0)$ is a simple point of ω, and the condition of contact there is independent; c passes once through O, which is an isolated F-point, while A is not.

The two components of ω, say

$$l_1,\ x = w = 0, \quad \text{and} \quad l_2,\ y = w = 0,$$

also play different parts. For l_1 meets c in one variable point, and is an F-line of first species which $\sim$ the P-plane y'. But l_2 contains O, and meets c there only; it is an F-line of second species, and $\sim$ the similar line l_2'; the passage of (ϕ) through l_2 is a necessary consequence of its passage through l_1 and contact at O, which cause ϕ to contain three points of l_2.

A general surface f_μ meets l_1 in μ points; let one be $Q\,(0, \eta, 1, 0)$. The corresponding P-line κ' is $y' = w' - \eta z' = 0$, and f' contains μ such lines. If $Q \rightarrow A$ along l_1, then $\eta \rightarrow 0$ and $\kappa' \rightarrow l_2'$, approaching it in the plane y', and in the limit, one of the μ sheets of f' touches y' along l_2'.

If one of the μ intersections of f, l_2 is $Q\,(\xi, 0, 1, 0)$, then f is the sum of multiples of $x - \xi z$, y, w, and f' of $f_2' \equiv x'y' - \xi z'w'$, w'^2, $y'w'$, of which the first is a quadric through l_2', and the last two pass twice through l_2'; hence the μ sheets of f' through l_2' touch μ quadrics such as f_2'. If $Q \rightarrow A$ along l_2, then $\xi \rightarrow 0$, and the sheet of f_2' through l_2' reduces to y', and as before when f passes through A, one of the sheets of f' touches y' along l_2'.

If more intersections of f, l_1 lie at A, one sheet of f' has higher contact with y' along l_2'; if more intersections of f, l_2 lie at A, more sheets of f' have simple contact with y' along l_2'.

If f passes simply through A,

$$f \equiv z^{\mu-1}(ax + by + cw) + (x, y, w\overset{2}{)\!(}x, y, z, w)^{\mu-2},$$

$$f' \equiv (z'w')^{\mu-1}(ax'y' + bw'^2 + cy'w') + (x'y', w'^2, y'w'\overset{2}{)\!(}x'y', w'^2, z'w', y'w')^{\mu-2};$$

the terms of lowest degree μ in y', w' are

$$(z'w')^{\mu-1}ax'y' + x'^2y'^2(x'y', 0, z'w', 0)^{\mu-2}$$

having y' as a factor, and one sheet of f' along l_2' touches y'.

If f touches l_1 at A without touching w, then $b=0$, the terms of degree $\mu+1$ in y', w' have y' as a factor, and y' osculates one sheet of f'. If instead f touches l_2, then $a=0$, and the terms of degree μ have y'^2 as a factor, and there is a cuspidal sheet of f' touching y'. If f touches w, then $a=b=0$, and the sheet of f' is tacnodal.

The form of the equations of transformation of p. 197 must be modified if O is the point of intersection of l_1, l_2. If we first put $q \equiv yz$, and then make $O \to A_1$ along A_4A_1 as on p. 194, we find that ϕ acquires a d.p. at O and becomes a cone, the coordinate x drops out of the equations, and the transformation fails. The homaloids already touch w at the d.p. of ω, and cannot be made to touch a different plane.

But if $O \to A_1$ along a path touching w, we obtain in the limit a homaloidal family with a point of osculation at O, for example

$$\phi = c_1(xw+yz)+w(c_2y+c_3z+c_4w).$$

If the two components of ω coincide, (ϕ) touches w at every point of ω, and cannot touch a different plane at any point of it. If O lies on ω, it is again a point of osculation. In the above expression for ϕ, we can replace yz by z^2.

18. Application to surfaces of higher degree (134, 261, 265).

As an example of the use of Cremona transformations to discover surfaces of higher degree, let us find what surfaces $f'_{\mu'}$, of degree $\mu'>3$, can be obtained by T_{2-2} from a cubic surface f_3. Since the theory of cubics is well known, we can obtain properties of the higher surface by transforming those of the lower. An example is worked out below.

The nature of f' depends on the multiplicities on f_3 of O, ω or the components l_1, l_2 of ω if we choose it to be a line-pair. The cases which give $\mu'>3$ are as follows:

F-elements on f_3	μ'	F-elements on f'
	6	$O'^3\omega'^3$
O	5	$O'^3\omega'^2$
l_1	5	$O'^2l_1'^2l_2'^3$
O^2	4	$O'^3\omega'$
Ol_1	4	$O'^2l_1'l_2'^2$
l_1^2	4	$O'l_1'l_2'^3$
ω	4	$O'\omega'^2$

As a more detailed example (134), consider what properties of the last quartic surface, on which ω' is a double conic and O' any simple point, follow from the existence of the 27 lines on the cubic. Of these:

(i) one, l_1, lies in the plane w and meets ω twice;

(ii) sixteen, $l_2, \ldots l_{17}$, meet ω once;

(iii) ten, $l_{18}, \ldots l_{27}$, meet l_1 and not ω.

None of them passes through O, which does not lie on f_3.

(i) All the points of $l_1 \sim O'$, and are associated with lines through O' lying in a plane; the property of f_4' is that it has a tangent plane at each ordinary point such as O'.

(ii) Since l_2 contains an F-point of ω, it $\sim$ a line meeting ω' and not passing through O'. Hence f_4' contains 16 lines meeting the double conic.

(iii) Since l_{18} contains no F-point, it $\sim$ a conic through O' meeting ω' twice. Hence there are 10 conics on f_4' through a general point O' of the surface.

Any line on f_4' must meet ω', and any conic must meet it twice; hence either can be transformed into a line of an f_3; and f_4' has no other lines or conics than those just found.

19. Application to a binode (353).

The use of Cremona transformations to study higher singularities of surfaces is discussed in Chapter XVI; as a preparatory example, consider the resolution of a binode by T_{2-2}.

Let f_μ have a d.p. O (0, 0, 0, 1)

$$f_\mu \equiv w^{\mu-2}u_2 + w^{\mu-3}u_3 + (x, y, z\overset{4}{)}(x, y, z, w)^{\mu-4}.$$

Apply a general T_{2-2} with O as F-point

$$x : y : z : w = x'w' : y'w' : z'w' : q',$$

where w, q have no special relation to f. After dropping w'^2, the homologue is

$$f'_{2\mu-2} \equiv q'^{\mu-2}u_2' + q'^{\mu-3}w'u_3' + w'^2 f'_{2\mu-4}.$$

Associated with the quadric tangent cone u_2 to f at O, we have the conic k_2', $w' = u_2' = 0$, which, together with the F-conic ω' taken $\mu - 2$ times, makes up the total intersection of f' with the P-plane w'.

If O is a conical d.p., the tangent cone u_2 does not break up, and k_2' is a proper conic; the section of f' by w' has no singular point on k_2' other than its intersections with ω', and where the section is not singular, f' itself cannot be singular.

On the other hand, f' has acquired fresh singularities at O' and ω'; since w is general, it meets f in a proper curve, and the tangent cone to f' at O' is proper. Hence O' is an ordinary multiple point (p. 360) and similarly ω' is an ordinary multiple curve with distinct tangent planes. The intersections of ω', k_2' are points of ω' where one of the tangent planes to f' is w', and are not otherwise remarkable. O' and ω' are accidental singularities, arising from the nature of the transformation and not from the nature of the singularity of f at O, which has entirely disappeared.

As in the plane, we say that a T_{2-2} applied to a d.p. always removes a conic node, and that if O is a d.p. of higher species, it is compounded of a conic node and whatever singularity of $f' \sim O$. This convention is justified in Chapter XVI.

Now let O be a binode on f of the simplest type B_3, such as reduces the class of f by 3 and no more; the tangent cone breaks up into a pair of planes, $u_2 \equiv yz$ say, and the edge $y = z = 0$ of the binode is not a line of closest contact and does not lie on u_3, or on q since this is general: u_3 must contain a term in x^3, and q one in x^2.

Then k_2' breaks up into a pair of lines, whose intersection $(1, 0, 0, 0)$ is associated with the edge. At this point, f' touches w', but is not singular; for a term $x'^{2\mu-3}w'$ occurs in the second group $q'^{\mu-3}w'u_3'$. Thus answering to the singularity of f at O, including the singularity of the tangent cone along the edge, we have a pair of lines on f', and at a point of these, which is simple for f', a d.p. of the section by the P-plane w', that is, a point of contact of f', w'. Hence the B_3 at O is compounded of a conic node, removed by T_{2-2}, and a point of contact with the P-plane, explicit on f'.

Now suppose that f' has a conic node which $\sim O$, and therefore lies at some point, $(1, 0, 0, 0)$ say, of k_2' other than its intersections with ω'. The conditions are

$$u_2 \equiv (y, z)^2, \quad u_3 \equiv (y, z)(x, y, z)^2.$$

Then the tangent cone at O is a pair of planes, O is a binode with $y = z = 0$ as edge; and this edge lies on u_3, and is a line of closest contact, showing that O is a binode of the next higher species B_4, which, as regards reduction of class, and also from other points of view, is equivalent to two conic nodes; T_{2-2} removes one of these and leaves the other explicit on f'.

Lastly, let O be a binode of any species B_{s+1}, $s \geqq 2$, such as reduces the class of f by exactly $s + 1$. The analytical expression of this is that f can be written (164):

$$f_\mu \equiv w^{\mu-2s+2}\phi_{s-1}\psi_{s-1} + w^{\mu-s-1}u_{s+1} + \dots + u_\mu,$$

where

$$\phi_{s-1} \equiv w^{s-2}y + w^{s-3}u_2 + \dots + u_{s-1},$$
$$\psi_{s-1} \equiv w^{s-2}z + w^{s-3}v_2 + \dots + v_{s-1}.$$

Here ϕ_{s-1}, ψ_{s-1} are surfaces with O as a simple point, approximations to the separate sheets of f. The condition that O is not a binode of higher species is that the edge $y = z = 0$ does not lie on u_{s+1}, which must contain a term in x^{s+1}.

After dropping w'^2, the homologue is

$$f'_{2\mu-2} \equiv q'^{\mu-2s+2}\phi'_{2s-3}\psi'_{2s-3} + q'^{\mu-s-1}w'^{s-1}u'_{s+1} + \dots,$$

where

$$\phi'_{2s-3} \equiv q'^{s-2}y' + q'^{s-3}w'u'_2 + \dots + w'^{s-2}u'_{s-1},$$
$$\psi'_{2s-3} \equiv q'^{s-2}z' + q'^{s-3}w'v'_2 + \dots + w'^{s-2}v'_{s-1}.$$

Now $O \sim$ the line-pair $w' = y'z' = 0$ on f'; the only singularity of f' on these, other than their intersections with q', is their common point O' (1, 0, 0, 0), associated with the edge at O. Here x', q', u'_{s+1} are finite, and y', z', w', ϕ'_{2s-3}, ψ'_{2s-3} are small of first order; the form of f' shows that O' is exactly a B_{s-1} on f', with $y' = z' = 0$ as edge. The B_{s+1} is resolved into a conic node, removed by T_{2-2}, and a binode of species lower by 2, explicit on f'.

By repeating the process, at each step removing a conic node and reducing the species by 2, we arrive at either B_4 or B_3, according as s is odd or even; both of these have been discussed. Hence B_{s+1} is resolved into $\frac{1}{2}(s+1)$ conic nodes if s is odd; or if s is even, into $\frac{1}{2}s$ conic nodes together with a point of contact with the last P-plane of the resolving series of T_{2-2}'s.

Thus in any case, if we apply a finite number of T_{2-2}'s to O and to those of its successive homologues that are singular for the homologues of f, we arrive at a surface on which all the points that $\sim O$ are simple, and all the multiple elements introduced by the transformations are ordinary.

The form assumed for the B_{s+1} is not valid if O is a point of a double curve, when the partial factorization $\phi_{s-1} . \psi_{s-1}$ can be carried on indefinitely, the terms of degree $> n$ in x, y, z arising from the factors being exactly destroyed by succeeding terms. Then there are ∞ d.p.'s adjacent to O (p. 244).

II. SPACES SUPERPOSED

20. Notation.

Return to the general T_{2-2}, and now let the two spaces be superposed. All the previous properties continue to hold, and we may also enquire into the invariant and involutory systems, as in the plane.

The simplest forms of the equations of the general T_{2-2} for separate spaces are (1), (2) of p. 184:

$$x' : y' : z' : w' = xw : yw : zw : q(x, y, z), \qquad \text{..............(1)}$$

$$x : y : z : w = x'w' : y'w' : z'w' : q(x', y', z'). \qquad \text{...........(2)}$$

We have two frames of reference, one in each space; in S, the four planes are any three, x, y, z, through the F-point O, and the plane w of the F-conic ω; and in S', the three planes x', y', z' which $\sim x$, y, z by T_{2-2}, and the plane w' of the F-conic ω'. In general we shall take $q(x, y, z) \equiv yz + zx + xy$; then three vertices of the tetrahedron lie on ω.

Now let the two spaces be superposed in any way. The two frames of reference do not in general coincide, so we take a fresh frame, the same for both spaces, in general position, which can be chosen later in any particular way that may be convenient. The effect of the change of frame in either space is to replace the coordinates by general linear functions, and we obtain from (1), (2) the equations of transformation for superposed spaces:

$$p_1' : p_2' : p_3' : p_4' = p_1p_4 : p_2p_4 : p_3p_4 : q(p_1, p_2, p_3), \qquad \text{.........(3)}$$

$$p_1 : p_2 : p_3 : p_4 = p_1'p_4' : p_2'p_4' : p_3'p_4' : q(p_1', p_2', p_3'),$$

where
$$\left.\begin{aligned} p_a &\equiv c_{a1}x + c_{a2}y + c_{a3}z + c_{a4}w \\ p_a' &\equiv c'_{a1}x' + c'_{a2}y' + c'_{a3}z' + c'_{a4}w' \end{aligned}\right\} \quad (\alpha = 1, 2, 3, 4).$$

The first F-system is now: O, $p_1 = p_2 = p_3 = 0$, and ω, $p_4 = q = 0$; and the first P-system is: the plane p_4, and the cone q.

We can use equations (3) as they stand, or we can solve for $x' : y' : z' : w'$, and obtain explicit equations in terms of base homaloids.

21. The invariant system.

We shall show that if O, O' are distinct the possible invariant systems are:

(i) 6 points;
(ii) 3 points and a line meeting ω, ω';
(iii) 2 points and a line OO';
(iv) two skew lines meeting ω, ω';
(v) 2 points and a conic meeting ω, ω' each twice;
(vi) 1 point and a conic through O, O' meeting ω, ω';
(vii) a cubic through O, O' meeting ω, ω' each twice;
(viii) 1 point and a plane through one line of each of ω, ω';
(ix) a line OO' and a plane;
(x) a quadric through ω, ω'.

(i) The invariant points are given by (3), if in them we put
$$x' : y' : z' : w' = x : y : z : w,$$
that is, by
$$p_4 f_{2,1} = p_4 f_{2,2} = f_3 = 0,$$
where
$$f_{2,1} \equiv p_1 p_2' - p_1' p_2, \quad f_{2,2} \equiv p_1 p_3' - p_1' p_3, \quad f_3 \equiv p_1 p_4 p_4' - p_1' q,$$
the accents here applying to the coefficients, and not to the variables.

If an invariant point D lies on p_4, either it is a general P-point, which coincides with its homologue O', and lies on p_4, p_1', p_2', p_3'; or it is an F-point of ω, which coincides with a point of its corresponding P-line: we can then choose p_2, p_3 to pass through D, which lies on p_2, p_3, p_4, p_2', p_3' q, q'. In either case D lies on $f_{2,1}$, $f_{2,2}$ and f_3.

Hence all the invariant points are among the intersections of $f_{2,1}$, $f_{2,2}$, f_3. These all contain the line l, $p_1 = p_1' = 0$, which does not in general contain any invariant point, since p_1 is a general plane through O. If the two quadrics $f_{2,1}$, $f_{2,2}$ have no common sheet, their residual intersection is k_3 meeting l twice and passing through O, O'; it is the curve generated by intersecting pairs of corresponding rays of the stars (O), (O').

Now f_3 is a cubic surface through l, O, ω; if no part of k_3 lies wholly on f_3, they meet in nine points, of which two are on l, one is O, and the other six are the invariant points D of T_{2-2}.

Though O is an additional common point of $f_{2,1}$, $f_{2,2}$, f_3, it cannot be a seventh invariant point; for if it coincides with one of its homologues, it lies on p_1, p_2, p_3, p_4', and is a d.p. of q, f_3, and a double intersection of f_3, k_3, absorbing one of the six points D, and not additional to them. There may however be a locus of invariant points, either a curve or surface, common to all of $f_{2,1}, f_{2,2}, f_3$; the number of isolated points is then modified.

(ii) The only lines which $\sim$ lines are those which contain just one F-point. A line meeting $\omega \sim$ one meeting ω' and a line through $O \sim$ one through O'. An invariant line δ_1 must be of one or other of these kinds in each space.

First, let δ_1 be of the first kind, meeting ω, ω'. Then δ_1 meets the pair of corresponding planes p_1, p_1' in the same point, and therefore meets l. The residual intersection of $f_{2,1}$, $f_{2,2}$ is a conic meeting δ_1 and l, and passing through O; it meets f_3 in these three points and in three points D. The invariant line replaces three isolated points.

Conversely, if three invariant points lie on a line δ_1, this is wholly invariant. For it $\sim$ a conic meeting it in three points, and therefore partly coinciding with it; thus δ_1 is self-corresponding. And the homography on δ_1 has three double points and is therefore identity; thus δ_1 is invariant.

(iii) Next, let $\delta_1 \equiv OO'$ be an invariant line of the second kind. Then O is invariant, and lies on p_4', and O' on p_4. Take O, O' and any two common points of p_4, p_4' as tetrahedron of reference; then δ_1 is $z = w = 0$, and

$$p_\alpha \equiv c_{\alpha 2}y + c_{\alpha 3}z + c_{\alpha 4}w, \quad p_\alpha' \equiv c'_{\alpha 1}x + c'_{\alpha 3}z + c'_{\alpha 4}w, \qquad (\alpha = 1, 2, 3),$$
$$p_4 \equiv x, \qquad\qquad p_4' \equiv y.$$

The equations of transformation (3) are satisfied if P, P' coincide at any point of δ_1, that is, if

$$z = w = z' = w' = 0, \quad x : y = x' : y';$$

the conditions are

$$c_{11}' : c_{21}' : c_{31}' : 1 = c_{12} : c_{22} : c_{32} : q\,(c_{12}, c_{22}, c_{32}).$$

These conditions are compatible, and do not lead to a reduction of degree of T_{2-2}. They express that δ_1 lies on $f_{2,1}$, $f_{2,2}$, f_3. It is skew to their other common line l; the residual intersection of $f_{2,1}$, $f_{2,2}$ is another pair of skew lines, each meeting δ_1 and l, and meeting f_3 in one of the two isolated invariant points D. The common point O of the three surfaces lies on δ_1 and not on the residual; it is invariant but not isolated.

Now the pencil of planes through $OO' \sim$ itself, and has two self-corresponding planes, in each of which T_{2-2} sets up a plane T_2 for which OO' is an invariant line; there is therefore one isolated invariant point D in each of these planes, and no other (p. 46). This invariant line δ_1 replaces four of the points D.

(iv) Two intersecting invariant lines are a particular case of a conic, see below. If there are two skew lines, one of each kind, say $\delta_{1,1}$ meeting ω, ω' and $\delta_{1,2}$ through OO', then l meets $\delta_{1,1}$ and not $\delta_{1,2}$. If one or both of the two quadrics, containing these three lines, are determinate, they break up and contain the plane $l\delta_{1,1}$, which is either part of f_3, and is an invariant plane, or meets it in l, $\delta_{1,1}$ and another invariant line, meeting $\delta_{1,1}$ and $\delta_{1,2}$, and forming with them a degenerate invariant cubic. Both these cases are treated below. Thus δ cannot consist only of a pair of skew lines of different kinds.

But it can consist of two skew lines $\delta_{1,1}$, $\delta_{1,2}$, both of first kind. Then l meets both, the quadrics are proper and their residual intersection is the line OO', meeting f_3 on $\delta_{1,1}$, $\delta_{1,2}$ and at O, and in no other point. There is no isolated invariant point, each line replacing three of the original six.

(v) Let there be an invariant conic δ_2. Since it $\sim$ a conic, it contains just two F-points of H, either two points of ω, or O and one point of ω.

First let δ_2 meet ω twice, and not pass through O; then it also meets ω' twice, and does not pass through O'. Now p_1 meets δ_2 in two points, which lie on p_1', and l is a chord of δ_2 and lies in its plane p. Hence $f_{2,1}, f_{2,2}$ meet p in the cubic curve δ_2, l, and break up, having p as a common sheet, meeting f_3 in the same cubic curve. The other sheets meet in OO', which does not meet l or δ_2, and meets f_3 in O and two isolated invariant points D.

(vi) Next, let δ_2 contain O, O' and meet ω, ω' each once. Then p_1 meets δ_2 in O and one other point, which lies on p_1', and on l. The residual intersection of $f_{2,1}$, $f_{2,2}$ is a line meeting δ_2 and l, and meeting f_3 in these and in the one point D.

(vii) A conic and a line meeting it form a degenerate cubic. If there were an invariant conic δ_2 and a line δ_1 not meeting it, any line in the plane p of δ_2 through the trace of δ_1 would contain three invariant points, and it and p would be wholly invariant. In the same way there cannot be a plane invariant curve of degree $\geqq 3$.

If there is an invariant twisted cubic δ_3, it forms with l the total intersection of $f_{2,1}$, $f_{2,2}$ and lies wholly on f_3; there is no isolated invariant point. Since O does not lie on l, it lies on δ_3; and since this $\sim$ a cubic curve, it contains three F-points of each system, O, O' and two points of each of ω, ω'.

If there were an invariant twisted curve of degree $\geqq 4$, then $f_{2,1}$, $f_{2,2}$, having this and l in common, would coincide, see (x).

(viii) If there is an invariant plane Δ_1, every line in it contains just one F-point of each system. This can only be if each of ω, ω' breaks up, and one line of each lies on Δ_1, also lying on the corresponding P-plane. The d.p.'s of ω, ω' coincide, being the intersections of Δ_1 with corresponding rays through OO'. Since Δ_1 is not principal, it does not contain O or O', and the F-lines on it are not an associated pair.

Each of $f_{2,1}$, $f_{2,2}$ breaks up into Δ_1 and another plane, the two others meeting in OO'. Then f_3 also breaks up, into Δ_1 and a quadric through O meeting OO' again in the one isolated invariant point.

(ix) This point can be replaced by an invariant line $\delta_1 \equiv OO'$, if O lies on p_4', and O' on p_4; then δ_1 is a self-corresponding line with three invariant points at O, O' and on Δ_1, and is therefore invariant.

(x) There remains the case when $f_{2,1}$, $f_{2,2}$ coincide,

$$p_1 p_2' - p_1' p_2 \equiv a\,(p_1 p_3' - p_1' p_3),$$

$$\frac{p_1'}{p_1} \equiv \frac{p_2' - a p_3'}{p_2 - a p_3},$$

for any choice of the three planes p_1, p_2, p_3 through O. Hence p_1, p_1' coincide, the stars (O), (O') are identical,

$$O \equiv O'; \quad q \equiv q'; \quad p_2', p_3' \equiv p_2, p_3; \quad f_{2,1} \equiv f_{2,2} \equiv 0;$$

f_3 breaks up into $p_1 \equiv p_1'$ and an invariant quadric through ω, ω', locus of d.p.'s of the homographies on rays through O.

There can be no other invariant point; for any line through it would contain three invariant points and be wholly invariant, and T_{2-2} would reduce to identity. For the same reason there cannot be an invariant surface of degree $\geqq 3$.

22. The involutory system.

The general T_{2-2} in one space has certain involutory pairs K, K' which are also homologues in T^{-1}_{2-2}. Take the planes of reference to suit H', so that

$$p_1' \equiv x', \quad p_2' \equiv y', \quad p_3' \equiv z', \quad p_4' \equiv w'.$$

Then for any pair that is either involutory or invariant

$$x' : y' : z' : w' = p_1 p_4 : p_2 p_4 : p_3 p_4 : q\,(p_1, p_2, p_3),$$

$$p_1(x', y', z', w') : p_2(x', y', z', w') : p_3(x', y', z', w') : p_4(x', y', z', w') = xw : yw : zw : q\,(x, y, z).$$

Eliminate x', y', z', w'; we replace $p_1(x', y', z', w')$ by

$$f_{2,1} \equiv (c_{11} p_1 + c_{12} p_2 + c_{13} p_3)\, p_4 + c_{14}\, q\,(p_1, p_2, p_3),$$

which is the quadric through O, ω which $\sim p_1$ in T^{-1}_{2-2}, and we obtain three equations for K,

$$w f_{3,1} = w f_{3,2} = f_4 = 0,$$

where $\quad f_{3,1} \equiv x f_{2,2} - y f_{2,1}, \quad f_{3,2} \equiv x f_{2,3} - z f_{2,1}, \quad f_4 \equiv x w f_{2,4} - q f_{2,1}.$

Now $f_{3,1}$, $f_{3,2}$ meet in O, O', ω, the conic k_2, $x = f_{2,1} = 0$, which meets ω twice, and a residual k_5 through O, O', meeting each of ω, k_2 four times. It meets f_4 in these ten points and ten others, of which six are the invariant points, and the remaining four fall into the two involutory pairs of T_{2-2}.

Since w is a P-plane, it does not in general contain any involutory point, neither does k_2, which lies in the general plane x through O'.

Now $f_{3,1}$, with w, is the M-surface of the general line $p_1 = p_2 = 0$ through O. It passes through O, O', ω, for which one of P', P_1 is indeterminate, and through O_1 the homologue of O in T^{-1}_{2-2}, for which P' is at O. So $f_{3,2}$ is the proper M-surface of $p_1 = p_3 = 0$. These both contain k_2, the locus of P when $P'P_1$ lies in p_1, which is the intersection of the plane and quadric homologues of p_1 in T_{2-2}, T^{-1}_{2-2}; and k_5 is the locus of P when $P'P_1$ passes through O.

If f_3' is the proper M-surface of a line l' through O', it meets k_5 in 15 points, namely O, O', the six invariant and four involutory points, and three others, for which $P'P_1$ lies in the plane Ol' and P in the plane homologue of Ol' in T^{-1}_{2-2}, which meets k_5 in O, O_1 and the three points in question.

The number of involutory pairs also follows from the fact that the invariant and involutory points of T_{2-2} are the ten invariant points of the $T_{4-4} \equiv T^2_{2-2}$ (p. 293).

The involutory pairs of T_{2-2} have been determined apart from the invariant points as follows. Let T_{2-2} be constructed by the homography between the stars (O), (O') and a correlation V (p. 191). It is first shown that if $OP \sim O'P'$ and also $OP' \sim O'P$, then P, P' are conjugate in one of a set of involutions, one on each generator λ of a certain regulus. Now V sets up on λ a homography which is involutory on just two generators; on each of these the two involutions have one common pair K, K', which is involutory for T_{2-2} (168).

The involutory system, consisting of those common elements of $f_{3,1}, f_{3,2}, f_4$ that are neither fundamental nor invariant, can have a great variety of forms. The maximum is a cubic surface, if T_{2-2} is not involutory.

23. F-points coincident (8).

Now let O, O' coincide at $(0, 0, 0, 1)$. The homography in the star (O) may have three invariant rays, self-corresponding for T_{2-2}, or an invariant plane and ray, or may be identity. In any case, the equations can be

$$x' : y' : z' : w' = axp_4 : byp_4 : czp_4 : \phi,$$

where

$$\phi \equiv wu_1 + u_2.$$

Each self-corresponding ray contains two invariant points, one of which may be O if both P-planes pass through O; or the ray may be wholly invariant. The possible invariant systems of this T_{2-2} are:

6 points,
O and 3 other points,
2 points and a line through O,
1 point and 2 lines through O,
3 lines through O,
2 points and a conic,
1 point and a conic through O,
a line and a conic, both through O,
1 point and a plane through O,
a line and a plane, both through O,
a quadric.

There is in general no proper involutory system; the common line of the two P-planes is improperly conjugate to O; but there may be an involution on any self-corresponding ray. If the homography is an involution, T_{2-2} has an involutory cubic surface. For $OP \sim OP'P_1$, containing both homologues of P. As P describes a general line l, so P', P_1 describe two conics c', c_1 through O, both lying in the plane corresponding to Ol in the involution, and therefore meeting in three points K' other than O, which $\sim$ three points K of l; for these, $K' \equiv K_1$, and K, K' are involutory. Since l contains three such points, the involutory locus is a cubic surface.

24. First type of quadric involution.

There remains the possibility, which has no parallel for the invariant system, that $f_{3,1}, f_{3,2}, f_4$ may all vanish identically without T_{2-2} degenerating. Then the involutory system is the whole space, every point belongs to an involutory pair, and T_{2-2} becomes an involution I_{2-2}, the direct and reverse transformations being the same, and the square identity, $I^2_{2-2} \equiv 1$.

In I_{2-2}, the two F-systems are identical, and also the two P-systems:

$$O \equiv O', \quad \omega \equiv \omega', \quad p_4 \equiv p_4', \quad q \equiv q'.$$

The homography of rays of the coincident stars (O), (O') is an involution, which may be proper or may reduce to identity.

In the first type of I_{2-2}, the involution of the star (O) is proper, and therefore has a plane, z say, of rays and an isolated ray, $x = y = 0$ say, which are invariant for the involution of rays, and self-conjugate for I_{2-2}. The equations of T_{2-2} reduce to

$$x' : y' : z' : w' = xw : yw : -zw : q(x, y, z),$$
$$x : y : z : w = x'w' : y'w' : -z'w' : q(x', y', z').$$

Since this is involutory, q is an even function of z, say

$$q \equiv u_2(x, y) - z^2.$$

These are equivalent to the three bilinear relations

$$\begin{aligned} x'z \qquad + z'x \qquad &= 0, \\ y'z + z'y \qquad &= 0, \\ x' \frac{\partial u_2}{\partial x} + y' \frac{\partial u_2}{\partial y} + 2z'z - 2w'w &= 0; \end{aligned}$$

hence this I_{2-2} is a degenerate case of the T_{3-3} given by three general bilinear relations (p. 294).

The relations are all symmetrical, and express that P, P' are conjugate with regard to the net of quadrics

$$(ax + by)z + c(u_2 + z^2 - w^2),$$

whose base consists of the conic δ_2, $z = u_2 - w^2 = 0$, and the pair of points D_1, D_2 $(0, 0, 1, \pm 1)$. We have thus a particular form of extension to space of the first type of plane I_2, where P, P' are conjugate with regard to a pencil of conics. The more natural extension uses a general net of quadrics with eight points as base, giving the I_{3-3} (p. 303) of which our I_{2-2} is a degeneration.

The base δ_2, D_1, D_2 is the invariant system, also given by

$$\frac{xw}{x} = \frac{yw}{y} = \frac{-zw}{z} = \frac{q}{w}.$$

It is the set of pairs of d.p.'s of the involutions on the self-conjugate rays through O; those on the single ray are the isolated pair, and those on the self-conjugate plane form the conic.

The two conics ω, δ_2 meet in the two points $z = w = u_2 = 0$; and lie together on the two cones $u_2 - (z \mp w)^2$, whose vertices are D_1, D_2. The generators of these cones each contain two invariant points and one F-point, and are self-conjugate lines, and are the only such besides those through O.

The self-conjugate planes are z and the pencil through OD_1D_2. Any line in z meets its corresponding c' in two invariant points; any line in a plane of the pencil meets its c' in a conjugate pair. Thus the complex associated with this I_{2-2} (p. 179) is the set of lines meeting OD_1D_2.

25. Second type of involution: quadric inversion (130).

If every ray through O is self-corresponding, the equations of I_{2-2} can be

$$x' : y' : z' : w' = xw : yw : zw : q(x, y, z),$$
$$x : y : z : w = x'w' : y'w' : z'w' : q(x', y', z'),$$

where q is any quadratic function. The invariant system is the quadric $\Delta_2 \equiv q - w^2$. Each ray through O bears an involution whose d.p.'s are the intersections with Δ_2. The P-plane w is the polar plane of O with regard to Δ_2, and ω is its intersection with Δ_2.

Every pair of conjugate points lie on a ray through O, and are conjugate with regard to Δ_2. These two properties completely define the transformation, which is central, and the associated complex reduces to the star (O). This is called *quadric inversion*, and is the natural extension of general plane inversion.

If Δ_2 is a sphere and O its centre, we have ordinary inversion with regard to Δ_2; the P-plane is at ∞, and ω is the absolute conic (16, 171, 172, 212, 241, 242, 277, 377, 388, 389). It effects a stereographic projection upon a plane, of any sphere with a radius of Δ_2 as diameter.

CHAPTER XI

POSTULATION AND EQUIVALENCE

1. Classification of F-elements.

We have to consider the different kinds of *F*-elements that may be used to define a homaloidal web. For each element we have first to find the postulation *P*, which is the number of independent conditions that it presents to the parameters of the family; this will be calculated on the understanding, first, that the degree of the family is high enough to avoid any special circumstances; and next, that any other *F*-elements, which may be present, have no special relations to the one under consideration. We have also, with the same assumptions, to find the equivalence *E*, which is the number of variable intersections of three general members of the web, which are absorbed by the *F*-element considered.

There are three principles of cross classification, giving eight kinds of *F*-element to be considered:

(i) the elements may be points or curves, and curves are subdivided according to degree, rank and so on;

(ii) they may be simple or multiple;

(iii) the conditions may be of incidence or of contact.

The four types of contact are treated in the next chapter; here we deal with simple and multiple points and curves of ordinary incidence. There are many difficulties, both analytical and geometrical, about the more complicated and general cases, and we shall therefore treat the simplest cases at some length.

2. Simple point, $P=1$, $E=1$.

The most elementary condition possible is that the homaloids shall pass through a fixed point O. If other *F*-elements are present, we assume them quite separate from O; then the variable intersection c of two general homaloids ϕ, ψ passes once through O. There is a single condition for the parameters, found by substituting the coordinates of O in the general equation $\phi=0$; if O is taken as origin of Cartesian coordinates, the constant term in ϕ must vanish: the postulation is 1.

The equivalence is also 1; for c has a certain number of free intersections with a third general homaloid χ; since χ as well as c passes through O, one intersection is fixed there, and the number of free intersections is reduced by 1. The tangent planes to ϕ, ψ at O meet in a line l, which is the tangent line to c; the tangent plane to χ passes through O but not through l.

3. Double point, $\mathbf{P}=4$, $\mathbf{E}=8$.

Let the origin O of Cartesian coordinates be an isolated d.p. on (ϕ). This means that there is no definite tangent plane, and that a general line through O meets ϕ in two points at O and in $n-2$ points elsewhere. Since O lies on ϕ, the constant term vanishes; if there were any linear terms, they would give a definite tangent plane; hence the coefficients of $1, x, y, z$ all vanish, giving four conditions. We assume

$$\phi \equiv u_2 + u_3 + \dots + u_n,$$

$$\psi \equiv v_2 + v_3 + \dots + v_n,$$

where u_α, v_α are homogeneous functions of x, y, z, of degree α.

To find E, consider the total intersection of ϕ, ψ, in the neighbourhood of O, given to a first approximation by the terms of lowest degree u_2, v_2. These represent two general quadric cones, the tangent cones at O to ϕ, ψ, and their intersection consists of four generators. The intersection has therefore four branches through O, touching these four lines, and these are all branches of c, for we assume that no F-curve passes through O. The four tangent lines to c are in general directions, and do not touch χ, but each branch meets it in two points at O; altogether the number of intersections of c, χ, that is, of ϕ, ψ, χ, absorbed at O is $E=4\times 2=8$.

4. Multiple point, $\mathbf{P}=\frac{1}{6}i(i+1)(i+2)$, $\mathbf{E}=i^3$.

If O is i-fold on (ϕ), a general line through O meets ϕ in i points at O and in $n-i$ points elsewhere. Put $x:y:z=a:b:c$, and eliminate y, z from $\phi=0$; the resulting equation for x has i zero and $n-i$ finite roots. All the terms of degree $\leqq i-1$ in x are absent for all values of $a:b:c$, and these arise from all the terms of degree $\leqq i-1$ in x, y, z in ϕ. The coefficients of all these terms vanish, and

$$\phi \equiv u_i + u_{i+1} + \dots + u_n.$$

The number of conditions is the number of vanishing coefficients, that is, of terms in $u_0, u_1 \dots u_{i-1}$. But u_α has $\frac{1}{2}(\alpha+1)(\alpha+2)$ terms, so

$$P=\sum_{\alpha=0}^{i-1} \tfrac{1}{2}(\alpha+1)(\alpha+2)=\tfrac{1}{6}i(i+1)(i+2).$$

This is the number of terms in a homogeneous function of four variables of degree $i-1$.

The intersection of ϕ, ψ in the neighbourhood of O approximates to that of their tangent cones u_i, v_i, each of degree i, which consists of i^2 generators; hence c has i^2 branches through O, one touching each of these lines. Since u_i, v_i are general functions, the branches of c are in general directions, and each meets χ in i points; the number of intersections of c, χ absorbed at O is $E=i^3$.

5. Set of distinct points.

If there are two or more distinct points $O_1, O_2 \ldots$ of multiplicities $i_1, i_2 \ldots$, we simply add their postulations and equivalences, but only if the degree of (ϕ) is high enough.

In homogeneous coordinates, let O_1 be $(1, 0, 0, 0)$ and O_2, $(0, 1, 0, 0)$; then the equations of $l \equiv O_1O_2$ are $z = w = 0$. The i_1-fold point O_1 requires the absence of all terms of degree $\geqq n - i_1 + 1$ in x, and O_2 requires the absence of all terms of degree $\geqq n - i_2 + 1$ in y. Now if $i_1 + i_2 > n + 1$, these sets of terms overlap, and the total postulation $<$ the sum of its values for the two points separately; for example, both require the absence of

$$x^{n-i_1+1}\, y^{i_1-1}.$$

If $i_1 + i_2 > n$, since the highest power in x, y jointly is $x^{n-i_1}\, y^{n-i_2}$, of degree $< n$, this and every other term has some power of z or w as a factor; if $z = w = 0$, every term vanishes for all values of $x : y$. Hence ϕ contains every point of l, which meets ϕ in $i_1 + i_2 > n$ points at O_1, O_2, and therefore lies wholly on the surface. Thus l is an F-line, of multiplicity $i_1 + i_2 - n$, and this must be considered in calculating E, which is quite different from the sum of its values for the separate points.

In the same way, if there are several points ΣO, lying on a curve k_m, with $\Sigma i > mn$, then ϕ contains k as a necessary consequence of its passages through ΣO, and E may be either $>$ or $<$ the sum of its values for the separate points. If $\Sigma i > mn + 1$, some of the conditions for the points cease to be independent, and $P <$ the sum of its values for the separate points.

6. Adjacent simple points.

When two isolated simple F-points approach each other and ultimately coincide, we do not have a case of contact, as the term is used in the next chapter. Here, the homaloids only have a common tangent line, but there, a common tangent plane.

Now at every simple F-point O, the two surfaces ϕ, ψ have a common tangent line, which is the line of intersection of their different tangent planes, and is the tangent line to c; they have already a second common point O_1, adjacent to O, namely the adjacent point of c. To regard O_1 as fixed does not alter the relative position of ϕ, ψ, and does not make them have contact at O. The speciality of the intersection only begins when we introduce the third surface χ; for the three tangent planes all have the line OO_1 in common, while in general they only have the point O.

We may think of O_1 as at first distinct from O; then c passes through both; when O_1 approaches and ultimately coincides with O, then c touches the limiting position of OO_1, which is a tangent line to every ϕ. Hence c touches χ, and meets it in two points at O, so that $E = 2$, as for two distinct points. Also $P = 2$; one condition is that O lies on ϕ, and the other is that the tangent plane contains a fixed line. If the equations of OO_1 are $y=z=0$, then the constant term and the term in x vanish, and we may take

$$\phi \equiv ay + bz + u_2 + \ldots + u_n.$$

If more than two simple points $O, O_1 \ldots O_{\beta-1}$ come together, there are two cases, according to the way in which they approach one another. There may exist a curve passing through them all, which has a limiting position k free from singularity at O. A necessary condition is that the radius of the circle OO_1O_2 should have a finite limit, for this is the radius of curvature of k. If this holds, and, for $\beta > 3$, further conditions for the existence of k, we have the limiting case which is for our present purpose strictly analogous to the case of which it is the limit; both P and E are the same as for β distinct points. There is only one branch of c through O, which has β-point contact with k. Regarded as a common point of ϕ, ψ, the point O is in no way special: the speciality is that χ also has β-point contact with k or c.

But the points may come together in such a way that k must be singular at O_1 as, for example, if O_1, O_2 approach O in perpendicular directions. Then c is also singular, and if ϕ, ψ are not singular also, they touch one another, for their tangent planes both contain the two or more tangent lines to c, and therefore coincide; O is at least a point of contact of (ϕ).

7. Simple points adjacent to a multiple point.

If a distinct simple F-point O_1 approaches a d.p. O, the tangent cone u_2 at O acquires a fixed generator, giving one condition more than the four presented by O. This line is one of the four common generators of the tangent cones to ϕ, ψ, and it also lies on the tangent cone to χ; the branch of c which touches it meets χ in three points instead of two, and E is increased by 1.

If β simple points $O_1 \ldots O_\beta$ approach O, and there exists a curve through $OO_1 \ldots O_\beta$ whose limit is not singular, then u_2 has one fixed generator, and there are also $\beta - 1$ other conditions involving the terms of higher degree. One branch of c meets χ in $\beta + 2$ points at O, and again E is increased by 1 for each simple point.

If O_1, O_2 approach O in different directions, then u_2 has two fixed generators, and two branches of c touch the tangent cone to χ, giving again two fresh conditions and two fresh intersections absorbed,

$$P = 4 + 2 = 6, \quad E = 8 + 2 = 10.$$

Similarly for three and four simple points, $P = 7$ and 8, $E = 11$ and 12 respectively. But if there are five points that approach O in general directions, the cone u_2 is entirely fixed, and coincides with the tangent cones to ψ and χ. Then O is a d.p. of contact, and it is found that though $P = 9$, as we might expect, E leaps to 18 (p. 235).

If more than five points approach O, in directions not lying on a quadric cone, then O becomes a triple point at least, and it is no longer convenient to regard it as arising from a d.p.

Whenever one branch of c has high contact with (ϕ), some of the conditions involve the terms of high degree, and unless n is great enough, both P and E are modified, and the nature of the intersection alters, an F-curve appearing which breaks away from c, reducing its degree, and c itself may not pass through O at all. For example, let three F-points of a quadric family come together at O, along a curve whose limit is a line l; then (ϕ) acquires l as an F-line, and c becomes a twisted cubic not passing through O. For other details see 36.

When other multiple points come to coincide with O, the tangent cone acquires multiple edges. The point may become one of higher singularity, and its properties are best treated by the methods of resolution described in Chapter XVI.

8. Simple line, $\mathbf{P} = n + 1$, $\mathbf{E} = 3n - 2$.

Let (ϕ) contain the whole of a fixed line l. In homogeneous coordinates, let the equations of l be $z = w = 0$. Then $\phi = 0$ when z and $w = 0$, for all values of x/y; every term of ϕ has either z or w as a factor, and all the terms vanish which involve x, y only. There are $n + 1$ of these, x^n, $x^{n-1}y \ldots y^n$, and $P = n + 1$.

It is convenient to arrange ϕ according to the degree of the terms in z, w jointly; the lowest terms are linear:

$$\phi \equiv (zu_{n-1,1} + wu_{n-1,2}) + (z, w \overset{2}{\big)\big(} x, y, z, w)^{n-2},$$

where $u_{n-1,\alpha}$ is a homogeneous function of x, y only.

The total intersection of ϕ, ψ breaks up into the F-line l and a variable residual c_{n^2-1} of degree $n^2 - 1$. The free intersections of ϕ, ψ, χ are among the $n(n^2 - 1)$ intersections of c, χ, but not all of these are freely variable. For c meets l in a certain number d of points, which, though variable on l, are restricted to lie on it, and are F-points, not general points of χ. The number of intersections of c, χ which are not fundamental is $n(n^2 - 1) - d$, and E is the number by which this falls short of the maximum n^3,

$$E = n^3 - \{n(n^2 - 1) - d\} = n + d.$$

We shall now show that $d = 2n - 2$, giving two distinct proofs of this important result, which can be generalized in different cases.

(i) An intersection of c, l is a d.p. of the total intersection of ϕ, ψ, and therefore a point of contact of the two surfaces. We have to show that the tangent planes to ϕ, ψ coincide at $2n - 2$ points of l. The tangent plane to ϕ at any point P is

$$t \equiv x\left[\frac{\partial\phi}{\partial x}\right] + y\left[\frac{\partial\phi}{\partial y}\right] + z\left[\frac{\partial\phi}{\partial z}\right] + w\left[\frac{\partial\phi}{\partial w}\right],$$

where in the square brackets the coordinates of P are substituted. At a point Q $(x, y, 0, 0)$ of l, only the first group of terms in ϕ contributes to t, and we may without ambiguity write the tangent plane

$$t \equiv zu_{n-1,1} + wu_{n-1,2},$$

it being understood that z, w are current coordinates, and x, y, which enter into $u_{n-1,\alpha}$, are the coordinates of the point of contact Q.

Let the tangent plane to ψ at Q be $zv_{n-1,1}+wv_{n-1,2}$. This is the same as t, provided

$$u_{n-1,1}/v_{n-1,1}=u_{n-1,2}/v_{n-1,2}, \quad \text{that is,} \quad u_{n-1,1}v_{n-1,2}-u_{n-1,2}v_{n-1,1}=0.$$

This is an equation in x/y of degree $2n-2$. It may be interpreted as a set of $2n-2$ planes, through the opposite edge, $x=y=0$, of the tetrahedron of reference, meeting l in the d points at which ϕ, ψ have the same tangent plane; hence

$$d=2n-2.$$

(ii) The second proof is more geometrical. Consider the system of intersections of ϕ, ψ and a general plane p through l. We may treat this as the intersection of p with the total intersection of ϕ, ψ, which breaks up into l and c_{n^2-1}; the system consists of l and the n^2-1 intersections of c, p, of which d lie on l; there are therefore n^2-1-d common points of ϕ, ψ, p which are free, not lying on the F-line.

But we may also treat the system as the intersection of ψ with that of ϕ, p, which is a plane curve of degree n, breaking up into l and a residual of degree $n-1$ meeting l in $n-1$ points and meeting ψ in $n(n-1)$ points of which therefore $n-1$ lie on l and the remaining $(n-1)^2$ are free. Equate these two expressions for the number of free intersections of ϕ, ψ, p:

$$n^2-1-d=(n-1)^2, \quad d=2n-2.$$

Thus for a simple line, $P=n+1$, $E=n+d=3n-2$. Unlike the expressions for F-points, both of these depend on the degree of the family, and not only on the nature of the F-element.

9. Reduction of P and E for incidence.

In the last chapter, p. 182, we found, for an F-conic on a quadric family, $P=5$, $E=6$; these are unaltered if the conic breaks up into two intersecting lines. But two skew lines l_1, l_2 have $P=6$, $E=8$, double the values for a single line; we can verify the value of P from the general quadric containing l_1, $z=w=0$ and l_2, $x=y=0$,

$$\phi \equiv c_1xz+c_2xw+c_3yz+c_4yw,$$

containing four arbitrary constants; the other six terms must vanish. Thus when the two lines intersect, their total postulation, as compared with that of two skew lines, is reduced by 1, and their total equivalence by 2.

Now to make ϕ contain l_1 and l_2, it is sufficient to make it pass through three points of each line. If l_1, l_2 are skew, these two sets of three points can have no point in common, and we must make ϕ pass through six points. But if l_1, l_2 intersect, their common point can serve in both sets, and we need only make ϕ pass through five points altogether; the total postulation is reduced by 1.

Again, each F-line has two intersections with its total residual in the intersection of ϕ, ψ. If the lines do not meet, each of these intersections is with c, and is one of the intersections of c, χ absorbed by the F-lines. But if l_1, l_2 meet, then of the two intersections of l_1 with its total residual, one is with l_2 and only one with c, one intersection fewer is absorbed by l_1, and also one fewer by l_2; the total equivalence is reduced by 2.

As we shall see later (p. 217) similar considerations hold for curves of any degree on any surfaces; for each intersection of two simple F-curves, P is reduced by 1 and E by 2.

10. Properties of curves.

A curve of higher degree can be specified by means of equations which represent surfaces passing through it. A distinction arises according to the number of equations required. If two are enough, the curve is the total intersection of the two surfaces; if more are required, it is the partial intersection of any two of them. Lines, conics and all other plane curves are total intersections; a twisted cubic is not, and a twisted quartic may or may not be one.

A partial intersection can be specified by three or more surfaces all passing through it, but not all through any other curve; or by specifying another curve with which it makes up a total intersection, so that the pair can be given by two equations.

We require two properties of curves (Vol. I, pp. 355–358 of 338). Let k be a curve of degree m, rank r, with δ actual and h apparent d.p.'s, and no other singularity. Then $r = m(m-1) - 2h - 2\delta$, and we write

$$\rho \equiv m(m+1) - 2h \equiv r + 2m + 2\delta.$$

(i) If k is the total intersection of two surfaces f_μ, f_ν,

$$m = \mu\nu, \quad h = \tfrac{1}{2}\mu\nu(\mu-1)(\nu-1), \quad \rho = \mu\nu(\mu+\nu).$$

(ii) If k is the partial intersection of f_μ, f_ν, it meets its total residual in d points, where

$$d = m(\mu+\nu) - \rho.$$

11. Simple curve, $\mathbf{P} = m(n+2) - \frac{1}{2}\rho$, $\mathbf{E} = 3mn - \rho$ (60, 294).

Let ω be an F-curve on a homaloidal family, and first let it be the total intersection of two auxiliary surfaces f_μ, f_ν. These must be given apart from the homaloids, since they serve to define ω, which itself enters into the definition of (ϕ); and there need be no connection between the degrees n, μ, ν. Then we obtain P as follows.

The general surface ϕ_n containing the total intersection of f_μ, f_ν is

$$\phi \equiv f_\mu \phi_{n-\mu} + f_\nu \phi_{n-\nu},$$

where the coefficients of $\phi_{n-\mu}$, $\phi_{n-\nu}$ are arbitrary, but those of f_μ, f_ν are assigned. This way of writing ϕ is not however unique; if n is large enough, we have identically

$$\phi \equiv f_\mu (\phi_{n-\mu} + f_\nu \phi_{n-\mu-\nu}) + f_\nu (\phi_{n-\nu} - f_\mu \phi_{n-\mu-\nu}),$$

where the coefficients of $\phi_{n-\mu-\nu}$ are also arbitrary, and can be used to give assigned values to certain of the coefficients in $\phi_{n-\nu}$, $\phi_{n-\mu}$, and so to reduce the number of parameters in ϕ.

The number of coefficients in a general homogeneous function of degree n in four variables is $\binom{n+3}{3} \equiv \frac{1}{6}(n^3 + 6n^2 + 11n + 6)$. Hence the number of parameters in ϕ is

$$\binom{n-\mu+3}{3} + \binom{n-\nu+3}{3} - \binom{n-\mu-\nu+3}{3},$$

and the postulation of ω is

$$P = \binom{n+3}{3} - \left\{\binom{n-\mu+3}{3} + \binom{n-\nu+3}{3} - \binom{n-\mu-\nu+3}{3}\right\},$$

which reduces to $\mu\nu \{n + 2 - \frac{1}{2}(\mu + \nu)\}$, or by (i),

$$P = m(n+2) - \tfrac{1}{2}\rho. \quad \text{.........................}(1)$$

We have assumed $n - \mu - \nu \geqq 0$, so that $\phi_{n-\mu-\nu}$ exists. If this is not so, the corresponding term must be omitted, and the correction $-\binom{n-\mu-\nu+3}{3}$ applied to (1). This term vanishes if $n - \mu - \nu = -1, -2$ or -3, and (1) is correct in these cases also; but if $n < \mu + \nu - 3$, the correction is positive and P is greater than is given by (1).

We have also assumed $n \geqq \mu$ and $n \geqq \nu$; let $\mu \leqq \nu$. If $\mu \leqq n < \nu$, then $\phi_{n-\nu}$ does not exist, and ϕ must break up into $f_\mu \phi_{n-\mu}$; the number of arbitrary coefficients is $\binom{n-\mu-3}{3}$, and

$$P = \binom{n+3}{3} - \binom{n-\mu+3}{3} = \tfrac{1}{6}\mu(3n^2 - 3n\mu + 12n + \mu^2 - 6\mu + 11).$$

In particular, if $n = \mu$, then $\phi \equiv f_\mu \phi_0$, and the surface is exactly determined, $P = \binom{n+3}{3} - 1$. If $n < \mu$, then ϕ cannot exist, all its coefficients vanish and $P = \binom{n+3}{3}$.

We can find E for any curve, whether it is a total or a partial intersection (Vol. I, p. 370 of 338). For c is of degree $n^2 - m$ and meets its residual ω in d points where $d = 2mn - \rho$ by (ii). Then c meets χ in $n(n^2 - m)$ points, of which d are points of ω and $n(n^2 - m) - d$ are free; hence

$$E = n^3 - \{n(n^2 - m) - d\} = mn + d = 3mn - \rho.$$

We now have the formulae

$$P = m(n+2) - \tfrac{1}{2}\rho, \quad E = 3mn - \rho,$$

the first proved so far only for the case in which ω is a total intersection, and the second without this restriction. Now let ω, being itself a total intersection, break up into two components ω_1, ω_2, of degrees m_1, m_2, etc., having d actual and therefore m_1m_2-d apparent intersections. For the compound curve we have

$$m=m_1+m_2, \qquad h=h_1+h_2+(m_1m_2-d),$$

whence

$$\rho=\rho_1+\rho_2+2d\,;$$

$$P=P_1+P_2-d, \quad E=E_1+E_2-2d,$$

where E_1, E_2 are the values of E for the components, but for the present P_1 is defined as the same function of the characteristics of ω_1 that P is of those of ω, the postulation not having yet been found for a partial intersection.

In the case when ω_1, ω_2 are themselves both total intersections of other auxiliary surfaces, we know that P_1, P_2 are their postulations, and then the last equation proves that the reduction of P, compared with the sum of its values for the components, is 1 for each common point O. This has been proved independently when ω_1, ω_2 are lines (p. 214); it cannot depend on the natures of the curves as a whole, but only on their behaviour in the neighbourhood of O; it is therefore always true (288). It expresses that the single condition of passing through O can be taken as one of the independent conditions presented to (ϕ) by each F-curve through O.

Then the postulations of ω_1, ω_2 are each the same function of the characteristics of its own curve, and their sum is $P+d$; it follows that they are P_1 and P_2, and the postulation formula is proved for a partial as well as a total intersection.

For example, let ω break up into m lines, with h apparent and therefore $\frac{1}{2}m(m-1)-h$ actual intersections: we have

$$P= m(n+1) - \{\tfrac{1}{2}m(m-1)-h\} = m(n+2)-\tfrac{1}{2}\rho,$$

$$E=m(3n-2)-2\{\tfrac{1}{2}m(m-1)-h\}=3mn-\rho.$$

The reduction of E can also be interpreted geometrically. There are a certain number of intersections of ω_1, with its total residual in the intersection of ϕ, ψ, namely d with ω_2 and the rest with c. For each intersection O of ω_1, ω_2 we lose one intersection of ω_1, c, that is, an intersection of c, χ absorbed by the first F-curve, and also one absorbed by the second; O reduces E by 2.

It has been assumed that ω has no singularities but ordinary d.p.'s. If it has higher singularities, we must find how these affect ρ and d. If ω has a multiple point O at which the branches do not all touch one plane, then O is necessarily multiple on ϕ, and it is best to treat O as a separate F-element with which ω is incident (p. 223).

12. Example: twisted cubic, $\mathbf{P}=3n+1$, $\mathbf{E}=9n-10$.

The simplest curve which is not a total intersection is a twisted cubic, for which $m=3$, $h=1$, $\rho=10$. We shall prove the results for this curve independently, using a parametric representation.

By a proper choice of the frame of reference, the coordinates of any point on the cubic are given in terms of a parameter τ by

$$x:y:z:w=1:\tau:\tau^2:\tau^3.$$

The conditions that ϕ should contain ω are that, after substitution for x, y, z, w in terms of τ, we have $\phi\equiv 0$ for all values of τ. Since the result of substitution is of degree $3n$ in τ, there are $3n+1$ conditions, namely that the terms in $\tau^{3n}, \tau^{3n-1} \dots \tau, 1$ all vanish. Thus

$$P=3n+1=3(n+2)-\tfrac{1}{2}.10.$$

We next verify that ω meets c in $6n-10$ points. The tangent plane to ϕ at the point τ is found by substituting for x, y, z, w in the square brackets in

$$t\equiv x\left[\frac{\partial\phi}{\partial x}\right]+y\left[\frac{\partial\phi}{\partial y}\right]+z\left[\frac{\partial\phi}{\partial z}\right]+w\left[\frac{\partial\phi}{\partial w}\right].$$

The coefficients become of degree $3n-3$ in τ. Now this plane contains the tangent line to ω, which is most easily written down as the intersection of the two tangent planes to two quadric cones through ω,

$$xz-y^2, \quad yw-z^2,$$

namely the planes

$$\tau^2x-2\tau y+z, \quad \tau^2y-2\tau z+w.$$

Hence t is the sum of multiples of these two expressions, the coefficients being two functions of τ of degree $3n-5$. The tangent plane to ψ is of the same form. At the d intersections of ω, c, where the tangent planes to ϕ, ψ coincide, we have an equation for τ of degree $2(3n-5)$. Hence $d=6n-10$, and

$$E=3n+d=9n-10=3.3n-10.$$

13. Double line, $\mathbf{P}=3n+1$, $\mathbf{E}=12n-16$.

The simplest example of a multiple curve is a double line upon a cubic family. Let the line l, $z=w=0$, be double on (ϕ_3). If a general cubic function is arranged according to powers of z, w, it has the form

$$\phi\equiv u_3+(zu_{2,1}+wu_{2,2})+(z^2u_{1,1}+zwu_{1,2}+w^2u_{1,3})+(z, w)^3,$$

where the u are functions of x, y only. Since l lies on ϕ, this is satisfied when $z=w=0$ for all values of x/y; the four terms of u_3 vanish. If there were no further conditions, the tangent plane to ϕ at any point O $(x, y, 0, 0)$ of l would be $zu_{2,1}+wu_{2,2}$, if in this x, y are constant and z, w variable. Since every point of l is double on ϕ, this tangent plane does not exist for any value of x/y, and the six terms of $u_{2,1}$, $u_{2,2}$ all vanish. Hence $P=10$.

At O, the first approximation to ϕ is given by the lowest surviving terms in z, w, that is, by $z^2u_{1,1}+zwu_{1,2}+w^2u_{1,3}$, which is a pair of definite tangent planes, each passing through l. At a point of intersection of l, c, one of these planes contains the tangent line to c, and coincides with one of the tangent planes to ψ, given by $z^2v_{1,1}+zwv_{1,2}+w^2v_{1,3}$ say. The values of x/y for which this happens are found by eliminating z/w between the equations of the two pairs of planes. The result is

$$(u_{1,1}v_{1,3}-u_{1,3}v_{1,1})^2=(u_{1,1}v_{1,2}-u_{1,2}v_{1,1})(u_{1,2}v_{1,3}-u_{1,3}v_{1,2}),$$

of degree 4 in x, y. Thus l meets c four times.

The common double line counts four times in the intersection of ϕ, ψ, for each sheet of ϕ meets each sheet of ψ along l. Hence c is of degree $3^2-4=5$; it meets χ in $5\times 3=15$ points; of these there are four lying on l, each a d.p. of χ and absorbing two intersections, and $15-4\times 2=7$ others, which are the free intersections of ϕ, ψ, χ. Hence $E=3^3-7=20$.

If l is double on a general family (ϕ_n),

$$\phi\equiv z^2u_{n-2,1}+zwu_{n-2,2}+w^2u_{n-2,3}+(z,\,w\overset{3}{\between}x,\,y,\,z,\,w)^{n-3};$$

the terms $u_n+zu_{n-1,1}+wu_{n-1,2}$ have vanished, and $P=n+1+2n=3n+1$. In the same way as before we find $4(n-2)$ points where one sheet of ϕ touches one sheet of ψ, which are intersections of c, l, at each of which c meets χ in two points. Now c is of degree n^2-4, and it meets χ in $4(n-2)\times 2$ points on l and $n(n^2-4)-8(n-2)$ other points, which are the free intersections of ϕ, ψ, χ. Hence

$$E=n^3-(n^3-12n+16)=12n-16.$$

14. Multiple line, $\mathbf{P}=\frac{1}{6}i(i+1)(3n-2i+5)$, $\mathbf{E}=i^2(3n-2i)$ (62).

In the same way, if l is i-fold on (ϕ_n)

$$\phi\equiv(z,\,w\overset{i}{\between}x,\,y)^{n-i}+(z,\,w\overset{i+1}{\between}x,\,y,\,z,\,w)^{n-i-1},$$

and the i tangent planes at $(x, y, 0, 0)$ are given by the first group of terms. All the terms of degree $\leqq i-1$ in z, w vanish, and

$$P=1(n+1)+2n+3(n-1)+\ldots+i(n-i+2)=\tfrac{1}{6}i(i+1)(3n-2i+5).$$

To find E, we must determine as before the number of intersections of l, c, where one of the i tangent planes $(z,\,w\overset{i}{\between}x,\,y)^{n-i}$ to ϕ coincides with one of the corresponding tangent planes to ψ. We have to eliminate z/w between two equations of degree i in z, w, the coefficients all being functions of x, y of degree $n-i$. The result is a determinant of $2i$ rows and columns, and an equation in x/y of degree $2i(n-i)$. This is therefore the number of intersections of l, c, and each is an i-fold point on χ. Now c is of degree n^2-i^2; the number of free intersections is

$$n(n^2-i^2)-2i^2(n-i)=n^3-i^2(3n-2i),$$

and

$$E=i^2(3n-2i).$$

15. Multiple curve, $\mathbf{P} = \frac{1}{12} i (i+1) \{6m(n+2) - (2i+1)\rho\}$, $\mathbf{E} = i^2(3mn - i\rho)$ (60, 288).

First, let the i-fold F-curve ω be the total intersection of f_μ, f_ν. The general form of ϕ_n is the sum of multiples of $f_\mu{}^\alpha f_\nu{}^{i-\alpha}$:

$$\phi \equiv f_\mu{}^i \phi_{n-i\mu} + \ldots + f_\mu{}^\alpha f_\nu{}^{i-\alpha} \phi_{n-\alpha\mu-(i-\alpha)\nu} + \ldots + f_\nu{}^i \phi_{n-i\nu}.$$

As before, this form is not unique, but may be written

$$\begin{aligned}\phi \equiv f_\mu{}^i (\phi_{n-i\mu} + f_\nu \phi_{n-i\mu-\nu}) &+ f_\mu{}^{i-1} f_\nu (\phi_{n-(i-1)\mu-\nu} - f_\mu \phi_{n-i\mu-\nu} + f_\nu \phi_{n-(i-1)\mu-2\nu}) \\ &+ \ldots + f_\mu{}^\alpha f_\nu{}^{i-\alpha} (\phi_{n-\alpha\mu-(i-\alpha)\nu} - f_\mu \phi_{n-(\alpha+1)\mu-(i-\alpha)\nu} + f_\nu \phi_{n-\alpha\mu-(i-\alpha+1)\nu}) \\ &+ \ldots + f_\nu{}^i (\phi_{n-i\nu} - f_\mu \phi_{n-\mu-i\nu}).\end{aligned}$$

The coefficients of $\phi_{n-i\mu} \ldots \phi_{n-i\nu}$ are arbitrary, but some of them can be given assigned values by properly choosing the other arbitrary functions $\phi_{n-i\mu-\nu} \ldots \phi_{n-\mu-i\nu}$. Hence

$$P = \binom{n+3}{3} - \sum_{\alpha=0}^{i} \binom{n - \alpha\mu - (i-\alpha)\nu + 3}{3} + \sum_{\alpha=0}^{i-1} \binom{n - (\alpha+1)\mu - (i-\alpha)\nu + 3}{3},$$

which reduces to

$$\begin{aligned} P &= \tfrac{1}{12} i (i+1) \mu\nu \{6(n+2) - (2i+1)(\mu+\nu)\} \\ &= \tfrac{1}{12} i (i+1) \{6m(n+2) - (2i+1)\rho\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)\end{aligned}$$

Let $\mu \leqq \nu$; we have assumed that none of the ϕ's are of negative degree, that is, $n \geqq \mu + i\nu$. If this is not so, some of the second series of ϕ's must be omitted, and (2) may require correction. Since $\binom{2}{3}$, $\binom{1}{3}$, $\binom{0}{3}$ all $= 0$, it remains correct if $n \geqq \mu + i\nu - 3$. If $n < i\nu$, some of the first series of ϕ's do not exist, and ϕ_n breaks up into f_μ, counted a certain number β of times, and a residual $\phi_{n-\beta\mu}$ on which ω is $(i-\beta)$-fold. If $i\mu < n < (i-1)\mu + \nu$, then ϕ_n breaks up into f_μ counted i times and a $\phi_{n-i\mu}$ subject to no restrictions, and $P = \binom{n+3}{3} - \binom{n - i\mu + 3}{3}$. If $n = i\mu$, then ϕ_n is simply $f_\mu{}^i \phi_0$ and the surface is exactly determined; and finally if $n < i\mu$, then ϕ_n does not exist.

To determine the number of intersections of ω, c, consider first the case when ω is i-fold on ϕ and simple on two surfaces ψ_1, χ_1 of degree n (Vol. I, p. 371 of 338). We know (p. 215) that in the intersection of ψ_1, χ_1, the residual k_{n^2-m} meets ω in $2mn - \rho$ points each i-fold on ϕ, which absorb $i(2mn - \rho)$ intersections of k_{n^2-m}, ϕ. The number of free intersections of k_{n^2-m}, ϕ, that is, of ϕ, ψ_1, χ_1, is $n(n^2 - m) - i(2mn - \rho)$.

The total intersection of ϕ, ψ_1 now consists of ω counted i times and a residual k_{n^2-im}; let this meet ω in d_1 points, which are simple on χ_1; the number of free intersections of k_{n^2-im}, χ_1, that is, of ϕ, ψ_1, χ_1, is $n(n^2 - im) - d_1$. Equate these two values:

$$n(n^2 - m) - i(2mn - \rho) = n(n^2 - im) - d_1; \quad d_1 = (i+1)mn - i\rho.$$

Next, let ω be i-fold on ϕ, ψ and simple on χ_1. By the last case, in the intersection of ψ, χ_1, the residual k_{n^2-im} meets ω in d_1 points, each i-fold on

ϕ; also the total intersection of ϕ, ψ consists of ω counted i^2 times and a residual $c_{n^2-i^2m}$. Let this meet ω in d points, which are simple on χ_1. Reasoning as before, we have

$$n(n^2-im)-id_1=n(n^2-i^2m)-d; \quad d=2imn-i^2\rho.$$

Finally, let ω be i-fold on all of ϕ, ψ, χ. By the last case, in the intersection of ϕ, ψ, the residual $c_{n^2-i^2m}$ meets ω in d points, each i-fold on χ; the number of free intersections is

$$n(n^2-i^2m)-id=n^3-3i^2mn+i^3\rho,$$

and E is the excess of n^3 over this number,

$$E=i^2(3mn-i\rho).$$

In this investigation of the equivalence, ω need not be a total intersection.

If a total intersection ω breaks up into any two curves ω_1, ω_2, intersecting in d points, then the total number of apparent d.p.'s of ω is unaltered and we find

$$P=P_1+P_2-\tfrac{1}{6}i(i+1)(2i+1)d, \quad E=E_1+E_2-2i^3d,$$

which suggest that the postulation formula is also valid for any curve, and that each intersection of two i-fold curves reduces P by $\frac{1}{6}i(i+1)(2i+1)$ and E by $2i^3$. This is already proved as regards E; and also as regards P in the case when ω, ω_1, ω_2 are all total intersections of various pairs of auxiliary surfaces. As before, this allows us to assume it generally.

16. Reductions for incidence (288).

To understand the reduction of P due to the intersection of multiple curves, consider the simplest case of two i-fold lines, l_1, $z=w=0$; l_2, $y=w=0$, intersecting at $O\,(1, 0, 0, 0)$. The conditions that l_1 is i-fold are that all terms such as $x^{n-\alpha-\beta-\gamma}y^\alpha z^\beta w^\gamma$ vanish when $\beta+\gamma<i$; the conditions that l_2 is i-fold are that all such terms vanish when $\alpha+\gamma<i$. The reduction in P is the number of conditions common to the two sets, that is, the number of terms for which

$$\alpha<i-\gamma, \quad \beta<i-\gamma, \quad \gamma<i, \quad \alpha+\beta+\gamma\leqq n.$$

The number is

$$\sum_{\gamma=0}^{i-1}(i-\gamma)^2=\tfrac{1}{6}i(i+1)(2i+1),$$

provided $n\geqq 2i-2$. If $n<2i$, every term for which $\gamma=0$ vanishes, and w is a factor; ϕ breaks up into w^{2i-n} and a surface ϕ_{2n-2i} on which l_1, l_2 are $(n-i)$-fold and for which their postulation can be calculated as above.

To understand the reduction in E, we consider that, at O, the surface must have an i-fold point, the tangent cone consisting of i planes through the tangent line to ω_1 and also through the tangent line to ω_2, that is, of i coincident fixed planes. There are i^2 intersections of ω_1 with its total residual absorbed at O; hence owing to the intersection of ω_1, ω_2, we lose i^2 intersections of ω_1, c, and also i^2 of ω_2, c; and c meets χ in i points at each intersection with either ω_1 or ω_2; the total reduction of E is $2i^3$.

In the case of the two lines l_1, l_2, every term of ϕ is of degree $\geqq i$ in z, w and also in y, w. The tangent planes along l_1 are given by (p. 219)

$$(z, w\overset{i}{)(}x, y)^{n-i} \equiv w^i (x, y)^{n-i} + w^{i-1} yz (x, y)^{n-i-1} + \dots + y^i z^i (x, y)^{n-2i}.$$

The intersections of l_1, c, other than O, are found by eliminating z/w between this and the similar expression derived from ψ, and excluding the solution $y=0$ corresponding to O. The determinant that arises is of degree $2ni-3i^2$, instead of $2ni-2i^2$ for a general i-fold line, the factor y^{i^2} being dropped, which shows that i^2 intersections of l_1, c are lost because of O.

These reduction terms are included in the general formulae given below, which allow for incidences of any number of branches of F-curves, of the same or different multiplicities.

In the same way, the actual d.p.'s of one curve ω give reductions of P and E, as compared with their values for a curve of the same degree and rank with no actual d.p.'s. These are already included in the standard formulae, since $\rho \equiv r + 2m + 2\delta$ contains the term 2δ.

17. Incidence of F-curve and F-point.

In the same way, there are reductions when an F-curve passes through an F-point of higher multiplicity.

Consider first the simplest case. Let O (1, 0, 0, 0) be a d.p. of ϕ lying on the simple F-line l, $z=w=0$. As on p. 213 we write

$$\phi \equiv z u_{n-1,1} + w u_{n-1,2} + (z, w\overset{2}{)(}x, y, z, w)^{n-2}.$$

Since O is a d.p., both $u_{n-1,1}$ and $u_{n-1,2}$ vanish when $y=0$; each has y as a factor, and the terms in x^{n-1} vanish; the d.p. imposes these two conditions on (ϕ) instead of four, and the reduction in P is 2. We have

$$\phi \equiv yt + (z, w\overset{2}{)(}x, y, z, w)^{n-2},$$

where $$t \equiv z u_{n-2,1} + w u_{n-2,2}.$$

At any point $(x, y, 0, 0)$ of l other than O, we have $y \neq 0$; the tangent plane is that of yt, from which the finite factor y can be dropped. If x, y are constant, t is the tangent plane to ϕ; if x, y, z, w are all variable, t is a ruled surface of degree $n-1$, with l as a simple line and $x=y=0$ as an $(n-2)$-fold directrix, touching ϕ at every point of l other than O, and having at O an ordinary point and a definite tangent plane, which is the limit of the tangent plane to ϕ at an adjacent point Q of l, as $Q \rightarrow O$. It is a useful convention that ϕ touches any plane through its d.p. (pp. 12, 227); we need not except this point. Thus, if a surface contains a line l and a d.p. lying on it, then along l it touches a surface of lower degree (p. 248).

Every d.p. of ϕ on l causes another linear factor such as y to detach itself, and lowers the degree of t. If ϕ has β d.p.'s on l, it touches a surface of degree $n-\beta$ along l; if $\beta = n-1$, then t is a plane; if $\beta \geqq n$, then l becomes a double line.

The intersections of l, c other than O are the points of contact of t and the corresponding surface derived from ψ; if O is the only d.p. of ϕ on l, their number is $2n-4$, and O absorbs two of the variable intersections of c, l, at each of which c meets χ once. Also of the four branches of the total intersection of ϕ, ψ through O, one is l and only three are c; and l absorbs one of the branches of c through O, each of which meets χ in two points at O. The reduction of E is $2+2=4$.

If we are to think of l as still meeting its residual in $2n-2$ points, we must say that it meets two branches of c at O and misses the third; but to the question: which two? there is no answer.

The same reductions hold when ϕ has a d.p. O lying at an ordinary point of simple F-curve ω of any degree. Of the four conditions necessary to make O a d.p., one is satisfied because O is a point of ω and so lies on ϕ already, and another because the tangent plane to ϕ at O is already restricted to pass through the tangent line to ω, and has only one degree of freedom instead of two.

The number of variable intersections of ω, c is found to be decreased by 2, and the number of branches of c through O is decreased by 1.

By similar reasoning from the case of lines, if two simple F-curves ω_1, ω_2 pass through the d.p. O, the number of variable intersections of c with each of ω_1, ω_2 is decreased by 2, and the number of branches of c through O is also decreased by 2; the reductions are double of their values in the last case. This includes the reductions due to O regarded as a point of intersection of ω_1, ω_2; or as a d.p. of ω_1, if the two fixed branches through O belong to the same F-curve. The passage of (ϕ) through ω_1, ω_2 fixes the tangent plane at O, and, without further condition, O is a point of contact of (ϕ).

If there are three branches of ω through O, there are two distinct cases according as their tangent lines are or are not coplanar. If not, O is a d.p. of (ϕ) without further condition; it absorbs two variable intersections of c with each branch of ω, and only one branch of c passes through O. But if the three branches of ω touch one plane p, the passage of (ϕ) through ω makes O a point of osculation on (ϕ) (p. 234). One further condition makes O a d.p., not a conical point, but a binode, one sheet of the tangent cone being the fixed plane p. This case is fully discussed in the next chapter, p. 238. If four branches of ω pass through O, in general directions, then O is a d.p. and the tangent cone has four fixed generators and one degree of freedom; the tangent cones to ϕ, ψ have for their total intersection the four tangents to ω, and c does not pass through O.

If ω is multiple, the simplest example is when O is a triple point on a double line l, $z=w=0$. The form of ϕ is

$$\phi \equiv yt + (z, w \overset{3}{)\!(} x, y, z, w)^{n-3},$$

where

$$t \equiv (z, w \overset{2}{)\!(} x, y)^{n-3},$$

and t is a surface of degree $n-1$, with l as a double line and O as an ordinary point of it, touching ϕ along l. Through O there pass nine branches of the total intersection of ϕ, ψ, that is, l counted four times and five branches of c, which touch the simple intersection of the two cubic tangent cones with l as a common double edge. Now y^4 can be dropped from the equation for points of contact of ϕ, ψ, and the number of variable intersections of l, c is reduced by 4.

We can treat in the same way any hypermultiple points lying on a multiple line, and we find a generalization of the theorem of p. 222:

If a surface ϕ_n has an i-fold line l and on it certain points of multiplicity $i+\beta$ $(\beta=1, 2 \ldots, n-i-1)$, then along l it touches a surface of degree $n-\Sigma\beta$, on which l is i-fold and these points are ordinary.

More generally, let any number of branches ω_a of multiplicities $i_a < i$ meet at an i-fold point O. There is a certain lowest degree i_0 of a cone which has the tangent line to each ω_a as an i_a-fold edge; and $i \geqq i_0$. If the tangents are in general directions, i_0 is the lowest integer such that

$$\tfrac{1}{2}(i_0+1)(i_0+2) > \Sigma \tfrac{1}{2} i_a (i_a+1),$$

but if the directions are related, i_0 may be lower than this. If no sheet of the cone is completely determined, the number of branches of c through O is $i^2-\Sigma i_a^2$, provided this $\geqq 0$; if this < 0, the whole or part of the cone is fixed, O is a point of contact of some sort, and the conditions are altered (p. 246). For example, if $i_1+i_2>i$, each tangent cone consists partly of the plane containing the first two tangent lines.

The reductions for the general case are included in the next set of formulae.

18. Noether's general formulae for P, E and R (62, 288, 313).

We can now give general formulae for postulation, equivalence and reduction of genus for the whole F-system H, expressing that in consequence of containing H, the general n-ic family is reduced to a web, with one free intersection and of genus 0.

Let an F-curve ω be of multiplicity i, degree m, genus p, rank r, with δ actual and h apparent d.p.'s and no higher singularities, and let

$$\rho \equiv m(m+1)-2h \equiv 2(p-1)+4m+2\delta.$$

Let the $D_{\alpha\beta}$ intersections of branches of different F-curves ω_a, ω_β be all simple, and let $\sum\limits_{\omega,\,\omega}$ extend to these. At such a point, if the multiplicities of the two F-curves are $i_a \geqq i_\beta$, the tangent cone consists of i_a planes through the tangent line to ω_a, of which i_β coincide in the fixed plane also containing the tangent line to ω_β; and similarly at one of the $\Sigma\delta$ d.p.'s of F-curves.

Let H consist of F-curves $\sum\limits_{\omega} \omega_a^{i_a}$ and isolated F-points $\sum\limits_{O} O^i$, and let $\sum\limits_{\omega,\,O}$ refer to the $D_{\omega O}$ incidences of a branch of ω with an F-point O of higher multiplicity.

Let there be no F-points of contact; then no sheet of the tangent cone at any O is fixed by the branches of F-curves through it.

Finally, let n be great enough to avoid the special conditions explained above (pp. 216, 220), that entail other corrections.

Then Noether's formulae are

$$\begin{aligned}P &= \tfrac{1}{6}(n+1)(n+2)(n+3)-4\\ &= \sum_{\omega} \tfrac{1}{6} i_\alpha (i_\alpha+1)\{(3n+6)m_\alpha - \tfrac{1}{2}(2i_\alpha+1)\rho_\alpha\} + \sum_{O} \tfrac{1}{6} i(i+1)(i+2)\\ &\quad - \sum_{\omega,\omega} \tfrac{1}{6} i_\beta (i_\beta+1)(3i_\alpha - i_\beta+1) D_{\alpha\beta} - \sum_{\omega,O} \tfrac{1}{6} i_\alpha (i_\alpha+1)(3i-2i_\alpha+2) D_{\omega O}.\end{aligned}$$

$$\begin{aligned}R &= \tfrac{1}{6}(n-1)(n-2)(n-3)\\ &= \sum_{\omega} \tfrac{1}{6} i_\alpha (i_\alpha-1)\{(3n-6)m_\alpha - \tfrac{1}{2}(2i_\alpha-1)\rho_\alpha\} + \sum_{O} \tfrac{1}{6} i(i-1)(i-2)\\ &\quad - \sum_{\omega,\omega} \tfrac{1}{6} i_\beta (i_\beta-1)(3i_\alpha - i_\beta-1) D_{\alpha\beta} - \sum_{\omega,O} \tfrac{1}{6} i_\alpha (i_\alpha-1)(3i-2i_\alpha-2) D_{\omega O}.\end{aligned}$$

$$\begin{aligned}E &= n^3-1\\ &= \sum_{\omega} (3i_\alpha^2 m_\alpha n - i_\alpha^3 \rho_\alpha) + \sum_{O} i^3 - \sum_{\omega,\omega} i_\beta^2 (3i_\alpha - i_\beta) D_{\alpha\beta} - \sum_{\omega,O} i_\alpha^2 (3i-2i_\alpha) D_{\omega O}. \quad \text{...(1)}\end{aligned}$$

From these, by adding and subtracting, we find

$$2(n^2-1) = \sum_{\omega} \{i_\alpha^2(2m_\alpha - \tfrac{1}{2}\rho_\alpha) + i_\alpha n m_\alpha\} + \sum_{O} i^2 - \sum_{\omega,\omega} i_\alpha i_\beta D_{\alpha\beta} - \sum_{\omega,O} i_\alpha i D_{\omega O} \quad \text{......(2)}$$

$$11(n-1) = \sum_{\omega} i_\alpha (6m_\alpha - \tfrac{1}{2}\rho_\alpha) + 2\sum_{O} i - \sum_{\alpha,\beta} i_\beta D_{\alpha\beta} - 2\sum_{\omega,O} i_\alpha D_{\omega O}, \quad \text{......(3)}$$

and from the degree of the homaloidal curves

$$n^2 - n' = \sum_{\omega} i_\alpha^2 m_\alpha. \quad \text{......(4)}$$

Certain other relations are proved by considering the genus of a plane section of ϕ, the genus of J and the number of d.p.'s in a pencil of homaloids. The following expressions have the same value in each space (310, 311, 313):

$$\sum_{\omega} i_\alpha m_\alpha - 4n, \quad \text{......(5)}$$

$$4\sum_{\omega} m_\alpha + 3\sum_{O} 1 - \sum_{\omega,\omega} D_{\alpha\beta} - 2\sum_{\omega,O} D_{\omega O}, \quad \text{......(6)}$$

$$\sum_{\omega} (p_\alpha - 1) - \sum_{O} 1. \quad \text{......(7)}$$

A combination of these also arises by considering the Jacobian curve of a net of homaloids (312).

We shall prove later (p. 277) that the sums Σp, $\Sigma p'$ are not only equal, but consist of the same terms. It follows from (7) that the total number of F-elements is the same in each space (400).

If this is to hold in more complicated cases, an F-element of contact must be considered to be several distinct F-elements, at least as many as there are distinct types of incidence of the element with c. For example, if O is a binode, there are two types of incidence with the total intersection of ϕ, ψ, four branches touching the fixed and one the variable biplane; thus in general O counts as two F-elements (pp. 239, 275).

As an example, consider the $T_{(2\mu-1)-(2\mu-1)}$ whose F-system in each space contains no isolated points, and consists of a $(\mu-1)$-fold twisted cubic ω and a set of multiple lines, chords of ω, say $\sum_{\beta=1}^{\sigma} l_\beta^{i_\beta}$, where $\mu;\ i_1 \ldots i_\beta \ldots i_\sigma$ is the characteristic of any plane Cremona transformation (273, 274, 391).

For ω, we have $m=3$, $\rho=10$ and for a line, $m=1$, $\rho=2$. Since $i_\beta \leqq \mu-1$, for each intersection, $i_\alpha=\mu-1$, $D_{\alpha\beta}=2$. Then all the above equations are satisfied by virtue of the plane equations of condition,

$$\Sigma i_\beta = 3\mu-3, \quad \Sigma i_\beta{}^2 = \mu^2-1.$$

These formulae may not apply as they stand if there are multiple intersections of two F-curves. For example, let two simple F-lines l_1, l_2 meet a double curve ω_α in the same point O. This absorbs two intersections of ω_α with its residual, and three of each of l_1, l_2, just as if these were skew lines meeting ω_α in distinct points. But if, instead of l_1, l_2, we have two branches of the same F-curve ω_β, then each branch already has one intersection at O, with the other branch, and the incidence with ω_α absorbs only two additional intersections on each branch. In the same way, if the F-curves are multiple, let ω_β have γ branches through O, where $\gamma \leqq i_\alpha$, all touching a plane which does not touch ω_α; this is equivalent to $\gamma-1$ intersections of each branch of ω_β with other branches, and if we include a reduction term for γ intersections of ω_α, ω_β, the formula for E requires the correction $+\gamma(\gamma-1)\, i_\beta{}^2$.

19. Isolated invariant points.

As an example, we write down the number of isolated invariant points of a general transformation in one space. With the notation of p. 175, this is $(n+1-\mu)^3-E$, where E is the equivalence on $(\Omega)_p$ of the F-elements $\Sigma\omega^{i_{\Omega\omega}}$, $\Sigma O^{i_{\Omega O}}$ that lie on (Ω), of the invariant curve δ, and of a plane $k_{n-\mu}$ in p, the latter with ν simple intersections with δ, and m sets of coplanar branches through points of ω_m.

$$\begin{aligned}
E = \sum_{\omega} \{3 i^2{}_{\Omega\omega} m\,(n+1-\mu) - i^3{}_{\Omega\omega}\,\rho\} \\
+ \sum_{O} i^3{}_{\Omega O} - \sum_{\omega,\,\omega} i^2{}_{\Omega\beta}\,(3 i_{\Omega\alpha} - i_{\Omega\beta}) - \sum_{\omega,\,O} i^2{}_{\Omega\omega}\,(3 i_{\Omega O} - 2 i_{\Omega\omega}) \\
+ 3\nu\,(n+1-\mu) - \rho_\delta - \sum_{\delta,\,\omega} (3 i_{\Omega\omega} - 1) - \sum_{\delta,\,O} (3 i_{\Omega O} - 2) - 2\nu \\
+ 3\,(n-\mu)(n+1-\mu) - (n-\mu)(n-\mu+1) \\
- \sum_{\omega} m\,\{(3 i_{\Omega\omega} - 1)\, i_{\Omega\omega} - i_{\Omega\omega}\,(i_{\Omega\omega} - 1)\},
\end{aligned}$$

all sums extending to F-elements or incidences, as indicated by the suffixes, that occur on Ω.

If there is no invariant locus, $\mu=\nu=0$, $i_\Omega = i$, and Ω contains the whole of H. The leading terms are the equivalence of H on (ϕ), that is, n^3-1, and $\Sigma m i_\Omega{}^2 = n^2 - n'$. Then E reduces to

$$(n+1)^3 - (n+n'+2),$$

and the number of isolated invariant points is $n+n'+2$.

CHAPTER XII

CONTACT CONDITIONS

I. POINTS OF TOTAL CONTACT

1. Simple point, simple contact, $\mathbf{P}=3$, $\mathbf{E}=4$.

A point of simple contact of the web (ϕ) is a simple point on the general surface, which has there a fixed tangent plane, the same for all the web. Take the point as the origin $O(0, 0, 0)$ of rectangular Cartesian coordinates, and the fixed tangent plane as z; then

$$\phi \equiv z + u_2 + u_3 + \ldots + u_n,$$

where u_α is a homogeneous function of x, y, z of degree α. The terms in $1, x, y$ are absent, and the number of conditions is $P=3$.

Let another general surface satisfying the same conditions be

$$\psi \equiv z + v_2 + v_3 + \ldots + v_n.$$

The intersection c of ϕ, ψ is that of either with the surface found by subtracting,

$$\Lambda \equiv \psi - \phi \equiv (v_2 - u_2) + (v_3 - u_3) + \ldots + (v_n - u_n),$$

which has a d.p. at O, the tangent cone being $v_2 - u_2$, which is a general variable quadric cone. The first approximation to c near O is

$$z = v_2 - u_2 = 0,$$

consisting of two lines, lying in the fixed plane z, but otherwise in arbitrary directions, varying with the parameters in v_2.

There are therefore two branches of c through O, both touching z, but lying in directions which vary in z, as ϕ, ψ vary in the web (ϕ). Since χ also touches z, each of these branches of c touches χ and meets it in two coincident points at O; the number of intersections of ϕ, ψ, χ absorbed at the point of contact is $E=4$.

For an example, see p. 292.

2. Convention as to exceptional members of a family.

The general ϕ has a definite tangent plane z at O, but there are exceptional members of the linear family (ϕ), such as Λ, for which O is a d.p. It would be inconvenient always to have to except these; as for the plane, we make the convention that Λ is still said to touch z; then if ϕ touches ψ, it touches every member of the pencil $c_1\phi + c_2\psi$. But Λ has no special relation to the plane z, and we must also when necessary regard it as touching every plane through O.

Consider intersections with lines through O. A general line meets ϕ in one point only at O, a tangent line meets it in two points, and z is the locus of tangent lines. A general line meets Λ in two points at O, and a general plane is a locus of such lines.

The section of ϕ by a general plane is a curve having a simple point at O, and the section by the tangent plane has a d.p.; the section of Λ by a general plane has a d.p. We might have chosen to say that the only planes touching Λ at O are those touching its tangent cone; or that Λ is only touched by a plane if this is a component of the tangent cone; but the corresponding sections of Λ have a cusp in the first case and a triple point in the second; and these definitions would not be so convenient as the convention adopted for exceptional members of a linear family having O as a simple point in general.

3. Three adjacent F-points.

An F-point of contact of a homaloidal web of surfaces is analogous to a double F-point of a plane homaloidal net of curves. In both cases, the point can in a certain sense be regarded as the limit of three distinct simple F-points, two of which approach the third in different directions; the correct value of P is obtained by summing its values for the distinct points, but E cannot be so obtained. We saw on p. 212 that there is a point of contact in the limit; we have to show how a fourth, variable intersection is absorbed.

A point of contact of (ϕ) is a common d.p. of the homaloidal twisted curves (c); the net of (c) lying on any one homaloid ϕ is analogous to a plane homaloidal net (k), and is represented by one in any plane representation of ϕ such as is used in Cremona's theorem (p. 167). If two F-points O_1, O_2 of (ϕ) approach a third O in different directions, the same is true of their images, which are F-points of (k). Then we know that (k) acquires a double F-point, and also that a fourth intersection of any two k's, variable before the limit, comes to coincide with the three fixed points (p. 13). This is the image of a fourth intersection of ϕ, ψ, χ, variable before the limit, which $\rightarrow O$ as $O_1, O_2 \rightarrow O$, and so becomes fixed in the limit, when O absorbs four intersections.

The directions of approach of O_1, O_2 disappear entirely from the final form of (ϕ).

4. F-point adjacent to F-curve.

Things are different in another case when a point of contact arises as a limit, namely when a distinct simple F-point $O_1 \rightarrow$ a point O of a simple F-curve ω; then both P and E are the same for the limit as for the system of distinct elements.

In the limit, (ϕ) touches at O the limit of the plane through O_1 and l the tangent line to ω at O. Before the limit, the tangent plane at O is already restricted to pass through l, and has only one degree of freedom; to fix it entirely gives only one fresh condition, which replaces the one condition of passage through O_1; thus P is unaltered.

In the limit, the *F*-point of contact O is a d.p. of the total intersection of ϕ, ψ, one branch being ω and one c; since c touches χ, two intersections of c, χ are absorbed at O. Before the limit, one branch of c passes through O, and one of its variable intersections with ω is an *F*-point Q near O, which ultimately becomes fixed at O instead of variable on ω. Now O_1 and Q are the only intersections of c, χ near O, and neither is free, one being absorbed by the *F*-point and the other by the *F*-curve; no free intersection $\rightarrow O$, and E is unaltered.

A point of intersection O of two simple *F*-curves, or a d.p. of one, is also an *F*-point of contact, but is not in general to be reckoned as a separate *F*-element. The contact is a necessary consequence of the passage of ϕ through the two branches of ω, and does not present any new condition; and no branch of c passes through O; the contact does not affect either P or E.

5. Simple point, s-point contact, $\mathbf{P} = \frac{1}{2}s(s+1)$, $\mathbf{E} = s^2$ (150, 327).

Contact of higher order of surfaces is an idea that requires careful study; many different definitions are possible and can be reconciled.

Let O (0, 0, 0) be a simple point on each of two surfaces ϕ_n, ψ_n. If there is contact of any order, they have at least a common tangent plane z:

$$\phi \equiv z + u_2 + u_3 + \dots + u_n,$$
$$\psi \equiv z + v_2 + v_3 + \dots + v_n.$$

If they have contact of order $s-1$, that is, s-point contact, then each of the following properties holds, and expresses, from a slightly different point of view, that the higher the contact, the closer the surfaces are together. Any one may be taken as definition, and we must show that the rest follow.

(i) The sections of ϕ, ψ by a general plane through O are two plane curves having s-point contact at O.

(ii) The total intersection of ϕ, ψ has s branches through O.

(iii) The first $s-1$ approximations to z in terms of x, y, of the forms

$$z=0,\ z=(x, y)^2,\ z=(x, y)^2+(x, y)^3+\dots,\ z=(x, y)^2+(x, y)^3+\dots+(x, y)^{s-1},$$

calculated from $\phi=0$, and valid at points of ϕ near O, are the same as those derived from ψ.

(iv) The distance between ϕ, ψ, measured normal to their common tangent plane, is small of order s compared with the distance of the normal from O.

(v) The quantity ψ is small of order s, compared with $OP = \sqrt{x^2+y^2+z^2}$, when in it are substituted the coordinates of any point P of ϕ near O.

Consider these properties first for the case of simple contact, $s=2$, already discussed; there is the common tangent plane z and no other particularity. Then (i) the general plane sections have simple contact, and (ii) O is a d.p. on the intersection.

At a general point P of ϕ near O, we have x, y, OP small of first order. The first approximation to z is $z=0$, and the second is $z=-[u_2]$, where square brackets denote that in them we have put $z=0$, leaving a function of x, y only. Thus (iii) the first approximations to z derived from ϕ, ψ are both 0, but the second, $-[u_2]$, $-[v_2]$, are different. Also (iv) the normal distance between the surfaces is z_2-z_1, where z_1 is calculated from $\phi=0$ and z_2 from $\psi=0$, for the same values of x, y. The first approximation to the difference arises from the second approximations to z_1, z_2 and is $[u_2-v_2]$, small of second order. Finally (v), the difference $\psi-\phi\equiv(v_2-u_2)+(v_3-u_3)+\ldots+(v_n-u_n)$ is small of second order compared with $\sqrt{x^2+y^2+z^2}$ at any point P near O; if the coordinates of P satisfy $\phi=0$, then ψ itself is small of second order.

Consider which of these properties hold if ψ has a d.p. at O, being an exceptional member of the linear family of surfaces touching ϕ at O. We can then take $\psi\equiv v_2+v_3+\ldots+v_n$. By virtue of the convention made for this case, (i) still holds, (ii) is true, and also (v), for now ψ is small of second order at any point whatever near O.

But (iii) and (iv) are false: at points of ψ near O, the first approximation to z_2 is found from $v_2(x, y, z_2)=0$, and is different from the first approximation to z_1; and z_2 is small of first order only, and so is z_2-z_1. The points of ψ do not lie nearer to the plane z than to any other plane.

With higher contact also, singularities may occur on exceptional members of the linear family of surfaces having s-point contact with ϕ at O. If our definition implied that both surfaces had a definite tangent plane, it would require continual supplementing; we therefore adopt the analytical definition (v) in the form:

A surface ψ has s-point contact at O, with a given surface ϕ on which O is simple, if ψ is small of order s compared with OP at any point P of ϕ near O.

It follows that ϕ has contact of the same order with any member of the family $c_1\phi+c_2\psi$, whether on it O is a simple or a multiple point.

As a consequence of the definition, the property (i) holds, whether ψ is singular or not, provided we adopt the corresponding definition for contact of plane curves. The property (ii) will be proved below (p. 234) when we deal with equivalence, and (iii) for the case when ψ is not singular, when we deal with postulation (p. 232). The modification of (iii), when ψ has a binode, is discussed fully under the head of partial contact (p. 241). We can prove (iv), also for the case when ψ is not singular, and more generally, that the distance between ϕ, ψ, measured at any finite angle with the tangent plane, is small of order s. The proof is as follows.

If ψ contained a term in x or y, it would be small of first order only; hence if it is not singular, its tangent plane is z, the same as that of ϕ.

Let P (x, y, z) be any point of ϕ near O; the adjacent point at a distance

ϵ from P in the general direction whose cosines are ξ, η, ζ has coordinates $(x+\epsilon\xi,\ y+\epsilon\eta,\ z+\epsilon\zeta)$. If this lies on ψ,

$$\begin{aligned} 0 &= \psi\,(x+\epsilon\xi,\ y+\epsilon\eta,\ z+\epsilon\zeta) \\ &= \psi\,(x, y, z) + \epsilon\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)\psi + \text{higher powers of } \epsilon \\ &\equiv \psi + \epsilon f, \text{ say}, \\ \epsilon &= -\psi/f. \end{aligned}$$

Now by definition, ψ is small of order s; and f is finite, its value at O tending, as $\epsilon \to 0$, to a finite multiple of the cosine of the angle between the normal to ψ and the direction ξ, η, ζ. Hence ϵ is small of order s.

If ψ is singular at O, the proof breaks down since $f \to 0$ with ϵ; the property (iv) ceases to hold.

6. Postulation of s-point contact, $\mathbf{P} = \frac{1}{2}s\,(s+1)$.

The parameters in ψ are the coefficients of all the linearly independent groups of terms, which are small of orders $\geqq s$ at points of ϕ near O, that is, by virtue of the equation $\phi = 0$. These will be called *admissible* groups for s-point contact. On the one hand, they include the single terms of degree $\geqq s$ in x, y, z, which will be called s-fold terms, and which are admissible without using the equation of ϕ. If these were the only groups admitted, ψ would have an s-fold point at O, and must be considered to have s-point contact with every surface passing simply through O.

On the other hand, every multiple of ϕ is admissible, being 0 by virtue of $\phi = 0$, and small of infinite order. We shall show that any admissible group is the sum of a multiple of ϕ and s-fold terms.

The order of smallness of a general function is the degree of its lowest terms; but by virtue of $\phi = 0$, we have x, y small of first order and z of second, and $x^{\alpha-\beta-\gamma} y^{\beta} z^{\gamma}$ of order $\alpha+\gamma$, greater than its degree if and only if $\gamma > 0$. If $\psi^{(\alpha)}$ is an admissible group, the degree of its lowest terms being $\alpha < s$, then these lowest terms are a multiple of z; for otherwise the order of smallness would be α not s. Hence $\psi^{(\alpha)}$ has the form

$$\psi^{(\alpha)} \equiv z v_{\alpha-1} + v_{\alpha+1} + \dots .$$

By subtracting a multiple of ϕ, we form another admissible group

$$\psi^{(\alpha+1)} \equiv \psi^{(\alpha)} - \phi v_{\alpha-1} \equiv v_{\alpha+1} + \dots - (u_2 + \dots + u_n)\, v_{\alpha-1},$$

whose lowest terms are of degree $\alpha+1$. If this $< s$, as before the lowest terms have z as a factor, and we go on subtracting multiples of ϕ and raising the lowest degree, until the remainder consists only of s-fold terms, when we have $\psi^{(\alpha)}$ expressed as the sum of a multiple of ϕ and s-fold terms. The process is definite and always possible; hence if $\psi^{(\alpha)}$ is of degree $\mu \geqq n+s-2$, we have it expressed in one and only one way in the form

$$\psi_{\mu} \equiv \phi f + \Lambda, \quad\dots\dots(1)$$

where $\qquad f \equiv v_{\alpha-1} + v_{\alpha} + \dots + v_{s-2}, \qquad \Lambda \equiv v_s + \dots + v_{\mu},$

where the v are all arbitrary.

But if $\mu < n+s-2$,

$$\Lambda \equiv v_s + \ldots + v_\mu + \hat{v}_{\mu+1} + \ldots + \hat{v}_{n+s-2},$$

where the $\hat{v}$ are not arbitrary, but exactly destroy all the terms of ϕf of degree $> \mu$. This includes the case when ϕ, ψ are general members of a homaloidal family, $\mu = n$, $\alpha = 1$.

Then if $n \geqq s-1$ we have

$$\psi \equiv \phi f + \Lambda, \quad f \equiv v_0 + v_1 + \ldots + v_{s-2}, \quad \Lambda \equiv v_s + \ldots + v_n + \hat{v}_{n+1} + \ldots + \hat{v}_{n+s-2}.$$

The postulation is the difference between the number of parameters in the general function of degree n, that is, in $v_0, \ldots, v_n$, and in ψ, that is, in $v_0, \ldots, v_{s-2}, v_s, \ldots, v_n$; the difference is the number of coefficients in v_{s-1}. For the general case of s-point contact therefore,

$$P = \tfrac{1}{2}s(s+1) \qquad \text{provided } n \geqq s.$$

In (1), f and Λ are two auxiliary surfaces, of which Λ has an s-fold point at O, and f does not pass through O if ψ is not singular there. If O is α-fold on ψ, it is $(\alpha-1)$-fold on f, and $v_0, v_1, \ldots, v_{\alpha-2}$ vanish; we have agreed to say that ψ still has s-point contact with ϕ.

A form similar to (1) has been used in an attempt to define the *sameness* of singularities on two different surfaces (95).

Our process gives a definite form for f, but the expression is still valid if we modify f by multiplying it by any function which is finite at O, and adding any s-fold terms, with corresponding modifications of Λ. This does not affect the number of parameters, and merely replaces f by another surface having s-point contact with it at O.

The property (iii) of p. 229 now follows, in the case when ψ is not singular; for the first $s-1$ approximations to z, calculated from $\psi = 0$, do not involve either Λ or f, and are the same as those derived from ϕ; but the sth approximation depends on Λ/f and is different for ϕ, ψ.

For simple contact, $s=2$, $P=3$; for osculation, $s=3$, $P=6$. If we put $s=1$, we have $P=1$, and O is an ordinary simple F-point without contact, which falls into its place naturally at the beginning of the series.

7. Modification for low values of n.

Let $\quad g_\alpha \equiv z + u_2 + \ldots + u_\alpha, \quad (\alpha = 1, \ldots, s-1), \quad$ with $g_1 \equiv z$;

by virtue of $\phi = 0$, we have

$$g_\alpha = -(u_{\alpha+1} + \ldots + u_n),$$

which is small of order $\alpha+1$. If we partly multiply out ϕf, and absorb all the terms of degree $\geqq s$ in those of Λ, we have

$$\psi \equiv g_{s-1}v_0 + g_{s-2}v_1 + \ldots + g_2 v_{s-3} + z v_{s-2} + v_s + \ldots + v_n,$$

where now all the v are arbitrary. The terms on the R.H.S. will be called *standard* groups.

So far we have assumed $n \geqq s-1$, high enough to admit all standard groups; if $n < s-1$ we can admit only those groups which, besides being small of order $\geqq s$, have their highest terms of degree $\leqq n$, and in any given case we must determine how many of these are linearly independent.

If ϕ is general, every standard group contains terms of degree $\geqq s-1$; but we can form certain admissible products of the g_α, which are altogether of lower degree. The argument given above shows that these are linear functions of standard groups, arranged so that the highest terms disappear.

With $s=4$, besides the standard groups

$$g_3 v_0, \quad g_2 v_1, \quad z v_2, \quad v_4, \quad v_5 \ldots,$$

all of which contain cubic or higher terms, we also admit z^2; hence 4-point contact, but not higher, is possible on a quadric family, namely the pencil $c_1\phi + c_2 z^2$. Now

$$z^2 \equiv g_2 z - z u_2,$$

and is a linear function of standard groups, from which the cubic terms disappear.

With $s=5$, besides the standard groups

$$g_4 v_0, \quad g_3 v_1, \quad g_2 v_2, \quad z v_3, \quad v_5, \quad v_6 \ldots,$$

all containing quartic or higher terms, there are two special cubic combinations

$$z g_2 \equiv g_3 z - z u_3 \text{ and } z^2 v_1 \equiv g_2 . z v_1 - z . u_2 v_1.$$

Hence we can have 5-point contact on a cubic family; and similarly 6-point contact on a cubic pencil, with z^3 as only admissible group besides ϕ. If ϕ is specialized, higher contact may be possible, see below.

If ϕ is a general n-ic, the admissible groups are built up of products of the g_α; for example, $v_{n-\gamma\alpha-\delta\beta \ldots} g_\alpha{}^\gamma g_\beta{}^\delta \ldots$ is admissible if its order of smallness $n+\gamma+\delta \ldots \geqq s$. Hence

$$n \geqq \gamma\alpha + \delta\beta + \ldots \geqq \gamma + \delta + \ldots \geqq s - n,$$

and there is no such term if $s > 2n$. The n-ic group of highest order of smallness is z^n, of order $2n$; and this is the highest contact possible on a family containing the general n-ic ϕ, the family being the pencil $c_1\phi + c_2 z^n$. If higher contact is required, then unless ϕ is specialized, ψ coincides with it.

For these lower values of n, the postulation may be decreased if we can replace any standard groups by others of altogether lower degree. This can only be done if the u are not general, so that the surface ϕ, with which s-point contact is required, is in some way specially related to the point O.

The simplest case is when u_2 has z as a factor:

$$\phi \equiv z(1+u_1) + u_3 + \ldots + u_n.$$

The approximations to z at points of ϕ near O are now obtained from

$$z = -(u_3 + \ldots + u_n)/(1+u_1),$$

showing that z is small of third order instead of second, and that ϕ osculates its tangent plane. Now z is admissible for 3-point contact; the standard group

$$g_2 v_{s-3} \equiv z v_{s-3} + z \, . \, u_1 v_{s-3},$$

of which the second term is now a particular case of the next group zv_{s-2}, can be replaced by the simpler admissible group zv_{s-3}, whose highest degree is $s-2$ instead of $s-1$. The postulation is not altered if $n \geqq s-1$, but for low values of n it may be decreased.

For example, let $s=6$; if n is high enough, we have $P=\frac{1}{2}.6.7=21$; if $n=3$, and ϕ is general, we have seen that the family is a pencil, $P=18$. In the present case we admit

$$g_3 \equiv \phi, \quad z^2 v_0, \quad z^2 v_1,$$

there are 5 parameters, and P is reduced to 15.

If z is also a factor of u_3, we have

$$\phi \equiv z(1+u_1+u_2)+u_4+\ldots+u_n,$$

and z is small of order 4. Then the standard groups $g_3 v_{s-4}$, $g_2 v_{s-3}$ can be replaced by the simpler forms zv_{s-4}, zv_{s-3}, of degrees $s-3$, $s-2$ respectively.

More generally, if z is a factor of $u_{\alpha+1}$ only, then the group g_α is small of order $\alpha+2$ instead of $\alpha+1$, and we admit $g_\alpha v_{s-\alpha-2}$, whose highest degree is $s-2$.

These and similar possibilities only affect P if $n<s-1$; each case must be discussed separately. In the extreme case, ϕ has the form $zf_{n-1}+u_n$, where f is finite at O; then z is small of order n, and n^2-point contact is possible on an n-ic family, namely the pencil $c_1\phi + c_2 z^n$.

8. Equivalence of s-point contact, $\mathbf{E}=s^2$.

The identity $$\psi \equiv \phi f + \Lambda$$

enables us to find the equivalence with great ease. The total intersection c of ϕ, ψ is the same as that of ϕ with Λ, which has an s-fold point at O. The first approximation to c is the intersection of the tangent plane z of ϕ with the general tangent cone v_s of Λ, and consists of s lines, lying in the fixed plane z and in general directions in that plane. Thus c has s branches through O, touching these s lines. This holds also if ψ is singular at O, for then f is specialized, but not Λ. Thus the property (ii) of p. 229 is established in all cases.

Now χ can also be put into the form

$$\chi \equiv \phi f_1 + \Lambda_1,$$

and the intersection of ϕ, ψ, χ is the same as that of c, Λ_1. Now Λ_1 has an s-fold point at O, with a general tangent cone; each of the s branches of c meets Λ_1 in s-points at O, and s^2 intersections are absorbed. Hence $E=s^2$.

9. Double point, simple contact, $\mathbf{P}=9$, $\mathbf{E}=18$.

If two surfaces have a common d.p., it is no longer convenient to think of them as each touching any plane and therefore touching each other. That convention applies only to those exceptional members of a linear family which have multiple points where the general member has a simple point. When the general member has a d.p., it is most natural to define simple contact to mean that the tangent cones coincide, and to extend this definition to include those exceptional members of the family which have a triple or higher multiple point.

We assume ϕ to be a general surface having a d.p. at O:

$$\phi \equiv u_2 \quad + u_3 + \ldots + u_n.$$

Then

$$\psi \equiv u_2 v_0 + v_3 + \ldots + v_n,$$

the tangent cone u_2 being the same for the whole family, except for those members which satisfy the single condition $v_0 = 0$, and have a triple point at O. The five ratios of the coefficients of u_2 are no longer parameters, and the additional postulation of contact, for a family already possessing a d.p., is 5; the total postulation is 9, which is one less than of a triple point.

The total intersection c of ϕ, ψ is the same as that of ϕ with

$$\Lambda \equiv \psi - \phi v_0 \equiv (v_3 - u_3 v_0) + \ldots + (v_n - u_n v_0),$$

which has a triple point at O, the tangent cone being arbitrary; there are six branches of c touching the six generators common to the cones u_2, $v_3 - u_3 v_0$. The intersection of ϕ, ψ, χ is the same as that of c with Λ_1 derived in the same way from ϕ, χ, which also has a triple point at O with an arbitrary tangent cone; each branch of c meets Λ_1 in three points at O, and $E = 18$.

The additional equivalence of contact, for a family already possessing a d.p., is 10, partly due to two extra branches of c through O and partly to one extra intersection of each branch with χ.

10. Multiple point, simple contact, $\mathbf{P}=\frac{1}{6}i(i^2+6i+11)$, $\mathbf{E}=i(i+1)^2$.

In the same way, we define simple contact at an i-fold point as meaning that ψ has the same tangent cone as ϕ. Then

$$\phi \equiv u_i \quad + u_{i+1} + \ldots + u_n,$$

$$\psi \equiv u_i v_0 + v_{i+1} + \ldots + v_n,$$

with the common tangent cone u_i. The additional postulation of contact is the number of ratios of coefficients in u_i, that is, $\frac{1}{2}i(i+3)$. The total postulation is one less than that of an $(i+1)$-fold point,

$$P = \binom{i+3}{3} - 1 = \tfrac{1}{6}i(i^2+6i+11).$$

The intersection of ϕ, ψ is the same as that of ϕ with $\psi - \phi v_0$, which has an $(i+1)$-fold point at O with a general tangent cone. There are $i(i+1)$ branches of c through O and each meets χ in $i+1$ points; hence

$$E = i(i+1)^2.$$

11. Multiple point, s-point contact,

$$\mathbf{P}=\tfrac{1}{6}i(i^2+3is+3s^2-1),\ \mathbf{E}=i(i+s-1)^2\ \ (399).$$

We define higher contact at a multiple point in the same algebraic way as at a simple point. We assume

$$\phi = u_i + u_{i+1} + \dots + u_n.$$

If ψ also has an i-fold point but there is no contact, then $s=1$, and ψ is small of order i at any point near O; if there is s-point contact, ψ is small of order $i+s-1$ at points of ϕ near O.

If ψ is not of multiplicity $>i$ at O, we can show that the distance between ϕ, ψ is small of order s. Let $P(x, y, z)$ be a general point of ϕ near O, where x, y, z are small of first order, and $(x+\epsilon\xi, y+\epsilon\eta, z+\epsilon\zeta)$ an adjacent point of ψ, at a distance ϵ from P in a general direction whose cosines are ξ, η, ζ. Then

$$\begin{aligned} 0 &= \psi(x+\epsilon\xi, y+\epsilon\eta, z+\epsilon\zeta) \\ &= \psi(x, y, z) + \epsilon\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)\psi + \text{higher powers of } \epsilon \\ &= \psi + \epsilon f, \text{ say,} \\ \epsilon &= -\psi/f, \end{aligned}$$

where f is small of order $i-1$ but not higher provided (ξ, η, ζ) does not lie on the tangent cone to ψ. But ψ is small of order $i+s-1$; hence ϵ is small of order s. This ceases to hold if ψ, and therefore f, is of higher multiplicity.

The admissible groups in ψ are those small of order $i+s-1$, by virtue of $\phi=0$; the standard groups are

$$g_\alpha v_{s-\alpha-1},$$

where
$$g_\alpha \equiv u_i + \dots + u_{i+\alpha-1}, \quad (\alpha = 1, \dots, s-1),$$

and
$$v_{i+s-1}, \dots, v_n.$$

If an admissible group has terms of degree lower than $i+s-1$, then if u_i does not break up, it is a factor of the lowest terms. This still holds if u_i breaks up. First let u_a be a simple factor of $u_i \equiv u_a u_{i-a}$, and therefore not a factor of u_{i-a}. At a general point P of the sheet touching u_a, we have u_a small of order $\geqq \alpha+1$ and u_{i-a} of order $i-\alpha$ only. Hence if

$$\psi \equiv v_i + v_{i+1} + \dots + v_n$$

is small of order $>i$ at P, then v_i has u_a as a factor, and similarly v_i contains every different factor of u_i at least once.

Next, if u_a is repeated, $u_i \equiv u_a{}^\gamma u_{i-\gamma a}$, and ϕ is not otherwise special, u_a is not a factor of $u_{i-\gamma a}$, or of u_{i+1}, and the sheet of ϕ is cuspidal; the first approximation is given by

$$u_a = (-u_{i+1}/u_{i-\gamma a})^{1/\gamma},$$

and u_a is small of order $\alpha + 1/\gamma$ only. If ψ is small of order $\geqq i+1$ at P, then v_i has $u_a{}^\gamma$ as a factor.

If ϕ is specialized, so that the sheet touching u_a is at least tacnodal, this argument breaks down. For example, let $\gamma = 2$, and let u_a be a factor of u_{i+1} but not of u_{i+2}; then u_a is small of order $\alpha+1$ at P, and if

$$v_i \equiv u_a u_{i-2a} v_a,$$

then ψ is small of order $i+1$ at every point of ϕ near O; but it is not convenient to consider this a case of total contact.

We may reverse the condition, and also require that ϕ is small of order $\geqq i+1$ at every point of ψ near O. Then v_i can have no factor v_a that does not occur in u_i; but even this is not sufficient; for example, if

$$\phi \equiv yz^2 + zu_3 + u_5,$$
$$\psi \equiv y^2z + yv_3 + v_5,$$

each of ϕ, ψ is small of order 4 at every point of the other near O, but still the tangent cones are not identical.

We require of our definition that every member of the pencil $c_1\phi + c_2\psi$ shall be small of order $i+s-1$ at every point of ϕ near O. Then the tangent cone $c_1u_i + c_2v_i$ of the general member can have no variable sheet, and it follows that u_i, v_i are the same.

We can now show as before that any admissible groups are linear functions of standard groups, and that ψ can be put into either of the forms

$$\psi \equiv \phi f + \Lambda, \quad \psi \equiv g_{s-1}v_0 + \ldots + g_2 v_{s-3} + u_i v_{s-2} + v_{i+s-1} + \ldots + v_n,$$

where $$f \equiv v_0 + v_1 + \ldots + v_{s-2},$$
$$\Lambda \equiv v_{i+s-1} + \ldots + v_n + \hat{v}_{n+1} + \ldots + \hat{v}_{n+s-2},$$

where the v are arbitrary, and the $\hat{v}$ are such as exactly to destroy all the terms of ϕf of degree $>n$. We assume $n \geqq i+s-1$; the contrary case requires special treatment.

Now P is the difference between the number of coefficients in a general function of degree n and the number in ψ, that is, in

$$v_0, \ldots, v_{s-2}, v_{i+s-1}, \ldots, v_n.$$

The difference is the number of coefficients in $v_{s-1}, \ldots, v_{i+s-2}$,

$$P = \binom{i+s+1}{3} - \binom{s+1}{3} = \tfrac{1}{6} i (i^2 + 3is + 3s^2 - 1),$$

and the additional postulation of s-point contact, for a family already having an i-fold point, is found by subtracting $\binom{i+2}{3}$, and is $\frac{1}{2} i (s-1)(i+s+1)$.

Now c is the total intersection of ϕ with Λ, which has an $(i+s-1)$-fold point at O with a general tangent cone. There are $i(i+s-1)$ branches of c through O. The intersection of ϕ, ψ, χ is the same as that of c with Λ_1, the surface corresponding to Λ, derived from ϕ, χ. Thus $E = i(i+s-1)^2$, and the additional equivalence of s-point contact is $i(s-1)(2i+s-1)$.

We have assumed that n is sufficiently high compared with both s and i. At the other extreme, we can have $n = i$; then ϕ is a cone, and if ϕ, ψ have even simple contact, they coincide altogether. If $n = i + 1$, then $\phi \equiv u_{n-1} + u_n$, and is a monoid; simple contact is possible for the family, and also osculation (399), the one admissible group for 3-point contact of degree as low as n being $u_{n-1}v_1$. Total contact of order $\geqq 3$ is not possible on a monoidal family; but if the tangent cone breaks up, there may be partial contact of order as high as $n + 1$, affecting one sheet only of ϕ; see below, p. 247.

II. POINTS OF PARTIAL CONTACT

12. Binode, simple contact of one sheet.

There is a case of partial contact, intermediate between that of a multiple point without contact, and one with a tangent cone altogether fixed. This occurs when the tangent cone breaks up, and part is fixed and part variable. With a d.p., the parts are both planes, and the singularity is called a *binode* or *biplanar point*, and the intersection of the two tangent planes or *biplanes* is the *edge* of the binode.

The condition that O is a binode on $\phi \equiv u_2 + u_3 + \ldots + u_n$ is that the quadric tangent cone u_2 breaks up into a pair of planes, and hence that its discriminant vanishes. This condition is cubic and not linear in the coefficients of ϕ, and cannot be one of the conditions determining a homaloidal web, unless it is combined with others in such a way as to reduce it to linear form.

Now the tangent cones to the linear family (ϕ) are a linear family (u_2), which is wholly degenerate, every member breaking up into a pair of planes. This can happen in two ways (28).

13. F-point of contact adjacent to double F-point.

In the first case, the two factors are both variable, and are members of a fixed pencil of planes; the binodes of (ϕ) all have a common edge, say l, $y = z = 0$; then $u_2 \equiv (y, z)^2$, the terms in x^2, xy, xz vanish, giving three additional conditions for the family already having an ordinary d.p. at O. The tangent cone to ψ is $v_2 \equiv (y, z)^2$, there is no contact of the sheets of ϕ, ψ, the intersection of u_2, v_2 consists of l counted four times, and c has still only four branches through O, all touching l, and touching but not osculating χ. Each branch meets χ in three points at O instead of two, and the additional equivalence is 4.

This additional postulation 3 and equivalence 4 suggest that we have an F-point of contact adjacent to the d.p. O. This can be proved by resolving the singularity by an auxiliary T_{2-2} with O as isolated F-point. As we saw on p. 200, this removes an ordinary conic node and transforms ϕ into a surface f' having simple contact with the second P-plane, w' say, of T_{2-2}, at a point Q' associated with the direction of the edge of the binode. Since all of (ϕ) have the same edge, the point Q' as well as the plane w' are the same for all of (f'), and the transformed family has at Q' an F-point of contact, which is implicit on (ϕ) in the first neighbourhood of O.

We can also investigate directly the limit when a distinct F-point of contact approaches a double F-point. Let (ϕ) have a d.p. at O,

$$\phi \equiv u_2 + u_3 + \dots + u_n,$$

and then make it have a point of contact at a neighbouring point O_1 $(\epsilon, 0, 0)$, the fixed tangent plane being the general plane through O_1,

$$p, \quad a_1(x - \epsilon) + b_1 y + c_1 z = 0,$$

and let $O_1 \to O$ as $\epsilon \to 0$.

If $$u_2 \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

the condition that ϕ passes through O_1 is to a first approximation

$$a\epsilon^2 = 0, \text{ that is } a = 0.$$

Write down the tangent plane to ϕ at O_1, by differentiating and substituting the coordinates of O_1, and express that this plane agrees with p, retaining only the lowest powers of the small quantity ϵ:

$$\frac{2a\epsilon}{a_1} = \frac{2h\epsilon}{b_1} = \frac{2g\epsilon}{c_1} = \frac{(n-2)\,a\epsilon^2}{-a_1\epsilon},$$

which, with the former condition, give

$$a = h = g = 0, \quad u_2 \equiv (y, z)^2,$$

which are the conditions found above for a binode.

The coefficients of the fixed tangent plane p disappear from the final result. In the limit, the tangent cone u_2 to ϕ has to pass through O_1 and touch p. It succeeds in passing through O_1, but it can properly touch only planes through O, and p does not pass through O; hence it touches p improperly, by acquiring O_1 as a d.p. and OO_1 as a double edge; and then it must equally be reckoned to touch every other plane through O_1.

An example of a family with this type of F-element will be found on p. 293.

14. Binode, $\mathbf{P} = 7$, $\mathbf{E} = 14$.

The second way in which every member of (u_2) can break up is when one plane is fixed and one variable. Every member of (ϕ) has a binode at O, the sheets of one set are in contact, and in general the other sheets are not. If the sheets of the second set are also in contact, the second tangent plane is fixed, and we have a particular case of total contact, with the whole tangent cone fixed, but happening to break up. The new case we have to consider is that of a pair of tangent planes, one fixed and one variable.

Take the fixed plane as z. The proper forms are

$$\phi \equiv zu_1 + u_3 + \dots + u_n,$$

$$\psi \equiv zv_1 + v_3 + \dots + v_n.$$

The ten arbitrary coefficients, of the terms v_0, v_1, v_2 of a general n-ic, are replaced by the three arbitrary coefficients of v_1 in ψ; hence $P = 7$.

Any particular ψ for which the parameters in v_1 all vanish has a triple or higher multiple point at O.

To study E, consider the first approximations zu_1, zv_1 to ϕ, ψ. They have in common the variable line $u_1 = v_1 = 0$, and the whole fixed plane z; therefore c has one branch through O touching this line, not touching χ but meeting it in two points at O; all the other branches of c touch z. Now c lies on

$$\Lambda \equiv u_1\psi - v_1\phi \equiv (u_1v_3 - v_1u_3) + \dots + (u_1v_n - v_1u_n),$$

formed from ϕ, ψ by eliminating z from the terms of lowest degree; Λ is the simplest sum of multiples of ϕ, ψ whose lowest terms do not have z as a factor. It has a 4-fold point at O, the tangent cone being the general quartic cone through the three fixed lines $u_1 = u_3 = 0$.

The total intersection of ϕ, Λ is that of ϕ, $u_1\psi$, and consists of c and the plane curve $\phi = u_1 = 0$, the latter having a triple point at O, the three branches touching the lines $u_1 = u_3 = 0$, and not touching z. All the branches that do touch z are therefore branches of c, and are given to a first approximation by

$$z = u_1v_3 - v_1u_3 = 0.$$

This shows that there are in general four such branches, each touching z but in general directions in that plane; each touches one sheet of χ and crosses the other sheet, meeting χ in three points.

Thus there are altogether five branches of c through O, one touching u_1 and four touching z; and $E = 1 \times 2 + 4 \times 3 = 14$.

The argument fails if z is a factor of $u_1v_3 - v_1u_3$; if this were to happen for all the family, z would be a factor of u_1 and of u_3, and O would be a tacnode on ϕ; if ϕ is general, it can only happen for exceptional members ψ which have higher contact with ϕ; see below, pp. 241, 243.

Consider what corresponds to the fourth property of contact of p. 229 concerning the distance ϵ between the two surfaces; let $P(x, y, z)$ lie on ϕ and $(x + \epsilon\xi, y + \epsilon\eta, z + \epsilon\zeta)$ on ψ. If P is a general point of the sheet of ϕ that touches u_1, then x, y, z are small of first order. We assume that ψ has a d.p. and not a higher point at O; then v_1 exists and is small of first order, ψ is of second, and $\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}$ are of first order. Expanding

$$\psi(x + \epsilon\xi, y + \epsilon\eta, z + \epsilon\zeta) = 0,$$

we see that ϵ is small of first order only. But if P lies on the sheet that touches z, or on the second sheet near the edge of the binode, then z is small of second order and ψ of third, but $\frac{\partial\psi}{\partial z}$ is still of first order. It follows that ϵ is small of second order.

By similar methods we can investigate an F-point of any multiplicity i, where the tangent cone breaks up into a fixed sheet, of simple contact for (ϕ), and a variable sheet of degree i_1. We find that c has i_1^2 branches touching the variable sheet and $(i-i_1)(i+i_1+1)$ branches touching the fixed sheet. The proper forms to assume are

$$\phi \equiv u_{i_1} u_{i-i_1} + u_{i+1} + \ldots,$$
$$\psi \equiv v_{i_1} u_{i-i_1} + v_{i+1} + \ldots .$$

15. Osculation of one sheet, P = 12, E = 24.

We can define osculation of one sheet as requiring ψ to be small of order 4 at those points of ϕ for which z is small of order 2. If ϕ is general, we still take

$$\phi \equiv zu_1 + u_3 + u_4 + \ldots + u_n;$$

the standard groups are

z, small of order 2, and its multiples, including zu_1 of order 3,

$zu_1 + u_3$ of order 4,

$zu_1 + u_3 + u_4$, of order 5, and so on.

The admissible groups for osculation are

$$z^2,\ zu_1 + u_3,\ zv_2,\ v_4,\ v_5,\ \ldots,\ v_n,$$

whence
$$\psi \equiv zv_1 + v_3 + v_4 + \ldots + v_n,$$

where now v_1, v_3 have the special forms

$$v_1 \equiv au_1 + bz, \quad v_3 \equiv au_3 + zv_2,$$

where a, b and the six coefficients of v_2 are arbitrary, these eight replacing the thirteen of v_1, v_3 in the last case. The additional postulation is 5, and altogether $P = 12$.

Here
$$u_1 v_3 - v_1 u_3 \equiv z(u_1 v_2 - bu_3),$$

and has z as a factor, the exceptional case noted on p. 240.

The plane v_1 passes through the line $z = u_1 = 0$; hence ϕ, ψ have the same edge; osculation of one sheet, with ϕ general, is only possible if all the binodes of the family have a common edge.

To find E, as before we deduce an auxiliary surface Λ by eliminating z from the terms of lowest degree of ϕ, ψ. The intersection c lies upon

$$\Lambda \equiv u_1^2 \psi - \{au_1^2 + u_1(bz + v_2) - bu_3\}\phi,$$

whose terms of lowest degree are

$$v_6 \equiv u_1^2(v_4 - au_4) - u_1(bzu_4 + v_2 u_3) + bu_3^2.$$

The total intersection of ϕ, Λ is that of ϕ, $u_1^2\psi$, and consists of c and the section of ϕ by u_1 counted twice. As we saw above, the latter has no branches through O touching z. There are therefore six branches of c through O which do touch z, the tangents being given by $z = v_6 = 0$, and in general directions in the plane z; each branch osculates one sheet of χ and crosses the other, meeting χ in four points. Since u_1, v_1 meet on z, there are no branches of c which do not touch z, and $E = 6 \times 4 = 24$.

Now consider the third property of p. 229. In our case of partial contact at the binode, the first approximation to z, at any point of any one of the sheets in contact, is $z=0$, deduced from the terms of lowest degree zu_1 or zv_1; but the first approximations to the second sheets are different. With osculation, the second approximations to the first sheets also agree, deduced by including the cubic terms and using in them the first approximation:

$$z=[-u_3/u_1], \quad z=[-v_3/v_1],$$

where square brackets denote that in them we have put $z=0$. These must agree for all values of $x:y$; hence if we admit no special assumption about $[u_1]$, $[u_3]$, so that ϕ is the most general surface with a binode at O, we must have

$$[v_1]\equiv a[u_1], \quad [v_3]\equiv a[u_3], \quad \text{..........................(2)}$$

where a is arbitrary. Now v_1, v_3 differ from $[v_1]$, $[v_3]$ only by multiples of z; hence

$$v_1\equiv au_1+bz, \quad v_3\equiv au_3+zv_2,$$

which are the forms we found above.

16. Reduction of P and E for a B_4.

The identities (2) can be satisfied in another way if we make a special assumption as to the nature of ϕ, namely that $[u_1]$ is a factor of $[u_3]$, that is

$$u_3\equiv zu_2+u_1t_2,$$

where u_2, t_2 are any fixed quadratic functions of x, y, z. Then ϕ can take the form

$$\phi\equiv(z+t_2)(u_1+u_2)+u_4+\ldots+u_n,$$

where the term $-t_2u_2$ has been included in the general quartic function u_4. This form shows that the binode O is of the type B_4 (p. 200).

Another interpretation is that there exist two distinct surfaces, $z+t_2$ and u_1+u_2, each having a simple point at O, and osculating one sheet only of ϕ. If ϕ is general, the second approximation to the first sheet of ϕ, namely $z=-[u_3/u_1]$, does not reduce, and the simplest surface osculating this sheet is the cubic $z[u_1]+[u_3]$, having a binode at O and the same edge as ϕ; this we saw above to be necessary for osculation. But with the present assumption, $[u_3/u_1]$ reduces to t_2, and there is the quadric $z+t_2$, on which O is simple, osculating one sheet of ϕ but with no special relation to the second sheet; and similarly there is the quadric u_1+u_2 osculating the second sheet and not touching the first. The singularity of ϕ at O is such that ϕ breaks up, into these two quadrics, if we neglect terms of degree $\geqq 4$, while in the more general case ϕ breaks up, into the two biplanes, only if we neglect terms of degree 3 as well.

The general form of ψ is found from

$$[v_3]=[v_1]t_2,$$

giving

$$v_3=v_1t_2+zv_2,$$

$$\psi\equiv(z+t_2)(v_1+v_2)+v_4+\ldots+v_n;$$

each of the family has one sheet osculating $z + t_2$, and one sheet osculating $v_1 + v_2$, which is a variable quadric through O. For osculation of one sheet at a B_4 we have $P = 11$, one less than for a B_3.

Here $$u_1 v_3 - v_1 u_3 \equiv z(u_1 v_2 - v_1 u_2),$$

giving a second form of the exceptional case of p. 240.

There is one branch of c touching $u_1 = v_1 = 0$, and meeting χ in two points; all the other branches osculate $z + t_2$. Their first approximations are given by the intersection of z with

$$\Lambda \equiv (u_1 + u_2)\psi - (v_1 + v_2)\phi \equiv (u_1 v_4 - v_1 u_4) + \dots,$$

a surface having a 5-fold point at O, formed by eliminating the common osculating quadric from ϕ, ψ. This shows that there are five such branches of c, each osculating one sheet of χ and crossing the other. For this case, $E = 2 + 5 \times 4 = 22$, two less than for a B_3.

17. Higher binode, $\mathbf{P} = \frac{1}{2}(s^2 + 3s + 4)$, $\mathbf{E} = s^2 + 3s + 4$.

If O is a B_{s+1}, a binode of the type which reduces the class by $s + 1$, we may take

$$\phi \equiv tf + u_{s+1} + \dots + u_n, \quad \dots\dots\dots\dots\dots\dots\dots\dots (3)$$

where $$t \equiv z + t_2 + \dots + t_{s-1}, \quad f \equiv u_1 + u_2 + \dots + u_{s-1}.$$

As for the B_3, if this singularity occurs on each member of a linear family, there are two cases possible. In the first, both f and t vary within a fixed pencil, and there is no contact. This is the limiting case that arises when a certain set of distinct F-points come together, moving along the fixed curve k, $t = f = 0$, which is the base of the pencil (t, f), and whose tangent at O is the edge of the binode. The set of F-points consists of $\frac{1}{2}(s + 1)$ d.p.'s if s is odd, and $\frac{1}{2}s$ d.p.'s and a point of contact if s is even. We saw on p. 199, how to resolve a B_{s+1} of any one surface by a series of T_{2-2}'s. The same series of transformations resolves the binodes of all the family (ϕ), the adjacent singularity always appearing at a point of the next space the same for every ϕ. As regards the singularities, explicit at O or implicit in its first $\frac{1}{2}\{s - (3 \text{ or } 2)\}$ neighbourhoods, ϕ behaves exactly as if it were degenerate, with k as the intersection of distinct sheets.

In the second possible case, one of the factors of the first group of terms in ϕ, say t, is fixed; one biplane is the fixed plane z, and there is s-point contact of the sheets of the family that touch z. The second biplane and the edge of the binode are variable. Then if a T_{2-2} is applied to O, the different surfaces (ϕ') have their explicit singularities, which were implicit in the first neighbourhood of O, at different points of the second P-plane of T_{2-2}. These adjacent singularities therefore are not F-points of (ϕ), but variable with the parameters of the family. They all lie on the line of the P-plane associated with the direction of the fixed biplane, which is a line of contact of (ϕ').

Now $$\psi = t(v_1 + \ldots + v_{s-1}) + v_{s+1} + \ldots + v_n,$$
and P is the number of coefficients in v_0, v_s, that is
$$P = \tfrac{1}{2}(s^2 + 3s + 4).$$

There is one branch of c touching $u_1 = v_1 = 0$; and we find $s + 2$ branches touching z and each meeting χ in $s+1$ points;
$$E = 2 + (s+2)(s+1) = s^2 + 3s + 4.$$

One sheet of ψ has s-point contact with t, which has a simple point at O, and the other sheet has s-point contact with a variable non-singular surface. It is not possible for any surface to have higher contact with one sheet of ϕ unless it has a binode with the same edge as ϕ; the proof of this is as follows.

The sth approximation to the first sheet of ϕ is
$$z = -(t_2 + \ldots + t_{s-1}) - [u_{s+1}/u_1].$$
If ϕ is the most general surface of its kind, the last term does not reduce, we have to multiply up, and obtain a surface whose lowest terms are $z[u_1]$ showing that it has a binode with the same edge as ϕ. But if ϕ is specialized, so that $[u_{s+1}/u_1] = t_s$ say, we obtain the non-singular surface $t + t_s$. But in this case,
$$u_{s+1} \equiv u_1 t_s + z u_s,$$
$$\phi \equiv (t + t_s)(f + u_s) + u_{s+2} + \ldots + u_n,$$
showing that the partial factorization of ϕ can be carried one step further, to include the terms of degree $s+1$, so that O is not merely a B_{s+1} on ϕ, but a B_{s+2}.

In the form (3) for ϕ, we assume $n > 2(s-1)$. If this does not hold, we must subtract terms of degree $> n$ so as to destroy those occurring in tf. For a given value of n, there is an upper limit for s. Since d.p.'s reduce the class of a surface, and also the number of double tangent planes, the consideration of each of these gives a function of n which cannot be exceeded by the number of conical or biplanar d.p.'s, whether these are distinct, or coincide in a B_{s+1}. But before s reaches the limit thus given, the singularity may change its nature, a double curve appearing.

If we apply the algebraical criterion for a B_{s+1}, contained in (3), to a general point O of a double curve, we shall show that the partial factorization can be continued indefinitely, and there is no limit to s. This is natural, as there are ∞ d.p.'s adjacent to O. We can find a non-singular surface having at O contact of any assigned order $s-1$ with one sheet only of ϕ, or we can find a surface ψ having a binode, one sheet having s-point contact with one sheet of ϕ, and the other touching an arbitrary plane; but if s is too high compared with the degree of ψ, this also acquires a double curve.

For example, let O lie on the double line l, $y = z = 0$. The biplanes both pass through l, and we take them to be y, z. Then O is a B_3 at least; assume that ϕ can be put into the standard form (3) for a B_{s+1}, with $u_1 \equiv y$. Now every term in ϕ is quadratic at least in y, z; in particular,

$$zu_2 + yt_2, \quad \ldots, \quad zu_{s-1} + t_2 u_{s-2} + \ldots + t_{s-1} y, \quad t_2 u_{s-1} + \ldots + t_{s-1} u_2 + u_{s+1}$$

are all quadratic in y, z. Hence $u_2, t_2, \ldots, u_{s-1}, t_{s-1}$ are all linear in y, z, and u_{s+1} is quadratic, say

$$u_{s+1} \equiv y^2 u_{s-1,1} + yzu_{s-1,2} + z^2 u_{s-1,3}.$$

Then $\quad \phi \equiv (t + yu_{s-1,1} + zu_{s-1,2})(f + zu_{s-1,3}) + \text{terms of degree} \geqq s+2,$

which is the standard form for a B_{s+2}, and shows that the partial factorization can be carried on indefinitely.

18. Higher binode, s_1-point and s_2-point contact.

As another example, let ψ have 4-point contact with one sheet of ϕ at a point O which is only a B_3 on $\phi \equiv zu_1 + u_3 + \ldots + u_n$. The admissible groups in ψ are those small of order 5 at points of ϕ for which z is small of order 2:

$$(zu_1 + u_3 + u_4)\, v_0, \quad (zu_1 + u_3)\, v_1, \quad zv_3, \quad z^2 v_1, \quad v_5, \ldots,$$

of which the only quadratic terms are zu_1v_0; hence ψ also touches the second biplane u_1 of ϕ, but on account of the term z^2v_1, small of third order at general points of the second sheet of ϕ, the contact of these sheets is of first order only.

More generally, if ψ is required to have s_1-point contact with one sheet of ϕ at a B_{s+1} of ϕ, where $s_1 > s+1$, then ψ must have s_2-point contact with the second sheet of ϕ also, where s_2 lies between certain limits depending on s, s_1, which we shall now find.

We have seen above that if $s_1 \leqq s$, the condition can be replaced by that of contact with the non-singular surface t and ψ need not have a second sheet at all. If it has, as an independent condition it may be required to have s_2-point contact with the second sheet of ϕ. If $s_2 < s_1$, then O is exactly a B_{s_1+1} on the general ψ, and

$$\psi \equiv t_\psi f_\psi + v_{s_1+1} + \ldots + v_n,$$

where $$t_\psi \equiv z + t_1 + \ldots + t_{s_1-1},$$

and f_ψ is subject to the conditions of s_2-point contact with the separate single-sheeted surface f. There are as before $s_1 + 2$ branches of c touching z, one of them now touching the edge, and there are s_2 branches touching u_1, in general directions in that plane, the same as for the intersection of f, f_ψ. If ϕ has a higher binode than ψ, this does not affect the results.

If $s_1 = s+1$, we have seen that ψ has a binode with the same edge as ϕ, and must have a second sheet, but still not one necessarily subject to any contact condition; for there is the admissible group zt, small of order $s_1 + 1$ at points of the first sheet of ϕ, but only of order 2 at points of the second sheet.

If $s_1 > s+1$, let $s_1 + 1 = \alpha s + \beta$, where $\beta = 1, 2, \ldots$ or s, and α is the least integer such that $\alpha + 1 \geqq (s_1+1)/s$. In ψ, the admissible group containing terms of the lowest degree is $t^{\alpha}(z+t_2+\ldots+t_{\beta-1})$, of lowest degree $\alpha+1$, which is also its order of smallness on the second sheet of ϕ. Hence

$$s_2+1=\alpha+1 \geqq (s_1+1)/s.$$

If ψ is not general, s_2 may be higher than this, but not lower; by the same argument, with the sheets reversed,

$$s_1+1 \geqq (s_2+1)/s.$$

Hence, for given values of s, s_1, the limits for s_2 are

$$(s_1+1)/s-1 \leqq s_2 \leqq s(s_1+1)-1.$$

Thus at a B_{s+1}, the surface tries to break up, and succeeds as far as concerns the first $s-1$ approximations to either sheet; for higher approximations, each sheet is obliged to drag the other after it, with a certain lag in the order of approximation involved. This is natural, since there is no sharp distinction between the sheets, but a continuous passage from one to the other near the edge. The relative orders of smallness, of z and u_1 in the above investigation, only hold for general points of the sheets, and not for a point near the edge, where z/u_1 may have any order of smallness or infinity, and it may be quite doubtful to which sheet the point belongs. If there are any points of ϕ near O actually lying on the edge, they do not differ essentially from the adjacent points. Generally speaking, the higher the suffix of the B, the sharper the fold of ϕ to which the edge is an approximation.

At a general point O of a nodal curve, on the contrary, the two sheets are absolutely distinct, and there is no continuous passage from one to the other. Every point of ϕ belongs definitely to one sheet or to the other, except the points of the double curve itself, which belong to both, and are quite different from the adjacent points, being d.p.'s of ϕ. As regards the neighbourhood of O of any order, ϕ behaves exactly as if it were degenerate, and the two sheets belonged to separate components.

This is not true, however, of the neighbourhood of the curve as a whole, for there may be certain special points of it at which it is possible to pass continuously from one sheet to the other.

19. Points of higher multiplicity (36, 161, 317).

At a point of higher multiplicity, when the tangent cone breaks up, the same principles and methods apply as for a binode, but general formulae are unwieldy. There are the same two main cases as for a double point. In one, all the sheets except one are fixed, and there is partial contact. In the other, the variable sheets are members of the same fixed pencil, and the symmetric functions of their parameters in the pencil are the linear parameters of the homaloidal web; this case is the limit of a set of distinct F-points, which have come together along the curve base of the pencil, and can be resolved by a series of auxiliary transformations.

As an example of contact of different orders of different sheets, consider a family of monoids vertex O. If there is simple partial contact,

$$\phi \equiv t_a u_{n-a-1} + u_n, \quad \psi \equiv t_a v_{n-a-1} + v_n.$$

There can be osculation of the first sheets if $v_n \equiv t_a v_{n-a}$. The highest partial contact possible is when

$$\phi = z u_{n-2} + u_n, \qquad \psi \equiv c_1 \phi + c_2 z^n;$$

then ψ is small of order $2n$ on the first sheet of ϕ, and of order n on the second; there is $(n+2)$-point contact of the one pair of sheets, and simple contact of the others. This family is only a pencil; but contact of the same order can occur on a homaloidal web (p. 315).

A particular case is that of a unode. If z is the fixed double tangent plane at O,

$$\phi \equiv z^2 + u_3 + \dots,$$
$$\psi \equiv z^2 + v_3 + \dots.$$

The branches of c touch $z^2 = v_3 - u_3 = 0$, and O is 6-fold on c; but there are only three distinct branches, each being cuspidal.

Choose the axes so that $y = z = 0$ lies on $v_3 - u_3$; then one branch of c touches this line, given by approximations of the form

$$x = \xi\epsilon^2, \quad y = \eta\epsilon^3, \quad z = \zeta\epsilon^3,$$

and meeting χ in six points.

Hence $P = 9$, $E = 18$, just as for an ordinary d.p. of total contact.

III. CURVES OF CONTACT

20. Simple line, simple contact, $\mathbf{P} = 3n - \beta$, $\mathbf{E} = 8n - 8 - 2\beta$ (87, 382).

Let the family (ϕ) have simple contact, at every point of the line l, whose equations in homogeneous coordinates we take to be $z = w = 0$. As on p. 213. we arrange ϕ in the form

$$\phi \equiv t + (z, w \overset{2}{)(} x, y, z, w)^{n-2},$$

where $$t \equiv z u_{n-1,1} + w u_{n-1,2},$$

and $u_{n-1,1}$, $u_{n-1,2}$ are homogeneous functions of x, y only. Then t is the tangent plane to ϕ at $(x, y, 0, 0)$, if we regard $u_{n-1,1}$, $u_{n-1,2}$ as constants and z, w as current coordinates.

The condition of contact is that ψ has the same tangent plane t as ϕ, except at points where either surface is singular, and therefore has no tangent plane. Hence $u_{n-1,1}/u_{n-1,2}$ is the same for ϕ, ψ for all values of x/y. If ϕ is general, $u_{n-1,1}$, $u_{n-1,2}$ have no common factor, their ratio does not reduce, and their $2n$ coefficients are in constant ratios. Thus the additional postulation of contact, for a family already passing through l, is $2n - 1$, and

$$P = (n+1) + (2n-1) = 3n,$$

being one less than for a double line.

Now $$\psi \equiv tv_0 + (z, w \overset{2}{\between} x, y, z, w)^{n-2};$$
the total intersection of ϕ, ψ is the same as that of ϕ with
$$\Lambda \equiv \psi - \phi v_0,$$
from which the terms linear in z, w disappear, and on which l is a double line. Hence l is to be counted twice as a fixed part of the intersection of ϕ, ψ, and the variable residual c is of degree n^2-2. Now c meets l wherever ϕ touches one of the sheets of Λ, that is, where t coincides with one of the tangent planes to Λ, given by an expression of the form
$$t_2 \equiv z^2 v_{n-2,1} + zwv_{n-2,2} + w^2 v_{n-2,3}.$$
The values of x/y at these points are given by eliminating z, w between t, t_2, that is, by
$$u^2_{n-1,2}v_{n-2,1} - u_{n-1,1}u_{n-1,2}v_{n-2,2} + u^2_{n-1,1}v_{n-2,3},$$
of degree $2(n-1)+n-2=3n-4$. There are thus $3n-4$ intersections of c, l, at each of which c touches t and therefore χ, and two intersections of c, χ are absorbed. The number of free intersections of c, χ is
$$(n^2-2)\,n - 2\,(3n-4) = n^3 - 8n + 8;$$
and for a line of contact, $E = 8n-8$.

So far we have assumed that $u_{n-1,1}$, $u_{n-1,2}$ have no common factor; then they never both vanish for the same value of x/y, the tangent plane to ϕ is never indeterminate, and ϕ has no multiple point lying on l. We have seen (p. 222) that if ϕ has β d.p.'s lying on l, then $u_{n-1,1}$, $u_{n-1,2}$ have a common factor u_β, the product of β linear factors that vanish one at each of the d.p.'s; and ϕ touches a surface of lower degree $n-\beta$, which has no d.p.'s on l. The proper form to assume is
$$\phi \equiv tu_\beta + (z, w \overset{2}{\between} x, y, z, w)^{n-2},$$
where now $$t \equiv zu_{n-1-\beta,1} + wu_{n-1-\beta,2}.$$

If we regard x, y, z, w as all variable, then t represents a surface of degree $n-\beta$, which touches ϕ along l. The condition of contact with ϕ can be replaced by that of contact with t.

The tangent planes to (ϕ) depend on t and not on u_β; we must take
$$\psi \equiv tv_\beta + (z, w \overset{2}{\between} x, y, z, w)^{n-2}.$$
The ratios of the coefficients of t are constants for the family, and those of v_β are additional parameters; P is reduced by β and becomes $3n-\beta$.

To find the intersections of c, l, consider the surface Λ formed by eliminating t between ϕ, ψ:
$$\Lambda \equiv \psi u_\beta - \phi v_\beta \equiv (z, w \overset{2}{\between} x, y, z, w)^{n-2+\beta}.$$
This passes through c and has l as a double line. The total intersection of ϕ, Λ consists of l counted twice, c, and also the intersection of ϕ, u_β. But u_β consists of β planes not passing through l, each meeting l in one of the

d.p.'s of ϕ, and meeting ϕ in a plane curve with a d.p. on l, neither branch touching t. Hence c meets l wherever Λ touches t, that is, where t coincides with one of the tangent planes to Λ given by

$$z^2 v_{n-2+\beta,1} + zw\, v_{n-2+\beta,2} + w^2 v_{n-2+\beta,3}.$$

The degree in x/y of the condition is

$$2(n-1-\beta)+(n-2+\beta)=3n-4-\beta;$$

this, instead of $3n-4$, is the number of intersections of c, l, each absorbing two of ϕ, ψ, χ. Hence E is reduced by 2β and becomes $8n-8-2\beta$.

The condition of touching ϕ along l obliges ψ to have the same number of d.p.'s on l as ϕ, but leaves their positions on l free, given by $v_\beta=0$; we can place them at any assigned points by taking for v_β the product of the forms linear in x, y that vanish one at each of the points.

If the position O on l of one of the d.p.'s is fixed for the whole of (ϕ), then through O there pass four branches of the total intersection of ϕ, ψ, namely l counted twice and two branches of c. Hence O is an isolated F-point and a separate element of H, being a point of higher multiplicity through which the line of contact passes. The reductions of P and E due to the incidence can be calculated in much the same way as for the corresponding case without contact (p. 222).

At a point O of l where ϕ is singular and ψ is not, ψ has a definite tangent plane t, which is the tangent plane to the non-singular surface touched by ϕ. The conditions of contact with ϕ along l assign a definite tangent plane to ψ at every adjacent point of l; at O there is only one plane, namely t, continuous with these adjacent planes; hence, since ψ is not singular, its definite tangent plane can be no other than t. The quadric tangent cone to ϕ has l as a generator and touches t along l; the first approximation to the intersection of ϕ and ψ is that of this cone and t, which consists solely of l taken twice. Thus c does not pass through O.

If $\beta>0$, we have a case where a multiple point of ϕ is not an F-point of the same multiplicity for the whole family. For example, the homaloidal web of singular cubics (158),

$$\{c_1(x+z)+c_2(y+w)\}(xw+yz)+(c_3z+c_4w)(xw-yz)$$

has a d.p. at the variable point $(c_2, -c_1, 0, 0)$ of l, and along l the cubics touch the fixed quadric $xw+yz$. We have a variable point of higher multiplicity lying on an F-line of contact of lower multiplicity.

Now β can have any value from 0 to $n-1$ inclusive. In the last case $u_{0,1}$, $u_{0,2}$ are independent of x/y, and t is the same plane at every point of l, touched by all of (ϕ). Then

$$P=2n+1\equiv 2(n+1)-1, \qquad E=6n-6\equiv 2(3n-2)-2,$$

both the same as for two simple distinct intersecting lines. If $\beta=n-2$, then t is a quadric, $P=2(n+1)$ and $E=2(3n-2)$, both the same as for two simple distinct skew lines. In these two cases, and these only, it may be possible to regard the contact as a limiting case, in which two distinct lines of intersection have come to coincide. In general,

$$P = 2(n+1) - (\beta + 2 - n), \quad E = 2(3n-2) - 2(\beta + 2 - n),$$

as for two distinct lines intersecting in $\beta + 2 - n$ points; but this has no obvious geometrical meaning unless $\beta = n - 1$ or $n - 2$ (p. 172).

When ϕ is given, the admissible groups in ψ are those small of order 2 at all points of ϕ near l; here x, y are finite and z, w small of first order. The order of smallness of a general function is its lowest degree in z, w; and t is a special group of second order, since, by virtue of $\phi = 0$, it is equal to a set of terms quadratic and higher in z, w.

If $\beta = 0$, in ψ we admit tv_0 and also terms of degree $\geqq 2$ in z, w; but if $\beta > 0$, the special group t is of degree $n - \beta$ in x, y, z, w, and instead of tv_0 we admit tv_β, of the same order of smallness as before, but containing $\beta + 1$ parameters instead of 1.

This example of a line shows that the conditions of contact along a curve are not fully determined, as regards either P or E, by merely giving the curve of contact ω and degree n of the family (ϕ). We must also know the set of tangent planes which ϕ is required to touch at the points of ω, forming a given circumscribed developable; or, instead, we may be given any other surface f which touches the same planes along ω. It is usually convenient to have f of the lowest possible degree; this $\leqq n$; if it were $> n$, then ϕ would vanish identically.

21. Simple line, osculation, $\mathbf{P} = 6n - 3 - 2\beta_2 - 3\beta_3$, $\mathbf{E} = 15n - 18 - 4\beta_2 - 6\beta_3$ (87).

Next, let l be a line of osculation. Let

$$\phi \equiv t_1 + t_2 + (z, w\overset{3}{)(}x, y, z, w)^{n-3},$$

where $\quad t_1 \equiv zu_{n-1,1} + wu_{n-1,2}, \quad t_2 \equiv z^2u_{n-2,1} + zw\,u_{n-2,2} + w^2u_{n-2,3}.$

In ψ we admit groups small of order $\geqq 3$ at points of ϕ near l; now z, w are small of first order, t_1 of second and $t_1 + t_2$ of third; thus

$$\psi \equiv (t_1 + t_2)\,v_0 + (z, w\overset{3}{)(}x, y, z, w)^{n-3};$$

and the conditions for osculation are that all the coefficients in $u_{n-1,1} \ldots, u_{n-2,3}$ are in fixed ratios; the additional postulation of osculation, for a family already having l as a line of contact, is $3(n-1)$ and

$$P = 3n + (3n - 3) = 6n - 3,$$

being one less than for a triple line, which ψ acquires with one more condition, namely $v_0 = 0$.

The auxiliary surface $\Lambda \equiv \psi - \phi v_0$ has l as a triple line; hence l counts three times in the intersection of ϕ, ψ, and the degree of c is $n^2 - 3$. Now c

meets l at the points where t coincides with one of the three tangent planes to Λ, given by a cubic in z/w, with coefficients of degree $n-3$ in x, y. This leads to an equation for x/y of degree $3(n-1)+n-3=4n-6$, which is the number of points at which c meets l and osculates χ, so that three intersections of c, χ are absorbed at each. The number of free intersections of ϕ, ψ, χ is $(n^2-3)n-3(4n-6)$, and $E=15n-18$.

If ϕ has a conical d.p. O on l, then ψ also has a d.p. at O. For the sections of ϕ, ψ by a general plane p have the same curvature at the point Q where p meets l. As $Q \rightarrow O$, the osculating quadric of ϕ at $Q \rightarrow$ a proper cone, and the curvature of the section $\rightarrow \infty$ for ϕ and therefore for ψ, and ψ is also singular. In this case, $u_{n-1,1}$, $u_{n-1,2}$ have, as a common factor, the linear function u_1 of x, y which vanishes at O; but this is not a factor of all the $u_{n-2,\alpha}$.

The proper form for ϕ is

$$\phi \equiv u_1 t_1 + t_2 + (z, w \overset{3}{)\!(} x, y, z, w)^{n-3},$$

where now t_1 is a special group of order of smallness 2 and total degree $n-1$; hence $t_1(z, w)^1$ is of order of smallness 3 and degree n and therefore admissible:

$$\psi \equiv (u_1 t_1 + t_2)\, v_0 + (z, w \overset{3}{)\!(} x, y, z, w)^{n-3} + t_1(z, w)^1,$$

there are two additional parameters, and P is reduced by 2.

The tangent cones to ϕ, ψ at O both have l as generator and along it they both touch t_1 and osculate $u_1 t_1 + t_2$. Through the common d.p. O of ϕ, ψ, there pass four branches of the total intersection, namely, l counted three times and one branch of c which touches the remaining generator of intersection of the two cones. Now this generator does not lie on the tangent cone to χ; and c passes through the d.p. of χ in a general direction and meets χ in two points at O. But we shall now show that owing to O there is a reduction of 2 in the number of intersections of l, c at simple points of ϕ, which are points of osculation of c, ψ. We gain one double intersection of c, ψ and lose two triple intersections; E is reduced by 4.

The auxiliary surface found by eliminating t_1 between ϕ, ψ is now

$$\Lambda \equiv \psi u_1 - \phi\,\{u_1 v_0 + (z, w)^1\} \equiv (z, w \overset{3}{)\!(} x, y, z, w)^{n-2}.$$

The equation for the values of x/y at the intersections other than O of l, c, which are the points of contact of t_1, Λ, is of degree $3(n-2)+n-2=4n-8$, being less by 2 than when ϕ is not singular.

Similarly, if ϕ has β_2 conic nodes on l, then t_1 is of degree $n-\beta_2$ and Λ of degree $n+\beta_2$; we admit $t_1(z, w \overset{1}{)\!(} x, y)^{\beta_2-1}$ in ψ, and P is reduced by $2\beta_2$, and E by $4\beta_2$.

If O is not a conic node but a binode, one of the biplanes contains l, and is continuous with the tangent planes to ϕ at adjacent points of l, being the plane t_1; the other biplane may be any plane u_1 through O. We assume that the edge is in a general direction, and does not coincide with l. Then ψ need not be singular at O; as an adjacent point $Q \rightarrow O$, the osculating quadric of ϕ at $Q \rightarrow$ a non-singular limit, and the curvature at Q, of the section of ϕ by a general plane p through Q, $\rightarrow$ the finite curvature at O of that branch, of the section of ϕ by the limit of p, which touches t_1. Then ψ touches one sheet of ϕ and crosses the other, and a general plane through O meets ϕ, ψ in two curves meeting in three points at O, and the incidence of the two surfaces in this case must still be included as osculation.

Now all the terms of ϕ of degree $\leqq 2$ in z, w have u_1 as a factor, and we may write

$$\phi \equiv tu_1 + (z,\ w\overset{3}{)(}x,\ y,\ z,\ w)^{n-3},$$

where

$$t \equiv t_1 + t_2,$$

and now each of t_1, t_2 is of total degree $n-1$ only. This shows that there exists the surface t of degree $n-1$, osculating ϕ along l, and having a simple point at O. In ψ we admit the term tf_1 where $f_1 \equiv (x,\ y,\ z,\ w)^1$, the three ratios of its coefficients being additional parameters. P is reduced by 3 owing to the binode at O. Now

$$\psi \equiv tf_1 + (z,\ w\overset{3}{)(}x,\ y,\ z,\ w)^{n-3};$$

it is not singular at O, but has instead a binode at the arbitrary point of l given by f_1.

Since the plane f_1 does not pass through O, the total intersection near O of ϕ, ψ is the same as that of ψ with $\Lambda \equiv \psi u_1 - \phi f_1$, on which l is triple. For each of ψ, Λ the point O is a general point of l, through which there passes no branch of their intersection except l counted three times. Hence c does not pass through O, but meets l only at simple points of ϕ. Since Λ is of total degree $n+1$ and t_1 of degree $n-1$, the number of these points is $4n-8$ as in the last case, and E is reduced by 6 owing to the binode O.

If the binode is of higher type, it does not affect the result, for the additional conditions fall only on the free terms $(z,\ w\overset{3}{)(}x,\ y,\ z,\ w)^{n-3}$. A triple or higher point on l would be a separate F-element. If we assume that the singularities of ϕ on l consist only of β_2 conic nodes and β_3 binodes of any type, then

$$P = 6n - 3 - 2\beta_2 - 3\beta_3; \quad E = 15n - 18 - 4\beta_2 - 6\beta_3.$$

Higher contact along a line can be treated by similar methods. Some examples will be found in **158**. The results are included in those for the more general case of contact along any curve, which now follow.

22. Simple general curve, s-point contact (150, 162, 163).

For a curve of higher degree, the work becomes laborious, unless there is a simple parametric representation of the curve. Details of the calculation are given in **162, 163**; the results are as follows.

If the problem is definite, we must be given the curve ω along which contact takes place, of degree m, rank r, with δ d.p.'s with distinct tangents; we assume it to have no other singularities; and also a surface f of degree μ passing through ω, and having β_a nodes of the type B_a ($a = 2, 3 \ldots s$; conic nodes are included as B_2's, the rest are binodes), which fall at simple points of ω, the edges not touching ω; and we require the postulation and equivalence of the condition of s-point contact with f along ω, for a family (ϕ) of degree n.

The general ϕ does not have any more complicated singularities; it has the same B's as f except for the B_s's, of which it has a different number, in variable positions. If f has a conic node, its section has infinite curvature, and so has that of ϕ; if ϕ has a simple point at a binode of f, it has $(s-1)$-point contact with one sheet of f, which shows that the binode is a B_s at least.

We admit in ϕ any multiple of f, and also any term which represents a surface on which ω is s-fold. Corresponding to each B_a of f, there exists a surface t_a, having a simple point instead of this binode, and therefore not a term already admitted, which has a-point contact with f along ω, and gives a special group, from which we can form other admissible groups. A careful reckoning shows that the reduction in P is $\frac{1}{2}\{s(s-\sigma)-\tau(\sigma+1)\}$, where σ is the quotient and τ the remainder in $s \div a$.

At a B_s of ϕ which is a simple point of ψ there is no branch of c. At a common B_a, if $\tau > 0$, there is one branch of c, touching the edge and meeting χ in $\tau(\sigma+1)$ points; if $\tau = 0$, there is no branch of c. There are also

$$(s+1)nm - s\rho - \sum_{a}(s-\sigma)\beta_a$$

other intersections of c, ω at simple points. The final expressions are

$$P = sP_\omega - \tfrac{1}{2}s(s-1)(\rho-\mu m) - \tfrac{1}{2}\sum_{a=2}^{s}\{s(s-\sigma)-\tau(\sigma+1)\}\beta_a,$$

$$E = sE_\omega - s(s-1)(\rho-\mu m) - \sum_{a=2}^{s}\{s(s-\sigma)-\tau(\sigma+1)\}\beta_a,$$

where P_ω, E_ω are the values of P, E for ω as a simple curve on (ϕ).

23. Double line, simple contact, $\mathbf{P} = 6n - 3 - \beta$, $\mathbf{E} = 21n - 36 - 3\beta$ (87).

The general surface having l as a double line is

$$\phi \equiv t_2 + (z, w\overset{3}{)\!(}x, y, z, w)^{n-3},$$

where
$$t_2 \equiv z^2 u_{n-2,1} + zw u_{n-2,2} + w^2 u_{n-2,3}.$$

The pair of tangent planes to ϕ at any point $(x, y, 0, 0)$ of l is given by t_2, with x, y as constants and z, w as current coordinates. If this pair of planes is the same for all members of (ϕ) at any point of l, then $u_{n-2,1} : u_{n-2,2} : u_{n-2,3}$ are fixed for all values of x/y. In general this implies that the $3(n-1)$ coefficients of $u_{n-2,a}$ are all in fixed ratios, giving $3n-4$ conditions more than for a double line. For a double line of contact,

$$P = (3n+1) + (3n-4) = 6n - 3.$$

But if all the $u_{n-2,\alpha}$ have a common factor u_β of degree β, a product of β linear functions of x, y, we have

$$\phi \equiv t_2 u_\beta + (z, w\overset{3}{\between} x, y, z, w)^{n-3},$$

where $$t_2 \equiv z^2 u_{n-2-\beta,1} + zw\, u_{n-2-\beta,2} + w^2 u_{n-2-\beta,3},$$

and it is only the coefficients of t_2 that are in fixed ratios, and the β ratios of the coefficients of u_β are additional parameters; P is reduced by β and becomes $6n-3-\beta$. Now if x, y, z, w are all variable, t_2 gives the simplest surface touched by (ϕ) along l; its degree is $n-\beta$. The point of l given by any linear factor of u_β is a triple point of ϕ; hence (ϕ) has β triple points on l, variable with the parameters of the family. We now assume

$$\psi = t_2 v_\beta + (z, w\overset{3}{\between} x, y, z, w)^{n-3}.$$

The total intersection of ϕ, ψ lies on $\Lambda \equiv \psi u_\beta - \phi v_\beta \equiv (z, w\overset{3}{\between} x, y, z, w)^{n-3+\beta}$, on which l is triple; hence l counts six times in the total intersection of ϕ, Λ, that is, of ϕ, ψ and of ϕ, u_β. Since l does not lie on u_β, it counts six times in the intersection of ϕ, ψ, and c is of degree n^2-6. Now c meets l where one sheet of ϕ osculates the corresponding sheet of ψ and touches one sheet of Λ; the condition that one of the two planes t_2 coincides with one of the three tangent planes to Λ is of degree

$$2(n-3+\beta)+3(n-2-\beta)=5n-12-\beta$$

in x/y. This is therefore the number of intersections of c, l, at each of which c touches one sheet of χ and crosses the other, and three intersections are absorbed. The number of free intersections of c, ψ is

$$(n^2-6)\,n-3(5n-12-\beta),$$

and $$E=21n-36-3\beta.$$

The β triple points of ϕ do not lie on c. Here the tangent cone to ϕ is a proper cubic cone with l as a double edge, along which it touches the pair of planes which is the tangent cone to ψ; the intersection of the two cones consists of l counted six times, and there is no branch of c. Neither does c pass through the pinch-points on l, at which the two tangent planes coincide and t_2 is a perfect square, for which the condition is of degree $2(n-2-\beta)$ in x/y. These $2n-4-2\beta$ pinch-points are the same for all the family; their number is reduced by 2 for each of the triple points. Since the two tangent planes coincide, it might appear at first sight as if each sheet of ϕ touched each sheet of ψ, giving eight branches of the intersection, but ϕ has not two distinct sheets, which would form a tacnode, but two half-sheets; the general section has a cusp, and a general curve touching the tangent plane meets ϕ in three points; there are still only six branches of the total intersection of ϕ, ψ, all coinciding with l, and c does not pass here. A tacnode would impose a condition on the coefficients of terms cubic in z, w, which are arbitrary in (ϕ); it does not occur on the general member.

The points of l of special interest are therefore

(i) $2n-4-2\beta$ pinch-points, the same for all of (ϕ) and for t the simplest surface touched by (ϕ); the two tangent planes coincide and the section has a cusp; c does not pass here.

(ii) β triple points, different for different members of (ϕ), ordinary points of t; the section of ϕ has a triple point; c does not pass here.

(iii) $5n-12-\beta$ points of intersection of c, l, different for every pair of surfaces ϕ, ψ; one sheet of ψ osculates one sheet of ϕ; ordinary points on each of ϕ, ψ, t; one branch of c passes through each.

24. Sheets of a surface along a double line (123).

At a general point O of a double line, the two sheets of ϕ are quite distinct. A short strip of ϕ along l near O can be transformed into two patches of another surface f', at finite distance from each other. Any path lying on the strip, joining a point P_1 of one sheet to a point P_2 of the other, has a d.p. on l, and P_1, P_2 lie on different branches. But if O is a pinch-point, P_1, P_2 can be joined by one branch of a curve, having a cusp at O, and the two patches of f' are merged in one.

If, however, O is of higher nature, the cusp may be replaced by a tacnode, and again the sheets may be distinct in the neighbourhood of O also. We shall see that O is then equivalent to two coincident pinch-points.

Thus though there are two distinct sheets of ϕ near a general point of l, when we consider the neighbourhood of the whole of l, there is only one sheet, or rather two connected half-sheets. In general, a surface on which l is simple cannot touch one half-sheet only of ϕ along l, unless the pinch-points all coincide in pairs, as for example when ϕ breaks up into two distinct surfaces through l (p. 258). The case of two distinct sheets bears much the same relation to a general double line that a binode does to a conic node.

If k is a multiple curve of ϕ of any nature, then, at a general point O of k, the section of ϕ by a general plane has a plane singularity of some kind, and the sheets of ϕ fall into cycles, answering to the distinct cycles of the section. If any surface approximates at O to one sheet of ϕ, it has a cycle of the same order as that to which the sheet belongs. There are a certain number of pinch-points of ϕ on k, where the singularity of the section is higher, two branches having contact of higher order than at O. In general, when ϕ is transformed into a surface f' and k into a set of curves simple on f', the neighbourhood of $O \sim$ as many distinct patches as there are cycles of the plane singularity at O. As O describes k, these patches describe strips which merge at places arising from pinch-points, and form a single connected sheet; but this may have distinct parts if the pinch-points are specialized.

Let ϕ have two distinct sheets, as regards the whole neighbourhood of a double line l, in the sense that there exists a surface f_μ on which l is simple, touching one of the tangent planes to ϕ but not the other, at all general points of l. Then for all values of x/y, the pair of tangent planes t_2 falls into two factors, one being the tangent plane to f_μ, say $t^{(1)} \equiv zu_{\mu-1} + wv_{\mu-1}$, which is rational in x, y. Then the second factor is also linear in z, w and rational in x, y, say $t^{(2)} \equiv zu_{\nu-1} + wv_{\nu-1}$ where $\mu + \nu = n$; and this expression gives another surface f_ν on which l is simple, touching the second sheet only of ϕ along l.

Now let the linear family (ϕ) be required to have this type of singular line l. As with a binode (p. 238) there are two alternatives. The first corresponds to a binode with two variable biplanes and a fixed edge; then $t^{(1)}$, $t^{(2)}$ are variable members of a fixed pencil. This is not a case of contact and l counts only four times in the intersection of ϕ, ψ. The form is

$$\psi \equiv (t^{(1)}, t^{(2)})^2 (x, y)^{n-2\mu} + (z, w\overset{3}{)(}x, y, z, w)^{n-3},$$

where $$t^{(1)} \equiv zu^{(1)}_{\mu-1} + wv^{(1)}_{\mu-1}, \qquad t^{(2)} \equiv zu^{(2)}_{\mu-1} + wv^{(2)}_{\mu-1}.$$

The three coefficients in $(t^{(1)}, t^{(2)})^2$ are parameters, and those in $u^{(1)}, \ldots, v^{(2)}$ are constants. Those in $(x, y)^{n-2\mu}$ are also constants: otherwise the family would not be linear; P is 3 less than for a triple line,

$$P = 2(3n-1) - 3 = 6n - 5.$$

The factor $(x, y)^{n-2\mu}$ determines $n - 2\mu$ fixed triple points of (ϕ) lying on l. Through each there pass five branches of c each meeting χ in three points. The other intersections of c, l are where one sheet of ϕ, touching say $t^{(1)} + at^{(2)}$, touches one sheet of ψ, touching say $t^{(1)} + bt^{(2)}$. Now $a \neq b$, since ϕ, ψ do not touch all along l; hence this can only be where $t^{(1)}$, $t^{(2)}$ are the same plane, at one of the $2\mu - 2$ fixed points given by $u^{(1)}v^{(2)} - u^{(2)}v^{(1)}$. Here ϕ, ψ each have a pinch-point, with the same double tangent plane.

Now the equation for the pinch-points, when ϕ has a general double line, reduces in our case to the square of this; so each of the points considered is a pair of coincident pinch-points both of ϕ and of ψ. All the lines of closest contact coincide with l; there are six branches of the total intersection, namely l counted four times and two branches of c, each touching l and osculating the tangent plane, and meeting χ in six points. The number of free intersections of c, ψ is

$$(n^2 - 4)n - 5 \, . \, 3 \, . \, (n - 2\mu) - 2 \, . \, 6 \, . \, (2\mu - 2) = n^3 - 19n + 6\mu + 24,$$

and $$E = 19n - 6\mu - 24.$$

Now for a general double line and $n - 2\mu$ triple points lying on it, we find by the usual methods

$$P_0 = 6n - 6\mu + 1, \qquad E_0 = 19n - 14\mu - 16.$$

In our case

$$P = P_0 + 3(2\mu - 2), \quad E = E_0 + 4(2\mu - 2);$$

thus each pinch-point contributes 3 to P and 4 to E, the same as for an F-point of contact. Our case can be regarded as the limit of a system consisting of a double line, $n - 2\mu$ triple points and $2\mu - 2$ points of contact, when all the points come to lie upon the line, the former at the points given by $(x, y)^{n-2\mu}$, and the latter at the points of contact of $t^{(1)}$, $t^{(2)}$. If n is odd, there is bound to be at least one triple point.

Now there are no variable intersections of c, l, so that l is an F-line of second species. An example of a triple F-curve of similar nature is given on p. 324.

25. Double line, partial contact, $\mathbf{P} = 4n - 2 + 2\mu$, $\mathbf{E} = 15n - 28 + 5\mu$ (87).

There is a second alternative, for the case when the pair of tangent planes falls into two factors rational both in z, w and in x, y; that is when only one is variable, say

$$t_2 \equiv t^{(1)} t^{(2)} \equiv (zu_{\mu-1,1} + wu_{\mu-1,2})(zv_{\nu-1,1} + wv_{\nu-1,2}),$$

where $\mu + \nu = n$, and the coefficients in $u_{\mu-1,a}$ are fixed and those in $v_{\nu-1,a}$ are parameters. Any factors corresponding to fixed triple points can be included in $t^{(1)}$; there are no variable triple points.

If x, y, z, w are all variable, $t^{(1)}$ represents a fixed surface of degree μ, with l as a simple line, touching one sheet of (ϕ) along l. The groups admissible in ψ are multiples of $t^{(1)}(z, w)^1$ and of $(z, w)^3$; we find

$$P = 4n - 2 + 2\mu.$$

Now l counts five times in the intersection of ϕ, ψ, and c is of degree $n^2 - 5$. The branches of c which do not touch $t^{(1)}$ meet l at the $2\nu - 2$ points of contact of two surfaces such as $t^{(2)}$, and each meets χ in two points. The branches touching $t^{(1)}$ meet l at the points of contact of $t^{(1)}$ with the surface $\Lambda \equiv (z, w \overset{4}{)(} x, y, z, w)^{n-4+\nu}$, formed by eliminating $t^{(1)}$ from ϕ, ψ; their number is $4(\mu - 1) + n - 4 + \nu = 2n + 3\mu - 8$, and, at each, c touches one sheet of χ and crosses the other. The number of free intersections of c, χ is

$$(n^2 - 5)n - 2(2\nu - 2) - 3(2n + 3\mu - 8) = n^3 - 15n - 5\mu + 28,$$

and

$$E = 15n + 5\mu - 28.$$

In this case also, the pinch-points of ϕ coincide in pairs, being given by $u_{\mu-1,1}v_{\nu-1,2} - u_{\mu-1,2}v_{\nu-1,1}$, of degree $n - 2$. Their positions are different on the different members of (ϕ), and c does not pass through them.

If ψ is to osculate the first sheet only of ϕ along l, then in ψ we admit multiples of $t^{(1)}(z, w)^2$ and of $(z, w)^4$; and also, if $n \geqq 2\mu$, of $(t^{(1)})^2$. If $n < 2\mu$, the second set of tangent planes t_2 becomes fixed also, and ψ touches but does not osculate the second sheet of ϕ. If ϕ has the special form

$$\phi \equiv f^{(1)} f^{(2)} + (z, w \overset{4}{)(} x, y, z, w)^{n-4},$$

where l is simple on $f^{(1)}$ and on $f^{(2)}$, then $f^{(1)}(z, w)^1$ are also admissible, and ψ can osculate one sheet of ϕ without touching the other. We have a variety of cases, corresponding in some sense to the different types of binodes, and distinguished by the increasing degree in z, w of the terms that must be neglected if ϕ is to be considered as degenerate. For an example see p. 290.

26. Examples, limiting cases.

As an example, consider some simple cases that arise as limits. Let (ϕ) contain a double line l and a simple line l_1 skew to l. For this system,

$$P=(3n+1)+(n+1)=4n+2, \quad E=(12n-16)+(3n-2)=15n-18.$$

The residual c_{n^2-5} meets l in $4n-8$ points and l_1 in $2n-2$ points. In the limit, when $l_1 \rightarrow l$, one sheet of ϕ touches the limit t of any quadric through l, l_1, so that $\mu=2$; the second sheet has a variable set of tangent planes. The $2n-2$ branches of c which met l_1 now meet l, lying on the sheet touching t, and the other $2n-6$ branches meeting l lie on the second sheet of ϕ. These values agree with those given above if we put $\mu=2$.

If l_1 meets l, the first sheet of ϕ touches the limit p of the plane through l, l_1, so that $\mu=1$. Of the $4n-8$ intersections of l with its total residual, one is with l_1 and $4n-9$ with c; of the $2n-2$ intersections of l_1 with its total residual, the intersection with the double line l absorbs three, and $2n-5$ are with c. In the limit, c has $4n-9$ branches meeting l, of which $2n-5$ touch p and $2n-4$ lie on the second sheet. These values again agree with those given above if we put $\mu=1$.

If ϕ contains two simple lines l_1, l_2 besides the double line l, all skew,

$$P=(3n+1)+2(n+1)=5n+3, \quad E=(12n-16)+2(3n-2)=18n-20.$$

The simplest surface containing the system is quartic, and in the limit the sheets of (ϕ) touch the pair of quadrics $t \equiv q_1 q_2$, limits of two passing through l, l_1 and l, l_2 respectively.

Now the general equation of a family touching t is

$$\phi_0 \equiv tu_{n-4}+(z, w\overset{3}{\between} x, y, z, w)^{n-3},$$

with $n-4$ variable triple points. This is a case of total simple contact; the fact that t breaks up does not affect the argument of p. 253; and $P=5n+1$, $E=18n-24$. Thus the surface ϕ_0 is too general, and the surface ϕ which arises in the limit is subject to two further conditions, absorbing four further intersections, beyond those of simple contact with t. We must look for a pair of points to account for the difference.

The only special pair are the two points of contact of q_1, q_2 lying on l; these are ordinary pinch-points of ϕ_0, through which c_0 does not pass; we shall show that they are tacnodes on ϕ.

Before the limit, l_1, l_2 lie on the two different sheets of ϕ; each of the $4n-8$ branches of c meeting l also meets one or both of l_1, l_2; but c meets each $2n-2$ times; hence there are just four branches of c meeting both l_1 and l_2. Each of these passes in the limit through a point of contact of q_1, q_2; hence two pass through each, and therefore eight branches of the total intersection of ϕ, ψ, namely l counted six times and these two branches of c. The point is therefore of a higher nature than a pinch-point, being a tacnode of both ϕ and t; there are two distinct sheets in contact. Now it requires one condition to convert a pinch-point into a tacnode; namely, that the tangent plane should be a component of the terms of third order; so the conversion of two pinch-points into tacnodes exactly accounts for the higher postulation. Also, each of the four branches of c through one or other tacnode meets χ in four points instead of three, which exactly accounts for the higher equivalence.

If there are d intersections of l with l_1 and l_2, and d_1 of l_1, l_2 with each other, $d=0$, 1 or 2, $d_1=0$ or 1; then before the limit,

$$P=(3n+1)+2(n+1)-2d-d_1, \quad E=(12n-16)+2(3n-2)-5d-2d_1.$$

If this leads in the limit to a case of simple contact of both sheets with β variable triple points, for which

$$P=6n-3-\beta, \quad E=21n-36-3\beta,$$

we must have

$$2d+d_1=\beta-n+6, \quad 5d+2d_1=3\beta-3n+16,$$

whence

$$d=\beta-n+4, \quad d_1=n-\beta-2, \quad d+d_1=2.$$

The only cases which can at all easily be interpreted geometrically are

$$d=2, \quad d_1=0, \quad \beta=n-2,$$

each sheet of ϕ touches a fixed plane; and

$$d=1, \quad d_1=1, \quad \beta=n-3,$$

one sheet of ϕ touches a fixed plane, and the other a fixed quadric.

But if ϕ touches two fixed quadrics, $\beta=n-4$, giving $d=0$, $d_1=2$, which has no obvious geometrical meaning (p. 298). Much work remains to be done in this region. For a few developments on higher contact along a multiple curve, see 163, 313.

CHAPTER XIII

THE PRINCIPAL SYSTEM

1. Principal elements.

In this chapter we consider the P-system in S that $\sim$ the F-system in S'; for convenience, we repeat some definitions and remarks given before.

An F-point O' has been defined as a point common to all the homaloids (ϕ'). It is an exception to the uniqueness of the transformation T, and its homologue in S is not a single point, but a locus, either a curve or a surface, called a *principal element* or P-element; the aggregate of these is the *P-system* and all its points are *P-points.* Every F-point is also principal (p. 162) but a P-point is not in general fundamental; it $\sim$ a definite unique point; and to all the ∞^1 or ∞^2 P-points of a P-element in S there corresponds the same definite point of S', which is an F-point.

A *P-surface* may either be a single P-element, or else a locus of curves, each of which is a P-element.

There are two important properties common to both kinds of P-element, which often give an easy way of discovering the P-system.

(i) If ϕ contains a general point of the element, it contains the whole of it.

(ii) It meets (ϕ) in F-points only.

These both hold because it $\sim$ a point, which presents only one condition to a plane through it, and does not lie on a general plane.

The converses also hold. (i) If ∞ points present the same condition to ϕ, this condition makes the corresponding plane pass through a single point, which $\sim$ the whole element and is fundamental. (ii) If the element has no free intersection with (ϕ), its homologue is not incident with a general plane, but is again a point.

Now the number of P-surfaces is finite, being $\leqq 4(n-1)$, the degree of J (p. 161); hence if j meets (ϕ) in F-curves only, it is completely determined by its degree and its multiplicities at the F-curves: for example, the P-plane of the F-conic in T_{2-2}. These may not present independent conditions to j, whose existence may constitute a special relation between them. For example, the F-curve ω_6 of the bilinear T_{3-3} may break up and include the three sides of a triangle; then their plane meets (ϕ_3) in F-lines only, and is principal. It is determined by any two of the three lines, and contains the third as a necessary consequence; and the existence of the plane through the three lines makes them meet in pairs.

In the same way, a curve κ meeting (ϕ) in F-points only either lies on a j meeting (ϕ) in F-curves only, and $\sim$ an isolated F-point O', or else $\kappa \sim$ a point O' of an F-curve ω', and κ generates a j which $\sim \omega'$, and which is completely determined by the degree of κ and its incidences with F-elements, whether fixed or variable on F-curves. For example, in the bilinear T_{3-3}, ω_6 may consist of four skew lines and their two transversals; a line meeting any three out of the four does not meet the general ϕ_3 again, and is a P-line. Its locus is the quadric containing the three lines, and there are four such P-surfaces, which however are not P-elements.

Now if the P-element is a surface j_μ of degree $\mu > 0$, a general line meets j in μ points; hence c' has μ branches through O', which is an isolated F-point, as defined on p. 162, whether it lies on any F-curve or not. But if the P-element is a curve κ, we can put $\mu = 0$, for κ does not meet a general line, and c' does not pass through O', which is a general point of an F-curve ω'.

If ω' is of first species, then as O' describes ω', the curve κ moves and describes a P-surface j_d, of the second kind mentioned above, which $\sim$ the whole of ω' considered as a single F-element. The degree d of j is the number of variable intersections of ω', c'; and $d > 0$, since ω' is of first species. But if ω' is of second species, it does not meet c', and $d = 0$; then κ does not describe a surface, but is fixed, all its points corresponding equally to all the points of ω'; hence κ is also an F-curve of second species.

Thus there are a finite number of P-elements which are surfaces, or none if H' has no isolated F-points; and ∞ P-elements which are curves, or a finite number if H' has no F-curves of first species, or none if H' has no F-curves at all. It follows that if H' has any isolated point O', then H has an F-curve, namely the intersection of j, ϕ; and, except for the linear transformation, which has no F-system at all, every Cremona transformation has at least one F-curve, in one or other space.

Now O' is $(4i' - 2)$-fold on J', and there is at least one j' through O'. This meets ϕ' in a curve through O' which is not fundamental, if no F-curve passes through O'; and $j' \sim$ an F-curve, not an F-point. Again we see that either H or H' has an F-curve (p. 160).

Thus if H' consists of isolated points only, and has no F-curve, then in S' no P-surface meets (ϕ') in F-curves, and in S there are no isolated F-points; H consists of F-curves only. This is the case when n' has its minimum value $\sqrt{n}$.

2. Symmetric groups of F- and P-elements.

The formulae of p. 225 are extensions to space of the equality of the total number of F-points in a plane transformation (p. 22). The incidence relations can also be extended, but are unwieldy, as the incidences of F- and P-elements are of so many different kinds, except for particularly simple types, such as ordinary and punctual transformations (pp. 274, 318). There is however an immediate extension to space of the geometrical treatment of the partitioning of symmetric groups of F-elements by associated P-elements.

A pair of F-elements O_1, O_2, either ordinary points or curves, or contact conditions, are symmetric if T is unaltered when O_1, O_2 are exchanged in its definition. The symmetry must extend to the elements themselves, and to their relations to (ϕ) and to the rest of H. For example, let the F-sextic of the bilinear T_{3-3} break up into a line of osculation l and three skew lines l_1, l_2, l_3 all meeting l (158, T 1). There are two distinct cases; in the first, the three are symmetric, all generators of the fixed quadric t_2 touched by (ϕ) along l, and meeting l in simple points of (ϕ). Now H' consists of a conic and four lines all meeting in a point, but the four are not symmetric; three lie in a plane through the tangent line to the conic and the fourth does not. We have one symmetric group of three elements in each space. In the second case, H' consists of a conic and two lines of contact, forming a symmetric pair; now l_1, l_2, l_3 are not symmetric; l_3 lies on the fixed tangent plane t_1 to (ϕ) along l, and meets l in a simple point of (ϕ); and l_1, l_2 are a symmetric pair, not lying on t_1, meeting l in d.p.'s of (ϕ).

Since O_1, O_2 are different, they $\sim$ P-elements j_1', j_2', which behave differently to at least one element O_1' of H', and $O_1' \sim$ a P-element j_1 which behaves differently to O_1, O_2. By exchanging these in the definition of j_1, we obtain a different P-element j_2; for the set of conditions are still compatible: if not, the fact would constitute a difference of relation of O_1, O_2 to the other F-elements, which would destroy their symmetry. Hence j_1, j_2 exist and behave alike to every other element of H except O_1, O_2.

In the same way, if there is a group of just α F-elements $O_1 \dots O_\alpha$, every pair of which are symmetric, we deduce a symmetric group in H' of exactly α elements, which may be of quite different nature from the first group. If $\alpha > 2$, each element O or O' selects an associated element from the associated group.

The proof breaks down if $O_1 \dots$ are F-curves of second species; then each $\sim$ an F-curve of second species of H', and we still have associated groups and elements.

There is a more complicated kind of symmetry possible, of sets instead of single elements. If each set is placed in a column, and the group of columns in an array

$$\begin{array}{c} O_{11} \ \dots \ O_{1\alpha} \\ \dots\dots\dots\dots \\ O_{\beta 1} \ \dots \ O_{\beta\alpha}, \end{array}$$

the group is symmetric if T is unaltered when each of the set in any one column is exchanged with the element in the same row in any other column. We deduce a symmetric group of α sets in H'; it does not follow that the number of elements in each of these sets is also β. It can be shown (p. 277) that the number of irrational elements, and their genera, are the same in both groups; but we say nothing about the numbers of rational elements. The elements in each array are those which enter into the definitions of the P-elements which $\sim$ the other array, in such a way as to distinguish them from one another.

3. P-surface corresponding to an isolated F-point.

Consider an isolated i'-fold F-point O' of H', through which there pass μ branches of c' and possibly branches of ω'. The corresponding P-element is a surface j_μ of degree μ. For example, j is a plane if $\mu = 1$; then O' may be a simple isolated point, or a point of contact lying on a simple F-curve, or a d.p. of (ϕ') through which there pass three branches of simple F-curves, and so on.

We can express i' in terms of numbers belonging to S as follows. Any line through O' meets a general ϕ' in $n' - i'$ variable points. Now n' is known from S, being the degree of c; we want to express the number of free intersections of ϕ' with any two planes through O'. A plane through O' $\sim$ a ϕ_n which breaks up into $j_\mu \phi_{n-\mu}$, where $(\phi_{n-\mu})$ is a net. Let ν be the total degree of its base curves, which can be found when (ϕ) and j are known. Then $(n-\mu)^2 - \nu$ is the degree of the variable intersection of $\phi_{n-\mu}$, $\psi_{n-\mu}$, that is, the number of free intersections of ϕ, ψ and a general plane.

Hence
$$n' - i' = (n-\mu)^2 - \nu,$$
$$i' = n' - (n-\mu)^2 + \nu.$$

Now if any F-curve ω_m is i_j-fold on j, it is at least $(i - i_j)$-fold on $\phi_{n-\mu}$. Also, if O' lies on any F-curve ω', the P-curve κ, which $\sim$ O' regarded as a point of ω', lies on $j\phi_{n-\mu}$; and if it or any part of it does not lie on j, it is a base curve of $(\phi_{n-\mu})$; or if it coincides with an ω, it may increase the multiplicity of ω in the base. Hence
$$\nu \geqq \Sigma (i - i_j)^2 m,$$
summed for all the F-curves. Since j meets ϕ in F-curves only,
$$\mu n = \Sigma i i_j m;$$
also
$$n^2 - n' = \Sigma i^2 m;$$
hence
$$i' \geqq \Sigma i_j^2 m - \mu^2.$$

For example, if j is a plane and O' an isolated simple point,
$$\mu = 1, \quad i' = 1, \quad 2 \geqq \Sigma i_j^2 m, \quad i_j = 0 \text{ or } 1.$$
The fundamental plane section of ϕ by j_1 is a curve of degree $\leqq 2$; it cannot consist of one line only; it is therefore a conic or a pair of lines, each repeated a certain number of times. The conic can occur only if n is even.

4. P-plane corresponding to a simple F-point.

First consider the simplest case. Let O' be an isolated simple F-point $(0, 0, 0, 1)$, without contact; then c' has one branch through O', and ω' has none. The term in w'^n is absent from ϕ', but those in $w'^{n-1}(x', y', z')$ are present, and give the variable tangent plane to ϕ' at O'. We can choose the frame of reference in S so that the terms linear in x', y', z' in

$$\begin{array}{lcccc} & \phi_1', & \phi_2', & \phi_3', & \phi_4' \\ \text{are} & x'w'^{n-1}, & y'w'^{n-1}, & z'w'^{n-1}, & 0. \end{array}$$

Consider a definite line l' through O', say $x' : y' : z' = \xi : \eta : \zeta$. Let $P'(\xi\epsilon, \eta\epsilon, \zeta\epsilon, 1) \to O'$ along l'; then $\epsilon \to 0$ and the homologue $P \to Q(\xi, \eta, \zeta, 0)$, which is a definite point of the plane w, and is associated with the direction of l' at O'. Thus $O' \sim$ the whole plane w, which is the corresponding P-surface j_1, whose points Q are in 1, 1 relation to the star of lines l' at O'. As P' moves on l', so P moves on a curve c; hence every c meets j_1 in just one free point Q, the other intersections being F-points.

Consider a plane p', $ax' + by' + cz' = 0$, through O', which is a locus of lines l'. The homologue is a ϕ which breaks up into j and a ϕ_{n-1} which is a locus of curves c, and whose intersection with j consists of F-curves and the line $ax + by + cz = w = 0$. This line is double on the degenerate ϕ, and every Q is a d.p. on ∞^1 degenerate ϕ's, which $\sim$ the ∞^1 planes through l'.

Next, let $P' \to O'$ along any path, and let the tangent at O' to the path be l'; then $P \to$ the same point Q as before. In particular, if the curve is c', to touch l' presents two conditions, and determines an ∞^2 set of the ∞^4 curves c'; these $\sim$ the ∞^2 lines through Q.

Since the P-plane meets ϕ in F-curves only, (ϕ) has a fundamental plane section.

Since c' has one branch through O', any surface through O', say

$$f_\mu' \equiv (x', y', z' \between x', y', z', w')^{\mu-1},$$

loses one free intersection with c'; its homologue f loses one intersection with any line, and its degree is lowered by 1. From the result of substitution

$$(\phi_1, \phi_2, \phi_3 \between \phi_1, \phi_2, \phi_3, \phi_4)^{\mu-1},$$

the P-plane j_1 always breaks away; hence j_1 is a common factor of ϕ_1, ϕ_2, ϕ_3, and is their H.C.F.; it is not a factor of ϕ_4, since the transformation is not degenerate.

5. P-surface corresponding to an F-point of contact.

Next, let O' be a simple F-point of contact, not lying on ω', and therefore with two branches of c' through it. Let x' be the fixed tangent plane; then

$$\phi' \equiv w'^{n-1}x' + w'^{n-2}u_2(x', y', z') + \dots .$$

By a suitable choice of the frame of reference in S, we may take ϕ_4' of this form, and

$$\phi_\alpha' \equiv w'^{n-2}u_{2,\alpha}(x', y', z') + \dots, \quad (\alpha = 1, 2, 3).$$

Then if $P'(\xi\epsilon, \eta\epsilon, \zeta\epsilon, 1) \rightarrow O'$ along the general line l', the coordinates of P are to a first approximation

$$(u_{2,1}(\xi, \eta, \zeta)\,\epsilon^2,\ u_{2,2}\epsilon^2,\ u_{2,3}\epsilon^2,\ \xi\epsilon),$$

and $P \rightarrow$ the fixed point $A_4(0, 0, 0, 1)$, provided ξ is finite.

But if $\xi = 0$, and l' lies in x', all the coordinates of P are small of the same order as ϵ^2. If ξ is small of the same order as ϵ, say $\xi \equiv \xi_1\epsilon$, then $P'(\xi_1\epsilon^2, \eta\epsilon, \zeta\epsilon, 1) \rightarrow O'$ along a conic touching x', and $P \rightarrow$ a definite point

$$Q(u_{2,1}(0, \eta, \zeta),\ u_{2,2},\ u_{2,3},\ \xi_1 + u_2(0, \eta, \zeta)),$$

which is a general point of a quadric cone j_2, vertex A_4, which is the P-surface which $\sim O'$, and whose points are in 1, 1 relation to the set of conics determined by the two parameters $\xi_1 : \eta : \zeta$.

Now P has the same limit Q if P' moves on any path osculating the associated conic. In particular, if P' moves on c', then P moves on a line whose two intersections with j_2 are associated with the two passages of c' through O'.

If f' passes through O', it loses two free intersections with c', the degree of f is lowered by 2, the P-quadric j_2 being dropped; hence j_2 is the H.C.F. of ϕ_1, ϕ_2, ϕ_3. Since both branches of c' touch x', if f' also touches x', it touches each branch of c', two more free intersections are lost, and another quadratic factor, q say, is dropped.

Now
$$f' \equiv w'^{\mu-1}x' + (x', y', z' \overset{2}{\between} x', y', z', w')^{\mu-2}$$

and the result of substitution is

$$j_2 q f \equiv \phi_4^{\mu-1}\phi_1 + (\phi_1, \phi_2, \phi_3 \overset{2}{\between} \phi_1, \phi_2, \phi_3, \phi_4)^{\mu-2}.$$

Hence q is a common quadratic factor of ϕ_1, ϕ_2, ϕ_3, and therefore identical with j_2; and since it is not a factor of ϕ_4, it is a repeated factor of ϕ_1.

Regarded as a P-surface, j_2 plays a double part, corresponding both to the F-point O', and also to the contact condition at O'. For an example, see p. 292.

If O' lies on ω', then of the two branches through O' of the total intersection of ϕ', ψ', one is ω' and one c'; and j_1 is a plane. The system ω', O' is in many ways equivalent to ω' and a distinct simple F-point O' lying near it; as $O' \rightarrow$ a point of ω', one of the variable intersections of ω', c' becomes fixed at O', and the degree of the P-surface j_d, which $\sim$ all the general points of ω', is reduced by 1, the plane j_1 breaking off from it.

For example, let ω' be the line $x' = y' = 0$. With the above notation, each u_2 lacks the term in z'^2; the coordinates of Q are linear in $\eta, \zeta, \xi_1/\eta$. If $P' \rightarrow O'$ along a path touching ω', then both ξ and η are small, say $\xi \equiv \xi_1\epsilon$, $\eta \equiv \eta_1\epsilon$. Since $u_2(0, 0, \zeta) \equiv 0$, we find that Q is $(0, 0, 0, \xi_1)$ and coincides with A_4 unless ξ_1 is small. But if $\xi_1 \equiv \xi_2\epsilon$, and $P'(\xi_2\epsilon^3, \eta_1\epsilon^2, \zeta\epsilon, 1) \rightarrow O'$ along a path touching ω' and osculating x', then u_2 approximates to $a\eta_1\zeta\epsilon^3$, and Q becomes $(a_1\eta_1\zeta,\ a_2\eta_1\zeta,\ a_3\eta_1\zeta,\ \xi_2 + a\eta_1\zeta)$, a definite point of a line κ_1

through A_4, lying on j_1 and also on j_d, and is the limit, as $Q' \rightarrow O'$, of the P-line which $\sim$ a general point Q' of ω'.

If ω' is of higher multiplicity i', a similar analysis shows that as $Q' \rightarrow O'$, the corresponding P-curve $\kappa_{i'}$ breaks up into a line κ_1 lying on j_1 and a residual $\kappa_{i'-1}$, which in general does not. If a surface f' passes through O', then j_1 is dropped, and the proper homologue f contains $\kappa_{i'-1}$. Thus to O' there corresponds, not a single P-element, but a P-system composed of j_1, $\kappa_{i'-1}$.

If f' touches x' at O', then j_1^2 is dropped, and f contains $\kappa_{i'-1}$. If f' contains the whole of ω', and therefore also O', both j_1 and j_d are dropped, and f need not contain $\kappa_{i'-1}$; and if f', like ϕ', passes through ω' and also touches x' at O', then $j_d j_1^2$ is dropped. This case is exemplified by T_{2-2}, when the F-point lies on the F-conic.

If two branches of ω' pass through O', and therefore none of c', it is no longer an isolated F-point, and the corresponding P-element is no longer a surface but a curve κ, being a particular member of the sets of P-curves which $\sim$ the two branches of ω'. It is either a double curve on a P-surface, or a curve of intersection of two distinct P-surfaces, according as the two branches of ω' belong to the same or different F-curves. The contact at O' is a necessary consequence of the passage of (ϕ') through ω', and does not present any further conditions to either (ϕ') or (c'); it has no effect on either postulation or equivalence, and is not to be regarded as a separate F-element, but rather as entirely incidental to ω'. This case is exemplified by T_{2-2}, when the F-conic breaks up.

6. P-surface corresponding to a multiple F-point.

If O' is a double F-point, not lying on ω', then c' has four branches through O', and the corresponding P-surface is a quartic j_4. We have

$$\phi_a' \equiv w'^{n-2} u_{2,a}(x', y', z') + \dots \quad (\alpha = 1 \dots 4),$$

and if $P' \rightarrow O'$ along l', then $P \rightarrow Q$, whose coordinates are $u_{2,a}(\xi, \eta, \zeta)$, quadratic functions of the two parameters $\xi : \eta : \zeta$, and whose locus is therefore a rational quartic, its points being in 1, 1 relation to the star of lines at O'.

If O' lies on ω', one of the four branches through O' of the total intersection of ϕ', ψ' belongs to ω', and three to c'; hence j is cubic. The four cones $u_{2,a}(\xi, \eta, \zeta)$ have a common generator, namely the tangent to ω', and the degree of the eliminant is reduced by 1. If there are two branches of ω' through O', then j is a quadric; in this case, the passage of (ϕ') through ω' would make O' an F-point of contact; one further condition makes it a d.p.

If there are three branches of ω' through O', not all touching the same plane, j is a plane. Now the d.p. at O' is a necessary consequence of the passage of (ϕ') through ω'; but though O' does not affect the postulation, it is still an isolated F-point, affecting the equivalence, and it still $\sim$ a P-surface. But if all four branches belong to ω', then O' ceases to be an isolated F-point, and it $\sim$ a P-curve κ which is common to all the sheets of P-surfaces which $\sim$ the four branches of ω'.

In the same way if O' is an i'-fold F-point, not lying on ω', then $O' \sim$ a rational surface j of degree i'^2, whose points are again associated with the star of lines at O'. If ω' has α branches through O', whose directions are not such as to make the tangent cone have a fixed part, then the degree of j is reduced by α.

The P-curve which $\sim$ a general point Q' of any of the α branches has a definite limit as $Q' \rightarrow O'$; if any component of one of these does not lie on j, it must lie on the proper homologue of any surface through O', and is part of the P-system which $\sim O'$. In the extreme case, ω' has i'^2 branches through O', whose tangents lie on ∞^1 proper cones of degree i'; then O' ceases to be an isolated F-point and j is replaced by a P-curve which is i'^2-fold on the aggregate of P-surfaces which $\sim \omega'$.

Different features arise if the tangent cone has a fixed part. For example, let O' be a binode with a fixed biplane x' and a variable biplane u_1':

$$\phi' \equiv w'^{n-2} x' u_1(x', y', z') + w'^{n-3} u_3' + \dots.$$

If $P'(\xi\epsilon, \eta\epsilon, \zeta\epsilon, 1) \rightarrow O'$ along l', then $P \rightarrow Q$ whose coordinates have the form $\xi u_1(\xi, \eta, \zeta)$, and whose locus is a plane j_1, provided $\xi \neq 0$. But if $\xi \equiv \xi_1\epsilon$ is small, and $P'(\xi_1\epsilon^2, \eta\epsilon, \zeta\epsilon, 1) \rightarrow O'$ along a path touching x', then $P \rightarrow Q$ whose coordinates are $\xi_1 u_1(0, \eta, \zeta) + u_3(0, \eta, \zeta)$, and whose locus is a quartic surface j_4. This answers to the fact that there is one branch of c' touching u_1', and four touching x'.

If f' passes through O', both P-surfaces are dropped, and $j_1 j_4$ is a factor of ϕ_1, ϕ_2, ϕ_3. If f' also touches x', then $j_1 j_4^2$ is dropped, and this is a factor of ϕ_1.

Any branch of ω' through the multiple point O' reduces the degree of the P-surface corresponding to the sheet of the tangent cone to ϕ' touched by ω'. If all the branches touching this sheet belong to ω' and not to c', the P-surface disappears. If any part κ of a P-curve, which $\sim O'$ regarded as a point of a branch of ω', does not lie on the surviving aggregate of P-surfaces j, then κ is a separate element of the P-system $\Sigma j, \Sigma\kappa$ which $\sim O'$. An example is worked out on p. 342.

7. P-curve corresponding to a point of an F-curve of first species.

Let O' be a general point of an F-curve $\omega'_{m'}$, of first species and of multiplicity i'. The corresponding P-element is a curve κ_β of degree β whose locus is a P-surface j_d which $\sim$ the whole of ω'.

The simplest case is when $O'(\xi, \eta, 0, 0)$ is a general point of a simple F-line ω_1', $z' = w' = 0$. Then

$$\phi' \equiv (z', w' \between x', y', z', w')^{n'-1}.$$

If $P'(\xi + \xi_1\epsilon, \eta + \eta_1\epsilon, \zeta\epsilon, \theta\epsilon) \rightarrow O'$ as $\epsilon \rightarrow 0$, then $P \rightarrow Q$, whose coordinates are of the form $(\zeta, \theta \between \xi, \eta, 0, 0)^{n-1}$, which are rational in the two parameters

ξ/η and ζ/θ. If O' and therefore ξ/η are fixed, as ζ/θ varies Q describes a line κ_1, whose points are in 1, 1 relation to the pencil of planes through ω_1', and whose locus as ξ/η varies is a surface j_d, of degree $d = 2n-2$ unless some reduction takes place; this being the number of intersections of ω_1' with its residual. If these are all variable, there is no reduction; but if α of them are fixed, being either intersections with other curves ω', or fixed intersections with c', at points of contact or of higher multiplicity lying on ω_1', then d is reduced by α. Since ω_1' is of first species, some of its intersections with c' are variable, $d > 0$, and j is always a surface.

More generally, let O' (0, 0, 0, 1) be a general point of an ordinary i-fold curve ω', and let λ', $y' = z' = 0$, be the tangent to ω' at O'. Then

$$\phi_a' \equiv w'^{\,n-i}\, u_{i,a}(y', z') + (x', y', z' \overset{i+1}{)(} x', y', z', w')^{n-i-1}.$$

Let $P' \rightarrow O'$ along the line l', $x' : y' : z' = \xi : \eta : \zeta$; then $P \rightarrow$ the definite point Q, whose coordinates are $u_{i,a}(\eta, \zeta)$. This is fixed if η/ζ is fixed, that is, if P' lies on a fixed plane p' through λ', and the locus of Q is a curve κ of degree i as η/ζ varies, that is, as p' rotates round λ'.

If P moves on a plane through Q, then P' moves on a homaloid having one sheet touching p' at O'.

This fails if O' is an intersection of two simple F-curves ω_1', ω_2'; here every ϕ' touches the plane t' containing the tangent lines λ_1', λ_2' to ω_1', ω_2'; all the functions $u_{1,a}$ are multiples of t', and Q is fixed. Now $O' \sim$ a P-line κ, part of the intersection of j_1, j_2, and a common particular position of κ_1, κ_2. The association of points of κ_1 with planes through λ_1' is improper at O', and also that of points of κ_2 with planes through λ_2'; every point of κ is associated with t', and every plane through either λ_1' or λ_2' with one special point of κ.

In the same way, if ω_1', ω_2' are of equal multiplicities $i_1' = i_2'$, the tangent cone to (ϕ') is $t'^{i_1'}$, and is fixed; the P-curves κ_1, κ_2, both of degree i_1', coincide in κ, every point of which is associated with t'.

But if $i_1' > i_2'$, the tangent cone consists of $t'^{i_2'}$ and $i_1' - i_2'$ variable planes through λ_1'; and κ_1 breaks up into κ_2, all of whose points are associated with t', and a residual of degree $i_1' - i_2'$, whose points are in a proper 1, 1 relation to the planes through λ_1'.

If ω' is a general curve, since any plane meets it in m' points, any ϕ contains m' curves such as κ. Its remaining intersection with j consists of F-curves only; for if ϕ contains any general point Q of j, it contains the whole of the particular κ through Q, all of whose points present the same condition to ϕ. The degree d of j is, as we have seen, the number of variable intersections of ω', c'.

Each κ is rational; if ω' is ordinary, the points of κ are in 1, 1 relation to the pencil of planes through the tangent line to ω'; but j need not be rational.

The degree β of κ is the number of times that O' must be considered incident with any ϕ'. This is a loose phrase that requires careful interpretation. It is only in the simpler cases that it is the same as the multiplicity i' of O' on (ϕ').

The intersection of κ with a general plane p consists of β points Q, each of which $\sim O'$. The neighbourhood of κ on p consists of β quite separate patches, with no continuous passage from one to the other, which $\sim$ the same number β of patches of ϕ', all adjacent to O', but distinct, with no continuous passage from one to the other, belonging to different sheets of ϕ'. Corresponding to a line l in p through any particular Q, there is a c' through O' lying on the corresponding patch of ϕ', and therefore there is a ψ' meeting ϕ' in c', one of whose i' sheets touches this sheet of ϕ'. In general, l meets κ in one point Q only, and there is not another pair of sheets of ϕ', ψ' in contact at O'. As l rotates round Q in p, this c' rotates round O', describing this sheet of ϕ'; the other sheets of ϕ' correspond not to Q but to the $\beta-1$ other intersections of p, κ. If ω' is an ordinary curve, the i' tangent planes to ϕ' at O' are distinct, there are i' distinct patches of ϕ'; hence in this case $\beta=i'$.

If p touches κ, two of the patches in S are merged into one; hence at O', two sheets of ϕ' coalesce, and O' is a pinch-point of this ϕ'.

But if ω' is a cuspidal F-curve of simplest type, then at the general point O', two tangent planes are coincident for every ϕ', and the number of separate patches of ϕ' is reduced by 1. Then as l rotates once about Q in p, so c' rotates twice about O' on ϕ', always touching the double tangent plane, and describing the two half-sheets of ϕ', there being a continuous passage from one to the other.

A similar description applies to cycles of more than two sheets; in such cases we have $\beta<i'$.

For example, in the T_{3-3},

$$\begin{aligned} x:y:z:w &= x'z'^2 : y'z'^2 : z'w'(z'+w') : w'^2(z'+w'),\\ x':y':z':w' &= xw(z+w) : yw(z+w) : z^3 : z^2w, \end{aligned}$$

there is a cuspidal F-line ω_1', $z'=w'=0$, with z' as fixed double tangent plane to every ϕ' at every point of ω_1'. The homologue of a general plane through a fixed point $O'(\xi, \eta, 0, 0)$ of ω_1' is

$$\phi \equiv c_1(\eta x-\xi y)\, w(z+w)+(c_3z+c_4w)\, z^2,$$

containing the P-line κ_1, $z=\eta x-\xi y=0$, along which it touches the plane $\eta x-\xi y$, which is fixed when O' is fixed. Here

$$i'=2, \quad \beta=1.$$

The locus of κ_1 as O' varies is the plane j_1, $z=0$.

Since any two ϕ's through κ_1 touch the same plane along it, and it counts twice in their intersection, we can regard κ_1 taken twice as the P-conic which $\sim$ the point O' of the double F-line.

In general, consider two different sheets of ϕ' at O', corresponding to Q_1, Q_2, two of the β intersections of p, κ. In S, any fixed pencil of planes meets p in a pencil of lines l through a fixed point P of p; let l rotate about P from the position PQ_1 to PQ_2; then one of its intersections Q with j describes a continuous arc Q_1Q_2 of the curve of intersection of p, j, and in each position determines one κ through Q.

In S', we have a pencil of homaloids ψ', each meeting ϕ' in a c'; in the first position, one of the intersections of c', ω' lies at O', and on the first sheet of ϕ'. As l rotates, this intersection moves away from O' through a series of other positions Q' on ω', and then returns to O'; the adjacent part of c' moves continuously on ϕ', starting from the first sheet of ϕ' at O' and returning to the second sheet. It has therefore, at some other point Q' of ω', passed from one sheet to the other, Q' being a pinch-point or point of higher multiplicity of ϕ', and corresponding to a position of PQ which touches, properly or improperly, the κ through Q. Thus any two sheets of ϕ', on which it is possible for a branch of c' to lie, become merged at some other point of ω'.

There may be other sheets of ϕ' through ω', but they cannot contain any position of c'; in respect to them, ω' behaves as if of second species. They contribute nothing to the degree of κ, and if there is such a sheet, again $\beta < i'$. This occurs in the example of Chapter XV. The strip along ω' of this sheet of ϕ' corresponds to a part of p not met by any general l, that is, a patch and not a strip, lying about an F-point, and whose locus as p varies is a strip along an F-curve also behaving as if of second species in regard to the sheet to which this strip belongs.

To find what circumstance in S does always answer to the multiplicity i' of ω', consider, as on p. 263, a general line l' through O', meeting ϕ' in $n' - i'$ other points. Their homologues are the free intersections of a general plane with the variable intersection, of degree $n' - i'$, of two homaloids of the net through κ. The homaloidal twisted curve $c_{n'}$ that $\sim l'$ breaks up into this and a fixed part, consisting of κ, which may be repeated, and possibly some F-curves; and i' is its total degree. Hence $i' \geqq \beta$.

8. F-curves of second species.

If $\omega'_{m'}$ is of second species, the corresponding P-element is also an F-curve ω_m of second species, and every point of each curve corresponds to each point of the other.

The degree of ω is the number of intersections with a plane, and is equal to the number of times that ω' must be considered incident with ϕ'; as before, this is not necessarily the multiplicity i' of ω' on ϕ'. It was thought to be so by Cremona and subsequent writers, till the other possibility was recently pointed out (272).

Since now the neighbourhood of any point O of $\omega \sim$ the whole neighbourhood of ω', two sheets of ϕ' may constitute a single incidence for the present purpose, if there is continuous passage from one to the other at any point of ω', without the two sheets coalescing at every point.

There is a simple example with $i' > m$, a common particular case of two types of T_{3-3} discussed by Cremona (85), where ω is a double line on a cubic homaloidal family.

Consider the T_{3-3} given by the symmetrical bilinear relations

$$xz' = zx', \quad yw' = wy', \quad zz' = ww',$$

or explicitly, $$x' : y' : z' : w' = xw^2 : yz^2 : zw^2 : z^2w.$$

The general homaloid

$$\phi \equiv c_1xw^2 + c_2yz^2 + zw\,(c_3w + c_4z)$$

has the double F-line ω_1, $z = w = 0$. As $\epsilon \to 0$, the point $P\,(\xi, \eta, \zeta\epsilon, \theta\epsilon) \to O\,(\xi, \eta, 0, 0)$ along the line

$$\eta x - \xi y = \theta z - \zeta w = 0.$$

The homologue of P is $P'\,(\xi\theta^2, \eta\zeta^2, \zeta\theta^2\epsilon, \zeta^2\theta\epsilon)$, which $\to O'\,(\xi\theta^2, \eta\zeta^2, 0, 0)$, a general point of ω_1', $z' = w' = 0$, along the line

$$\eta\zeta^2x' - \xi\theta^2y' = \zeta z' - \theta w' = 0.$$

Hence ω_1 is a P-curve of second species, and each of its points corresponds to every point of ω_1'.

The symmetry of the bilinear relations shows that the nature of ω_1' in S' is just the same; we have a pair of F-lines of second species, each of degree 1 and multiplicity 2, so that $i' = 2m$, $i = 2m'$.

If O is fixed, and P rotates round O in the fixed plane p, $\eta x - \xi y$, then ξ/η is fixed, and ζ/θ takes every value once and only once; P' describes the whole of ω_1' twice, passing any point O' for two equal and opposite values of ζ/θ; but the paths of approach of P' to O' are different for the two passages, lying on the two different sheets of the ϕ' which $\sim p$. The homologue of a small closed curve in p surrounding O is a curve close to ω_1' and covering it twice.

9. Relation between multiplicity of ω and degree of ω' (225·1, 272).

More generally, let ω be of degree m and multiplicity i; first let there be no contact conditions, so that at a general point O of ω, all directions, except that of the tangent to ω, are on the same footing for the transformation. Let a plane p meet ω in m points O. As $P \to O$ by a path in p, so $P' \to$ some point O' of ω' by a path lying on the definite ϕ' which $\sim p$. If $P \to O$ in turn from all directions in p, so $P' \to$ each point O' of ω' in turn, but the correspondence between directions round O and points of ω' need not be 1, 1. On account of the uniqueness of the transformation, one direction at O gives a unique point O', but the correspondence is not necessarily unique the other way. As OP rotates about O, so O' runs along ω', and may describe it more than once. A certain number, β say, of directions round O may correspond to the same O' counted β times, and approached along different paths lying on β different sheets of ϕ'.

On account of the generality assumed, β is the same for every pair of points O, O'; the i' sheets of ϕ' fall into m sets of β each, corresponding to the m intersections of p, ω. Thus $i' = \beta m$, and similarly $i = \beta' m'$, where β, β' are integers. We can further show $\beta = \beta'$. For the intersection of p with a general ϕ is a plane curve k having i branches through each of m points of ω; the homologue k' is the intersection of ϕ' with a general plane, and has i' branches through each of m' points of ω'. The mi points of k, adjacent to ω along mi different paths of approach, correspond to the $m'i'$ points of k' adjacent to ω'. Thus

$$mi = m'i'; \quad i/m' = i'/m = \beta = \beta'.$$

Hence we have the theorem:

For a pair of homologous F-curves of second species without contact, the multiplicity of each is the same integral multiple of the degree of the other.

Consider a general point O of ω, and an adjacent point P of S. Now $P \sim$ a definite point of P' of S' adjacent to ω', that is, to a point O' of ω', and determining a line l' through O', which $\sim$ a certain c through P, that is, meeting ω at O. Now since ω is of second species, no general c passes through O; so this c is special, and breaks up into ω counted β times and a residual of degree $n' - \beta m$. For the number of free intersections of c and a general plane is the number of intersections other than O' of l' and a general ϕ', which is $n' - i' = n' - \beta m$.

Thus any two homaloids ϕ, ψ, which $\sim$ two planes p', p_1' through l', meet in a c which breaks up in this way, and therefore ϕ, ψ have β sheets in simple contact along ω, and have higher contact at the one point O.

If ϕ, ψ could have simple contact at O and not at all general points of ω, then a proper $c_{n'}$ would pass through O, which cannot be. The condition that ψ touches ϕ at O ensures that ψ also touches it all along ω, and, as we have seen, with β sheets. We may think of P as determining a P-curve κ, lying on ϕ, adjacent to ω and covering it β times, once on each sheet of the set; ϕ contains m' such P-curves, and if ψ contains any one point P of any κ, it contains the whole P-curve.

If p_1' rotates about O' in any way, ψ changes but is always in contact with β sheets of ϕ, and, by continuity, with the same set of β sheets. But the point O of intersection of $c_{n'-\beta m}$, ω moves on ω, describing it β times as l' rotates once round O'.

Thus the points of ω are in $1, \beta$ relation to the lines through O' in p'; it follows that ω is rational, and similarly ω'.

If l' is not a chord of ω', then p', p_1' have no other point O' in common, and the remaining $i - \beta$ sheets of ψ do not touch ϕ, either along ω or at any of its general points. First let ω be a line, $z = w = 0$. Then the tangent planes to ϕ are given by a rational expression of the form

$$t \equiv (z, w)^{i}(x, y)^{n-i} \equiv \prod_{a=1}^{i} (z - \Lambda_a w),$$

where Λ is a function of x, y, only which may be irrational; then Π contains all the conjugate irrationals, and t falls into a certain number of rational factors $t = \Pi t_\sigma$ of degrees i_σ say in z, w.

If ψ touches one sheet of ϕ along ω, the similar expression derived from ψ has one factor $z - \Lambda w$ in common with t for all values of x/y; being rational, it has all the conjugate linear factors also, and therefore the whole of a factor t_σ, and ψ necessarily has i_σ sheets in contact with ϕ as a consequence of having one sheet. But we know that this number is β. Hence each $i_\sigma = \beta$, and t falls into m' factors each rational in x, y and rational of degree β in z, w. Each function t_σ represents an algebraic surface on which ω is β-fold only, touching one set of β sheets of ϕ; and no algebraic surface can touch some but not all of the β sheets along ω.

If ω is of degree $m > 1$, since it is rational, we can find an auxiliary transformation T, for which ω is an ordinary curve, and which converts it into a line ω_1''; for example, a T_M (p. 310). This T converts (ϕ) into another homaloidal family (ϕ'') for which ω_1'' is an F-line of second species, and for which we can determine t_σ'' as above. The homologue of t_σ'' is a surface f having β sheets only through ω and touching one set of sheets of ϕ, but it is not possible for any algebraic surface to touch fewer sheets. We have the result:

If ω_m is an i-fold F-curve of second species, the i sheets of ϕ fall into m' sets of β, such that if an algebraic surface touches one sheet of ϕ along ω, it must touch all the other sheets of the same set, but need not touch any other sheet of ϕ, or have any other sheet through ω.

This is not true if there is any contact condition for (ϕ) along ω. The whole argument, including the relation $i' = \beta m$, breaks down, for there are certain directions round O in p which are on a different footing from the rest, and may correspond to a different number of sheets of ϕ'.

Thus it is only in one class of cases that the degree of ω is equal to the multiplicity of ω': when each sheet of ϕ touches a separate surface having one sheet through ω.

10. Example: cubic F-curve in a type of involution.

Examples of all this are given by the involutory transformations, S, S' being superposed, in which the rays joining conjugate points are chords of a cubic ω_3, the involution on each chord being determined by two pairs, one being the intersections with ω_3 and the other given in any definite way. Then any two points O, O' of ω correspond to each other, in the involution upon the chord OO'; hence ω is an F-curve of second species; we can show that it is always at least triple, three sheets of any homaloid touching three separate quadrics. In this case

$$i = i' = m = m' = 3, \quad \beta = 1.$$

Let OO' be any chord of ω and p the plane through it and the tangent line to ω at O. If $P \rightarrow O$ by a path touching p, the chord of ω through $P \rightarrow OO'$, and $P' \rightarrow O'$. Hence if ϕ touches p at O, it $\sim$ a plane through O', and conversely any plane p' through $O' \sim$ a ϕ which touches p at O. If O' is fixed and O describes ω, then OO' describes the quadric cone q which projects ω from O'. Since ϕ touches OO' for all positions of O, it has one sheet touching q along ω. Now p' meets ω in three points O_1', O_2', O_3'; hence ϕ has three separate sheets through ω, touching three cones such as q, whose vertices vary with p'.

With the usual parametric representation of ω_3 (p. 218), the expression for p' is

$$c_1 x' + c_2 y' + c_3 z' + c_4 w',$$

where $$c_1 : c_2 : c_3 : c_4 = \tau_1\tau_2\tau_3 : -\Sigma\tau_1\tau_2 : \Sigma\tau_1 : -1.$$

At a general point of ω, the tangent plane to each q_a is a linear function of the parameter τ_a of its vertex, and ϕ is symmetric in τ_1, τ_2, τ_3 or linear in c_1, c_2, c_3, c_4, which enter effectively: (ϕ) can have no other parameter, and any other sheet of ϕ through ω must touch a fixed surface. If there is such a sheet, there may be branches of c on it meeting ω, and with respect to it, ω may be of first species. An example is afforded by the I_{9-9} described on p. 325.

11. An important property.

The F-curves ω of second species have another important property: the passage of (ϕ) through them is a necessary consequence of its passage through the isolated F-points and F-curves of first species. For the transformation is determined if, instead of (ϕ), we are given the ∞^4 set of curves (c) which $\sim$ the lines of S'; then a homaloid can be defined as the locus of a c which passes through a given point and meets a given c. Since (c) does not meet ω except at isolated F-points, it is completely determined by the other F-elements without reference to ω, and (ϕ) is determined also. Thus ω does not present any additional condition to the family already containing the other F-elements (192).

12. Incidence relations (167·2).

The incidences of the F- and P-elements of all the different kinds involve a host of incidence numbers and of complicated relations between them.

It will be shown later (p. 382) that, by composition with auxiliary transformations of certain simple types, any Cremona transformation can be reduced to *ordinary* type, for which each F-system has no isolated F-points or contact conditions, and consists of ordinary curves only, free from multiple points and multiple intersections.

For an ordinary transformation, the only incidences of F-elements are simple intersections of two ordinary curves; the incidence relations become manageable, and the formulae for any more general case can be deduced by tracing back the composition with auxiliary transformations.

We therefore assume H to consist of a set $\Sigma \omega_\alpha{}^{i_\alpha}$ of σ ordinary curves of first or second species, and H' of σ' curves $\Sigma \omega'_{\alpha'}{}^{i'_{\alpha'}}$, where ω_α is of multiplicity i_α, degree m_α, genus p_α, with d_α intersections, all variable on ω_α, with a general homaloidal curve c, and $D_{\alpha\beta}$ intersections with ω_β. If it is of first species, $\omega_\alpha \sim$ a P-surface j_α', of degree d_α, with $i'_{\beta'\alpha}$ sheets through $\omega'_{\beta'}$; and a point of $\omega_\alpha \sim$ a P-curve κ_α' of degree i_α. If it is of second species, $\omega_\alpha \sim$ an F-curve $\omega'_{\alpha'}$ of second species, and we make the convention that this is regarded as a P-surface of degree 0, with -1 sheets through itself, and none through any other P-curve; that is, in this case we define $i'_{\alpha'\alpha} = -1$, $i'_{\beta'\alpha} = 0$. This device is found to make the formulae extend to F-curves of either species.

Between these numbers there hold many relations, including the simplified forms of the equations of p. 225. The following are the simplest. Here i_l is the lower of i_α, i_β, and $i'_{l'\alpha}$ the lower of $i'_{\alpha'\alpha}, i'_{\beta'\alpha}$; also $D_{\alpha h} = D_{\alpha\beta}$ if $i_\beta > i_\alpha$, and $= 0$ if $i_\beta \leqq i_\alpha$.

To prove these, we combine various methods. Some directly express certain known geometrical relations; thus we obtain

(3) from the degree of J';
(4) from the multiplicity of $\omega'_{\alpha'}$ on J';
(5) ; (6) ; (7) ; (8) from the numbers of intersections of
ϕ, c ; j_α, c ; $\kappa_{\alpha'}, \phi$; $\kappa_{\beta'}, j_{\alpha'}$ respectively;
(12) ; (13) from the degree of the intersection of
$\phi, j_{\alpha'}$; $j_{\alpha'}, j_{\beta'}$ respectively.

Others arise from equating the numbers of free intersections, before and after transformation, of three surfaces incident or coincident in certain ways with the F- and P-elements; another set follows from applying the formulae already proved to the transformation compounded of two ordinary transformations with a common F-curve (cf. p. 20). Yet others are deduced by summation. In particular (1) is proved by summing (9) with regard to α'; it expresses that the total number of F-curves is the same in each space.

It follows that the total number of F-elements is the same in each space for a transformation T of any type, provided that any one F-point or F-curve, that is not of a kind admitted in an ordinary transformation, is reckoned as involving as many different elements as there are different F-curves arising from it, when T is reduced to ordinary type by a standard method.

$$(1)\quad \sigma = \sigma'.$$

$$(2)\quad \sum_{\alpha} p_{\alpha} = \sum_{\alpha'} p'_{\alpha'}.$$

$$(3)\quad \sum_{\alpha} d_{\alpha} = 4n' - 4.$$

$$(4)\quad \sum_{\alpha} i'_{\alpha'\alpha} = 4i'_{\alpha'} - 1.$$

$$(5)\quad \sum_{\alpha} d_{\alpha} i_{\alpha} = nn' - 1.$$

$$(6)\quad \sum_{\alpha} d_{\alpha} i_{\alpha\alpha'} = n'd'_{\alpha'}.$$

$$(7)\quad \sum_{\alpha} i_{\alpha} i'_{\alpha'\alpha} = n i'_{\alpha'}.$$

$$(8)\quad \sum_{\alpha} i_{\alpha\alpha'} i'_{\beta'\alpha} = d'_{\alpha'} i'_{\beta'}, \qquad (\alpha' \neq \beta').$$

$$(9)\quad \sum_{\alpha} i_{\alpha\alpha'} i'_{\alpha'\alpha} = d'_{\alpha'} i'_{\alpha'} + 1.$$

$$(10)\quad \sum_{\alpha} i_{\alpha} m_{\alpha} - 4n = \sum_{\alpha'} i'_{\alpha'} m'_{\alpha'} - 4n'.$$

$$(11)\quad \sum_{\alpha} i_{\alpha}^2 m_{\alpha} = n^2 - n'.$$

$$(12)\quad \sum_{\alpha} i_{\alpha} i_{\alpha\alpha'} m_{\alpha} = n d'_{\alpha'} - i'_{\alpha'} m'_{\alpha'}.$$

$$(13)\quad \sum_{\alpha} i_{\alpha\alpha'} i_{\alpha\beta'} m_{\alpha} = d'_{\alpha'} d'_{\beta'} - i'_{l'} D'_{\alpha'\beta'}, \qquad (\alpha' \neq \beta').$$

$$(14)\quad \sum_{\beta} i_{l} D_{\alpha\beta} = \sum_{\alpha'} i'_{\alpha'\alpha} (i'_{\alpha'\alpha} + 1)\, m'_{\alpha'} + 4 i_{\alpha} m_{\alpha} - d_{\alpha} (d_{\alpha} + 4).$$

$$(15)\quad \sum_{\beta} i_{\beta} D_{\alpha\beta} = \sum_{\alpha'} i'_{\alpha'\alpha} m'_{\alpha'} + n m_{\alpha} - 2 i_{\alpha} (p_{\alpha} - 1) - 4 d_{\alpha}.$$

$$(16)\quad \sum_{\beta} i_{l} i_{\beta} D_{\alpha\beta} = i_{\alpha} \sum_{\alpha'} i'^2_{\alpha'\alpha} m'_{\alpha'} + n i_{\alpha} m_{\alpha} - i_{\alpha} d_{\alpha}^2 - d_{\alpha}.$$

$$(17)\quad \sum_{\beta} i^2_{l} D_{\alpha\beta} = 2 i_{\alpha} \sum_{\alpha'} i'^2_{\alpha'\alpha} m'_{\alpha'} + i_{\alpha}^2 (4 m_{\alpha} + 2 p_{\alpha} - 2) - 2 i_{\alpha} d_{\alpha}^2 - d_{\alpha}.$$

$$(18)\quad \sum_{\beta} i_{\beta\alpha'} D_{\alpha\beta} - d'_{\alpha'} m_{\alpha} + 2 i_{\alpha\alpha'} (p_{\alpha} - 1)$$
$$= \sum_{\beta'} i'_{\beta'\alpha} D'_{\alpha'\beta'} - d_{\alpha} m'_{\alpha'} + 2 i'_{\alpha'\alpha} (p'_{\alpha'} - 1).$$

$$(19)\quad \sum_{\beta} i_{\beta} i_{\beta\alpha'} D_{\alpha\beta} - \sum_{\beta} (i_{\beta} - i_{\alpha})(i_{\beta\alpha'} - i_{\alpha\alpha'})\, D_{\alpha h}$$
$$= (n i_{\alpha\alpha'} + i_{\alpha} d'_{\alpha'})\, m_{\alpha} - i_{\alpha} i_{\alpha\alpha'} (4 m_{\alpha} + 2 p_{\alpha} - 2) - i'_{\alpha'\alpha} m'_{\alpha'}.$$

$$(20)\quad \sum_{\beta} i_{\beta\alpha'} i_{\beta\beta'} D_{\alpha\beta} - \sum_{\beta} (i_{\beta\alpha'} - i_{\alpha\alpha'})(i_{\beta\beta'} - i_{\alpha\beta'})\, D_{\alpha h}$$
$$= (d'_{\alpha'} i_{\alpha\beta'} + d'_{\beta'} i_{\alpha\alpha'})\, m_{\alpha} - i_{\alpha\alpha'} i_{\alpha\beta'} (4 m_{\alpha} + 2 p_{\alpha} - 2) - i'_{l'\alpha} D'_{\alpha'\beta'}, \qquad (\alpha' \neq \beta').$$

$$(21)\quad \sum_{\beta} i^2_{\beta\alpha'} D_{\alpha\beta} - \sum_{\beta} (i_{\beta\alpha'} - i_{\alpha\alpha'})^2 D_{\alpha h}$$
$$= 2 d'_{\alpha'} i_{\alpha\alpha'} m_{\alpha} - i^2_{\alpha\alpha'} (4 m_{\alpha} + 2 p_{\alpha} - 2)$$
$$+ d_{\alpha} m'_{\alpha'} - i'_{\alpha'\alpha} (4 m'_{\alpha'} + 2 p'_{\alpha'} - 2) - \sum_{\beta'} (i'_{\beta'\alpha} - i'_{\alpha'\alpha})\, D'_{\alpha'\beta}.$$

13. Tummarello's theorem (167·2, 400).

Equation (2) expresses that the sum of the genera of all the F-curves is the same in each space. We have a more precise theorem: the irrational curves of H, H' are associated in pairs of the same genus. This is proved on the following lines.

An intersection $O_{\alpha\beta}$ of two F-curves ω_α, ω_β, of equal multiplicities $i_\alpha = i_\beta$, is an i_α-fold point of total contact of (ϕ). Its nature is a necessary consequence of the ordinary multiplicities of ω_α, ω_β, and it does not present any contact *condition* to (ϕ). There is a net of (ϕ) having $O_{\alpha\beta}$ as an $(i_\alpha + 1)$-fold point; this $\sim$ the net of planes through a point which can be shown to be of the same nature as $O_{\alpha\beta}$, and symmetrically related to it.

If an F-curve ω_α, with $p_\alpha > 0$, acquires a d.p. O_α without breaking up, T ceasing to be ordinary to this extent, we can show in the same way that O_α is associated with a point $O'_{\alpha'}$ of H' of the same nature. One F-curve $\omega'_{\alpha'}$ has also acquired a d.p. without breaking up, and its genus was $p'_{\alpha'} > 0$. Now ω_α can acquire p_α d.p.'s without breaking up; since the equation $\sigma = \sigma'$ continues to hold, and σ remains unaltered, so does σ'; hence $\omega'_{\alpha'}$ does not break up on acquiring p_α d.p.'s, and $p'_{\alpha'} \geqq p_\alpha$. Similarly, $p_\alpha \geqq p'_{\alpha'}$, that is,

$$p_\alpha = p'_{\alpha'}.$$

Hence the irrational F-curves of H, H' are associated in pairs of the same genus, such that if one of a pair acquires a d.p., so does the other.

If T is not ordinary, a corresponding theorem holds, provided the F-genera, explicit and implicit, are properly defined (p. 383).

14. General analytical investigation (12, 209).

With Cartesian coordinates, let O' (0, 0, 0) be an i'-fold F-point, and for the present let us make no assumption as to the type of F-element to which it belongs. Then

$$\phi' \equiv u_{i'}(x', y', z') + u_{i'+1} + \dots + u_{n'},$$

and for the four base homaloids, at least one of the $u_{i'}$ does not vanish identically.

Let P' $(\xi\epsilon, \eta\epsilon, \zeta\epsilon)$ be any point which $\rightarrow O'$ along a path k' as $\epsilon \rightarrow 0$. The homologue P has coordinates, from which we can cancel $\epsilon^{i'}$, but no higher power of ϵ,

$$x : y : z : 1 = \phi_1' : \phi_2' : \phi_3' : \phi_4',$$

where $$\phi_\alpha' \equiv u_{i',\alpha}(\xi, \eta, \zeta) + \epsilon u_{i'+1,\alpha}(\xi, \eta, \zeta) + \dots.$$

As $\epsilon \rightarrow 0$, so $P \rightarrow$ a definite point Q whose coordinates are

$$x : y : z : 1 = u_{i',1}(\xi, \eta, \zeta) : u_{i',2} : u_{i',3} : u_{i',4}.$$

These functions $u_{i'}$ do not all vanish identically; they are explicit rational functions of degree i' of the three homogeneous parameters ξ, η, ζ.

(i) If the two parameters $\xi : \eta : \zeta$ enter independently, that is, if the Jacobians of the $u_{i',\alpha}$ do not all vanish identically, and if no reduction takes place, the locus of Q is a rational P-surface j of degree i'^2. The degree of j is reduced by γ, if the functions $u_{i',\alpha}(\xi, \eta, \zeta)$, interpreted as a family of plane curves, have common base points of equivalence γ. The set of values, say $\xi : \eta : \zeta = 1 : b : c$, at one of these points, gives an exceptional direction at O', and a fixed generator of the tangent cones $u'_{i'}$ to (ϕ'), due either to a branch of an F-curve ω' through O', or to an implicit isolated F-point in the first neighbourhood of O'.

To explore this further, we must assume that $\eta - b\xi$, $\zeta - c\xi$ are small of first order, equal to $\eta_1\epsilon$, $\zeta_1\epsilon$ say, so replacing η, ζ by the other parameters η_1, ζ_1. Then each $u_{i',\alpha}$ has ϵ as a factor; in the expansions for x, y, z, w we can cancel $\epsilon^{i'+1}$, and we must retain the next term of each series. This is equivalent to assuming that k' touches the fixed generator. This process may give another surface locus for Q, indicating an adjacent isolated F-point; or it may fail to give a surface, and lead to a P-curve, as explained below, see (ii), indicating a branch of ω'; and in either case it may also point to exceptional values of $\xi : \eta_1 : \zeta_1$, which must be explored in the same way by assuming that certain of their combinations are small, and retaining higher terms of the expansions. This is equivalent to fixing the osculating plane and curvature of k' at O', and examines the nature of certain points in the second neighbourhood of O'.

There is also a reduction in the degree of j if the $u_{i',\alpha}$ have a common factor f_β of degree $\beta < i'$; then if f_β is finite, the locus of Q is a P-surface of degree $\leqq (i' - \beta)^2$. There is a whole sheet of exceptional directions at O', and $u_{i'}$ has a fixed component, O' being a point of partial contact.

If $\beta = i'$, then if f_β is finite, ξ, η, ζ disappear entirely from $x : y : z : w$, Q is a definite point O, and there is no P-surface j. Then the whole of $u_{i'}$ is fixed and O' is a point of total contact.

We must now consider the assumption $f_\beta = \theta\epsilon$, arising when k' touches the fixed tangent sheet. We can then obtain $x : y : z : w$ as finite functions, free from ϵ, of the four parameters ξ, η, ζ, θ, connected by the relation $f_\beta(\xi, \eta, \zeta) = 0$. The locus may be a surface, curve or point; we can no longer assert that it is rational, or assign a limit to its degree depending only on i.

(ii) It may happen that, after any common factor of the $u_{i',\alpha}$ has been removed, the two parameters do not enter independently, but through a single rational combination $\theta \equiv f_{\beta,1}/f_{\beta,2}$. Then the Jacobians of the $u_{i',\alpha}$ vanish identically, and $u_{i',\alpha}$ reduces to a rational function of θ of degree γ say, where $\beta\gamma \leqq i'$; the locus of Q is a rational curve of degree $\gamma \leqq i'$. The exceptional directions at O' are now given by $f_{\beta,1} = f_{\beta,2} = 0$, and are fixed γ-fold generators of the $u_{i',\alpha}$. These must be explored by assuming that $f_{\beta,1}$, $f_{\beta,2}$ are both small, so introducing two new parameters instead of θ; but at the final stage, when $\epsilon = 0$, we have ξ, η, ζ connected by the two relations

$$f_{\beta,1}(\xi, \eta, \zeta) = f_{\beta,2}(\xi, \eta, \zeta) = 0,$$

and again the locus may be of any kind, degree and genus.

(iii) The parameters may disappear entirely if the $u_{i',a}$ are all multiples of the same function of ξ, η, ζ. This case arises in (i) if $\beta = i'$, and in (ii) if $\beta = 0$. Then O' is an F-point of total contact and Q is a fixed point, and has no locus unless we make special assumptions. It can be shown that, since the four functions ϕ_a are linearly independent, at some stage we find a set of assumptions for ξ, η, ζ, and a form of k', that give a surface or curve as the P-element which $\sim O'$.

By this process, we can explore the F-points existing in the successive neighbourhoods of O'. If O' is isolated, and if it has implicit isolated F-points adjacent to it, the corresponding P-surfaces emerge at the different stages, and their number is finite. If there are branches of F-curves ω' through O', then P-curves emerge, and in this case, however far we go in exploring paths k' touching ω', there is always a further exceptional assumption, of higher contact of k', ω'. A proof that a complete exploration is possible depends on the theory of resolution of singularities of a surface, treated in Chapter XVI.

15. The Jacobian (115, 310).

On p. 161, we stated without proof the theorem: the P-surfaces are the sheets of the Jacobian J of the family (ϕ). This usually gives the quickest way of obtaining the P-surfaces in one space, without having recourse to the other space.

The Jacobian is defined as the locus of variable d.p.'s of members of the family, which is the same thing as the locus of points where two linearly independent ϕ's touch one another. A d.p. at a given arbitrary point of space imposes four conditions, and is not possible on the ∞^3 family (ϕ). Exactly as in the plane (p. 15), we can prove that the condition, that the homogeneous coordinates of Q must satisfy in order that Q may be a possible position for a d.p. of some ϕ, is

$$J \equiv \begin{vmatrix} \frac{\partial \phi_1}{\partial x}, & \frac{\partial \phi_1}{\partial y}, & \frac{\partial \phi_1}{\partial z}, & \frac{\partial \phi_1}{\partial w} \\ \frac{\partial \phi_2}{\partial x}, & \frac{\partial \phi_2}{\partial y}, & \frac{\partial \phi_2}{\partial z}, & \frac{\partial \phi_2}{\partial w} \\ \frac{\partial \phi_3}{\partial x}, & \frac{\partial \phi_3}{\partial y}, & \frac{\partial \phi_3}{\partial z}, & \frac{\partial \phi_3}{\partial w} \\ \frac{\partial \phi_4}{\partial x}, & \frac{\partial \phi_4}{\partial y}, & \frac{\partial \phi_4}{\partial z}, & \frac{\partial \phi_4}{\partial w} \end{vmatrix} = 0,$$

where $\phi_1, \ldots \phi_4$ are any four linearly independent members of the family. This is therefore the equation of the Jacobian.

Its degree is $4(n-1)$. We have shown that an isolated i-fold F-point is $(4i-2)$-fold on J and that an i-fold F-curve is $(4i-1)$-fold on J (p. 161).

There may be certain points of S that are d.p.'s on two linearly independent ϕ's, and therefore on the ∞^1 ϕ's of the pencil which they determine; these are double on J. For if each of ϕ_1, ϕ_2 has a d.p. at $P(0, 0, 0, 1)$ every term in the first row of the determinant is of degree $\geqq 1$ in x, y, z, and J is of degree $\geqq 2$.

Conversely, a d.p. of J is either an F-point, or is double on two linearly independent ϕ's, or is a binode on one. Let $Q(0, 0, 0, 1)$ be double on $\phi_1 \equiv w^{n-2}u_2 + \ldots$. There are two other linearly independent ϕ's through Q. If these touch at Q, a linear combination of them, independent of ϕ_1, has a d.p. at Q. If not, take their different tangent planes as y, z:

$$\phi_2 \equiv w^{n-1}y + \ldots, \quad \phi_3 \equiv w^{n-1}z + \ldots.$$

If $\phi_4 \equiv aw^n + w^{n-1}u_1 + \ldots$ and $a = 0$, then Q is an F-point; if not, take $a = 1$. The only terms of J linear in x, y, z arise from

$$\begin{vmatrix} w^{n-2}\dfrac{\partial u_2}{\partial x}, & w^{n-2}\dfrac{\partial u_2}{\partial y}, & w^{n-2}\dfrac{\partial u_2}{\partial z}, & 0 \\ 0 \quad , & w^{n-1} \quad , & 0 \quad , & 0 \\ 0 \quad , & 0 \quad , & w^{n-1} \quad , & 0 \\ w^{n-1}\dfrac{\partial u_1}{\partial x}, & w^{n-1}\dfrac{\partial u_1}{\partial y}, & w^{n-1}\dfrac{\partial u_1}{\partial z}, & nw^{n-1} \end{vmatrix} = nw^{4n-5}\frac{\partial u_2}{\partial x}.$$

If Q is double on J, then $\dfrac{\partial u_2}{\partial x} \equiv 0$, $u_2 \equiv (y, z)^2$, and Q is a binode on ϕ_1, whose edge, $y = z = 0$ lies on the tangent plane to every ϕ through Q.

We shall first show that every P-point Q lies on J. Each of the ∞^2 ϕ's through Q contains the whole P-element to which Q belongs, either containing a P-curve κ or breaking up into a P-surface j and a second component. In the former case, the tangent planes at Q all contain l the tangent line to κ, and are ∞^1 instead of ∞^2; hence two linearly independent ϕ's have the same tangent plane, and one has a d.p. at Q, namely that one ϕ which can be made to contain Q and two points adjacent to Q in directions not coplanar with l. This $\phi \sim$ the plane associated with Q in the 1, 1 relation of p. 268. Hence Q lies on J, and so does its locus κ.

In the latter case, one more condition makes the second component also pass through Q, which is double on the ∞^1 ϕ's which can be made to contain Q and one point adjacent to Q in a direction not lying on the tangent plane to j; and Q is double on J, and so is its locus j; the whole sheet is reckoned twice in J.

Conversely, we shall next show that every point of J is principal. All the points of S are either ordinary points for the transformation, or P-points, the latter including all F-points. If ϕ contains an ordinary point P, the tangent plane is free to take up any position through P, for all points adjacent to P are ordinary, and ϕ can be made to contain any two of them without becoming singular. If the tangent plane is found to be restricted, the point is not ordinary but principal.

Now if ϕ_1 is the homaloid which has a d.p. at any point Q of J, and ϕ_2, ϕ_3 are two other linearly independent ϕ's through Q, we have

$$\phi \equiv c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4,$$

where

$$\begin{aligned}\phi_1 &\equiv \qquad\qquad\quad w^{n-2}u_2 + \dots,\\ \phi_2 &\equiv \qquad w^{n-1}u_{1,2} + \dots,\\ \phi_3 &\equiv \qquad w^{n-1}u_{1,3} + \dots,\\ \phi_4 &\equiv w^n + w^{n-1}u_{1,4} + \dots.\end{aligned}$$

If ϕ passes through Q, the condition is $c_4 = 0$; then the tangent plane is $c_2u_{1,2} + c_3u_{1,3}$, which contains the fixed line l, $u_{1,2} = u_{1,3} = 0$, and is not entirely unrestricted. Hence the point Q is not ordinary, but principal.

Any ϕ through Q also contains the point Q_1 adjacent to Q on l; and Q, Q_1 present the same condition $c_4 = 0$ to ϕ, and $\sim$ the same point A_4' in S'; thus A_4' is a point whose homologue is not unique, and is an F-point of S'; and Q and Q_1 are P-points of S.

Thus every point of J is principal, and J consists entirely of P-surfaces taken a proper number of times.

16. Multiplicity of the sheets of J.

If there are no contact conditions, we can verify that this accounts for the total degree of J. The ∞^4 family (c') is selected, from the ∞^{4n} family of all rational curves in S' of degree n, by $4n - 4$ conditions arising from incidences with H'. An F-point through which there pass μ branches of c' presents 2μ conditions to (c'), and $\sim$ a P-surface of degree μ reckoned twice in J. An F-curve meeting c' in d variable points presents d conditions to (c') and $\sim$ a P-surface of degree d reckoned once in J.

Hence the total degree of the P-surfaces in $J = 2\Sigma\mu + \Sigma d$

$=$ the number of conditions for (c') $\quad = 4n - 4$

$=$ the total degree of J.

When there are conditions of contact, particularly of partial contact, things are more complicated. For example, if the tangent cone at O' breaks up into a variable sheet of degree i'' and a fixed sheet, with simple contact, of degree $i' - i''$, then c' has i''^2 branches touching the variable sheet, each meeting χ' in i' points and subject to two conditions; and c' has $(i' - i'')(i' + i'' + 1)$ branches touching the fixed sheet, each meeting χ' in $i' + 1$ points and subject to three conditions (p. 240). The total number of conditions, for all the branches through O' of any one c', is not an integral multiple of the total number of branches, as it must be in the simpler cases, and O' $\sim$ two quite different P-surfaces.

It is a matter of taste and convenience whether, in any context, we regard O' as one complex F-element with a degenerate P-surface, or as two adjacent distinct F-elements, each with its own proper P-surface. The latter is on the whole the more satisfactory view, especially in regard to certain theorems on the total number of F-elements (p. 275).

In general, let O' be an i'-fold F-element, either a point or a curve of first species, and Q a general point of j, one of the set $\Sigma j_{O'}$ of P-surfaces which $\sim O'$. The ∞^2 lines l through $Q \sim$ an ∞^2 set of branches of curves c' incident with O', each meeting χ' in a certain number σ of points at O' and subject to a certain number, $\tau + 2$ say, of conditions, of which two are determined by the position of Q on j, and the remaining τ are characteristic of the whole sheet j; and may be replaced by a partly or wholly different set of conditions if Q lies on another P-surface of $\Sigma j_{O'}$.

Thus the branches of any one c' incident with O' fall into sets; let μ be the number in the set satisfying the conditions characteristic of j. This is equal to the number of intersections of l, j and therefore the degree of j is μ.

If two ϕ's pass through Q, they both contain the whole P-element on which Q lies, and their variable intersection c breaks up. The residual is the proper homologue of a line l' incident with O', and its degree is the number of the other intersections of l' with a general ϕ', which $\leq n' - i'$.

If any surface f' satisfies such conditions at O' that it has α intersections with each of the μ branches of c', then every Q is α-fold on the homologue, which breaks up, having j^α as a component. In particular, if j^ρ breaks away whenever f' contains O' simply, then f' has ρ intersections with each branch, which is superlinear if $\rho > 1$, having a ρ-fold point. If O' is the point $(0, 0, 0, 1)$, then j^ρ is a common factor of ϕ_1, ϕ_2, ϕ_3; if O' is the line $z' = w' = 0$, then j^ρ is a common factor of ϕ_3, ϕ_4.

If f' is a general ϕ', then $\alpha = \sigma$, and the complete factor that breaks away is G; hence the multiplicity of j in G is σ.

It is also true, but longer to prove, that the multiplicity of j on J is τ; it is best treated by eliminating the contact conditions by means of auxiliary transformations (p. 382), and so reducing this case to the simpler one already discussed.

If O' is a simple F-point of contact, $\mu = 2$, $\sigma = \tau = 3$. With the notation of p. 264, if a definite point Q of j is given, then the parameters ξ_1, η, ζ are uniquely determined, and the curves c' which $\sim$ lines l through Q are subject at O' to the conditions of touching the fixed plane $p' \equiv \zeta y' - \eta z'$, and osculating the fixed quadric $q_1' \equiv \eta\zeta x'w' - \xi_1 y'z'$. Consider the quadric family

$$q' \equiv a_1 q_1' + a_2 q_2' + a_3 q_3' + a_4 q_4',$$

where

$$q_2' \equiv p' p_1',$$

and p_1' is a general plane, and q_3', q_4' general quadrics. Then q_1' osculates and q_2' touches every c' which $\sim$ a line through Q; but q_1', q_2' touch different planes at O', no q' has a d.p. at O', which does not lie on the Jacobian of (q'):

$$J_{q'} \equiv \frac{\partial (q_1', q_2', q_3', q_4')}{\partial (x', y', z', w')}.$$

Now $(q') \sim$ a family (f_{2n}), with j a factor of f_1, f_2 but not of f_3, f_4; also f_1 has three intersections at Q with every l, that is, has a triple point at Q, and similarly f_2 has a d.p.

In $J_f \equiv \frac{\partial (f_1, f_2, f_3, f_4)}{\partial (x, y, z, w)}$, the point Q is double on each element in the first row and simple on each element in the second; thus Q is triple on J_f.

But
$$f(x, y, z, w) \equiv q'(\phi_1, \phi_2, \phi_3, \phi_4),$$
$$J_f \equiv \frac{\partial (\phi_1, \phi_2, \phi_3, \phi_4)}{\partial (x, y, z, w)} [J_{q'}],$$

where in the second factor we put $\phi_1 \ldots$ for $x' \ldots$. Now this factor does not vanish for $x' = y' = z' = 0$, and does not contain Q; hence Q is triple on the first factor, which is J. This holds for every point Q of j; it follows that j is a triple sheet of J.

Again, if O' is a binode of partial contact, then for the fixed biplane, $\mu = 4$, $\sigma = \tau = 3$, and for the variable biplane, $\mu = 1$, $\sigma = \tau = 2$ (p. 240). We can apply a similar argument to each set of branches in turn, and show that each of G, J has the factors $j_4^3 j_1^2$. For the set of four branches we can use a family of quadrics as above, and for the single branch we can use instead the family of planes.

A similar argument applies whenever the $\tau + 2$ conditions, satisfied by c' at O' when l passes through Q, can be interpreted as those of τ_1-point and τ_2-point contact with two surfaces, each passing once through O', and not touching each other there, with $\tau_1 + \tau_2 = \tau + 2$.

Another example is worked out in detail on pp. 348, 351.

17. General curves on P-surfaces.

If a general curve k' moves so as to pass through an F-point O', the degree of the homologue k is lowered; this means that k breaks up. If O' is a point of an F-curve, and $\sim$ a P-curve κ, this is the part that breaks away; if O' is isolated, and $\sim$ a P-surface j, the part that breaks away is not fixed, but depends on the way in which k' approaches O'.

For example, let k' be a line, p', p'' two planes through it, and first let neither pass through O'. The homologue c is the free intersection of ϕ, ψ. Now let p' be the plane $k'O'$; then ϕ breaks up into $j_\mu \phi_{n-\mu}$, and c does not break up, being the free intersection of ψ, $\phi_{n-\mu}$. The intersection of ψ, j is a set of F-curves, since j is a P-element and ψ is a general homaloid (p. 260). Next, make p'' pass through O'; then k' also passes through O', approaching it in p'. Now ψ breaks up also, into $j\psi_{n-\mu}$, and k breaks up, the proper homologue lying on $\phi_{n-\mu}$, $\psi_{n-\mu}$, and a part breaking away which lies on j, $\phi_{n-\mu}$. This does not consist of F-curves only, but contains a curve on j which $\sim$ the points of p' adjacent to O', and therefore depends on p', that is, the plane of approach of k' to O'.

If k' is a general curve, as it approaches O' the part near O' ultimately determines a plane p' through O', and from k there breaks away the curve on j which $\sim$ the neighbourhood of O' on p'.

If O' (0, 0, 0, 1) $\sim j_\mu$, then ϕ_1, ϕ_2, ϕ_3, which $\sim$ planes through O', have j as a factor, and ϕ_4 has not. If there is no contact at O', the factor is simple, and

$$\phi_a \equiv j_\mu \phi_{n-\mu, a} \quad (a=1, 2, 3).$$

Let f' have O' as an i-fold point, the tangent cone being $u_i(x', y', z')$; after substituting and dropping j^i, we have f a sum of multiples of j and $u_i(\phi_{n-\mu,1}, \phi_{n-\mu,2}, \phi_{n-\mu,3})$, and the form of u determines k the curve of intersection, residual to F-curves, of f, j, which selects from the ∞^2 points of j, all of which $\sim O'$, the ∞^1 which $\sim O'$ regarded as a point of f', and approached by paths which lie on f' and therefore touch u'; that is, $k \sim$ the neighbourhood of O' on u'.

Conversely if k is a given curve on j, it $\sim$ the neighbourhood of O' lying on a certain cone. For example, let f be the cone projecting k from a general point of S; then $f \sim$ an f' through O', whose tangent cone has one sheet u' associated with k, and possibly another sheet associated with a residual intersection of f, j. Then any surface through $k \sim$ a surface touching u' at O'.

Thus a family (f'), having O' as a base point of simple contact, is transformed into a family (f) with k as an ordinary base curve without contact. This is the simplest case of the theorem on dissipation of contact by auxiliary transformation, proved in Chapter XVI.

18. Determination of the reverse transformation.

We can now determine the second homaloidal family from considerations in the first space alone. We are given either (ϕ) or H; each determines the other.

The first step is to find the P-system. All the P-surfaces j are given if we form J. If the intersection of j with a general ϕ consists of F-curves only, then j is a P-element; if not, it consists of a certain number of elementary P-curves. These elements, together with the F-curves of second species, constitute the P-system.

We can usually discover most if not all of its elements without calculating J, from the property of meeting (ϕ) in F-curves only. Also, if any ϕ is degenerate, only one of its components can be the proper homologue of the corresponding plane, and all the rest are P-surfaces. A common sheet of two degenerate ϕ's is always a P-surface, for it cannot be the proper homologue of two different planes.

Thus we can find all the P-elements; they are of three kinds:

(i) P-surfaces, meeting (ϕ) in F-curves only;

(ii) P-curves, meeting (ϕ) in F-points only; their loci are surfaces meeting (ϕ) in F-curves and P-curves;

(iii) F-curves of second species.

From these we have to deduce the nature of H'.

(i) A P-element which is a surface j_μ indicates an isolated F-point O', through which there pass a set of μ branches of c'. The multiplicity i' of O' is the reduction in degree of the effective part of c when the two homaloids ϕ, ψ pass through any point Q of j, and therefore break up, having some power of j as a factor.

Let ϕ_1, ϕ_2, ϕ_3 be the three linearly independent homaloids having the highest powers, say ρ_1, ρ_2, ρ_3, of j as factors, where $\rho_1 \geqq \rho_2 \geqq \rho_3$. Then a general ϕ through Q has j^{ρ_3} as a factor and O' is ρ_3-fold on each of the μ branches of c'.

If $\rho_1 = \rho_2 = \rho_3$, then all directions round O' are alike for the set, and there is no contact.

If $\rho_1 > \rho_2 = \rho_3$, there is a fixed tangent plane p_1' which $\sim \phi_1$, and all the μ branches of c' have contact with p_1' of order $\rho_1 - \rho_3$.

If $\rho_1 = \rho_2 > \rho_3$, there is a fixed generator $p_1' = p_2' = 0$ of the tangent cone to ϕ' at O', and contact of order $\rho_2 - \rho_3$ with this, but all planes through this line are alike for the set.

If $\rho_1 > \rho_2 > \rho_3$, the μ branches have a fixed tangent line and osculating plane.

Conditions of higher contact at O' can be detected by considering quadratic and higher functions of ϕ_1, ϕ_2, ϕ_3 that contain additional factors j; but the method is not often profitable.

The multiplicity τ of j in J gives the number of conditions at O' for each branch of c'; two of these are of incidence at O', the $\tau - 2$ others are conditions of contact with fixed surfaces or curves through O'.

If ϕ, ψ always have another common sheet, this indicates that (ϕ') has another set of sheets through O', containing a set of curves c' subject to other conditions; then O' is a point of partial contact. If ϕ, ψ have in common a P-curve not lying on a common sheet, this is a sufficient but not necessary condition for an F-curve through O', see (ii).

(ii) A P-surface j_d, which is τ-fold on J, and is not a P-element but a locus of P-curves κ_ν, of which m lie on every ϕ, indicates a P-curve ω' of first species of degree m, through which ϕ' has a set of ν separate sheets, and which meets d branches of c' at variable points, each branch subject to τ conditions, of which one is of incidence with ω', and $\tau - 1$ are of contact with a fixed surface through ω'.

When ϕ, ψ contain a point Q of j, they contain a whole κ, and c breaks up and contains κ, taken i'/ν times; the reduction in degree of the effective part of c is the multiplicity i' of ω' on ϕ', and i'/ν is the multiplicity of each separate sheet.

(iii) An F-curve ω of second species, of degree m and multiplicity i, indicates an F-curve ω' of second species, of degree i/β and multiplicity $m\beta$, where β is the number of times that ω breaks away from c, when ψ is made to contain a point of ϕ adjacent to a general point of ω.

If a P-curve has a point in common with a different P-element, there must be a component in common, which however need not be the whole curve. If the common part is not an F-curve, then the two F-elements in H' are incident.

By these and similar considerations, H' can be completely determined, and therefore (ϕ'), and the reverse transformation in all its details.

CHAPTER XIV

SPECIAL SPACE TRANSFORMATIONS

I. TRANSFORMATIONS OF LOW DEGREE

1. T_{2-3}, F- and P-systems (61, 66, 85, 287, 413).

The transformation next lowest in degrees, after T_{2-2}, is T_{2-3}, whose first homaloidal web consists of quadrics, and the second of cubics. There is only one such type, and the nature of each *F*-system is completely determined by the degrees of the transformation.

The *F*-curves in S are of total degree $2^2-3=1$; the quadrics have therefore one *F*-line, and the rest of H consists of points. Now for a line on a quadric web, $P=3$, $E=4$ (p. 213); since a general quadric has 9 parameters and three such meet in 8 points, the set of isolated points has

$$P=9-3-3=3,\quad E=8-4-1=3.$$

This is given by three simple points, and by no other system; therefore H consists of a line l and three points O_1, O_2, O_3.

The variable intersection of two of the quadric homaloids ϕ, ψ, is a twisted cubic c_3 through O_1, O_2, O_3, meeting l twice; it meets a third homaloid χ in these five *F*-points and one free point, and $c_3 \sim$ a general line. Thus O_1, O_2, $O_3 \sim$ *P*-surfaces meeting a line once, and therefore planes, and $l \sim$ a *P*-surface meeting a line twice, and therefore a quadric. These are the only sheets of the second Jacobian J'; for since the three planes $\sim$ isolated *F*-points, they are reckoned twice in J' (p. 281) whose degree $4(3-1)=8$ is thus accounted for.

To determine H', we find its homologue, the first *P*-system, that is, the set of surfaces and curves meeting (ϕ_2) in *F*-points only. Now the line and three points of H suggest four planes to examine.

The plane $O_1O_2O_3$ meets ϕ in a conic κ_2 through O_1, O_2, O_3 and a fourth *F*-point X, the trace of l on the plane; and κ_2 meets ψ in these four *F*-points only, and is a *P*-curve of degree 2, one such lying on each ϕ; it therefore $\sim$ a point of a double *F*-line l' of H'.

The plane lO_1 meets ϕ in l and another line κ_1 through O_1, meeting l, and meeting ψ in these two *F*-points only; κ_1 is therefore a *P*-line, one such lying on each ϕ, and $\sim$ a point of a simple *F*-line l_1' of H'. Similarly, the *P*-planes lO_2, $lO_3 \sim$ two other *F*-lines l_2', l_3'.

The planes $O_1O_2O_3$, lO_1 have in common the line O_1X, which both is a *P*-line κ_1 of lO_1, and also forms with O_2O_3 a *P*-conic κ_2 of $O_1O_2O_3$; hence l_1' meets l' in one point, and so do l_2', l_3'. But the planes lO_1, lO_2 have nothing in common except the *F*-line l; hence l_1', l_2' do not meet.

Therefore H' contains a double line l' and three simple lines l_1', l_2', l_3', meeting l and skew to each other.

The total degree of F-curves in H' is $3^2-2=7$, and since the double line counts for 4, the four lines exhaust it. The degree of J is $4(2-1)=4$; the four planes exhaust this, J has no other sheet, and there are no isolated F-points in H'.

The postulation of the double line l' on a cubic family is 10; that of l_1', by itself, is 4, but this is reduced by 2 on account of the intersection of l_1' with l' (p. 225). The total postulation of the F-lines is

$$10+3(4-2)=16=\binom{6}{3}-4,$$

which is the required total for (ϕ_3').

The equivalence of l' alone on (ϕ_3') is 20; that of l_1' is 7, reduced by 5 on account of the intersection with l'. Hence the total equivalence of H' is

$$20+3(7-5)=26=3^3-1,$$

as required.

The variable intersection of ϕ', ψ' is a conic meeting each of l', l_1', l_2', l_3' once; it meets χ' in these four F-points, the first counted twice, and one free point.

One treatment (66) of the resolution of higher singularities of a surface f aims at the elimination of all isolated multiple points; an auxiliary transformation, such as T_{2-2}, with an isolated F-point O' in S', would introduce an accidental isolated singularity of f' at O', which must then be eliminated. This is avoided by the use of T_{2-3} instead of T_{2-2}, since H' consists of lines only.

2. Equations of $\mathbf{T}_{2-3}$.

Take O_1, O_2, O_3 and a general point A_4 of l as tetrahedron of reference, and take the unit point on l, whose equations are then $x=y=z$. Then all the base homaloids can be sets of planes:

$$x':y':z':w' = x(y-z) : y(x-z) : w(y-z) : w(x-z),$$

$$x:y:z:w = x'w'(z'-w') : y'z'(z'-w') : z'w'(x'-y') : z'w'(z'-w').$$

$$J=2w(y-z)(z-x)(x-y),\quad J'=-3z'^2w'^2(z'-w')^2(x'w'-y'z').$$

The F- and P-systems are:

$$H\left\{\begin{array}{llllll} F\text{-line } l & x=y=z & \sim P\text{-quadric} & q'\equiv x'w'-y'z', & \text{containing} & l', l_1', l_2', l_3' \\ F\text{-point} & O_1(1,0,0,0) & \sim P\text{-plane} & z', & ,, & l', l_1' \\ \quad ,, & O_2(0,1,0,0) & \sim \quad ,, & w', & ,, & l', l_2' \\ \quad ,, & O_3(0,0,1,0) & \sim \quad ,, & z'-w', & ,, & l', l_3' \end{array}\right\}J'$$

$$J\left\{\begin{array}{llllll} P\text{-plane } O_1O_2O_3 & w & \sim \text{double} & F\text{-line} & l', & z'=w'=0 \\ \quad ,, \quad lO_1 & y-z & \sim \text{simple} & ,, & l_1', & x'=z'=0 \\ \quad ,, \quad lO_2 & z-x & \sim \quad ,, & ,, & l_2', & y'=w'=0 \\ \quad ,, \quad lO_3 & x-y & \sim \quad ,, & ,, & l_3', & x'-y'=z'-w'=0 \end{array}\right\}H'$$

A general point $(\xi, \eta, 0, 0)$ of $l' \sim$ the P-conic $\kappa_2 \equiv \eta x(y-z) - \xi y(x-z)$, in w, through O_1, O_2, O_3 and $X\,(1, 1, 1, 0)$. A plane $\theta z' - \zeta\omega'$ through $l' \sim$ the plane $\theta(y-z) - \zeta(x-z)$ through l; the two pencils of planes are projective, and between corresponding planes T_{2-3} sets up the homography

$$\begin{aligned} x : y : z : w &= \theta(\zeta-\theta)x' : \zeta(\zeta-\theta)y' : \zeta\theta(x'-y') : \theta(\zeta-\theta)z', \\ x' : y' : z' : w' &= \zeta x : \theta y : \zeta w : \theta w. \end{aligned}$$

In this, a general point $Q\,(\xi, \xi, \xi, 1)$ of $l \sim Q'\,(\xi\zeta, \xi\theta, \zeta, \theta)$, and $l \sim$ the line $\theta x' - \zeta y' = \theta z' - \zeta w' = 0$, a generator of q' of the opposite system to l', which describes q' as the planes rotate round l, l'. If Q remains fixed, Q' describes the P-line $x' - \xi z' = y' - \xi w' = 0$, a generator of q' of the same system as l'.

A general point $(0, \eta, 0, \theta)$ of $l_1' \sim$ the P-line κ_1, $\theta y - \eta w = y - z = 0$ through O_1; a plane $\zeta x' - \xi z'$ through $l_1' \sim$ the plane $\zeta x - \xi w$ through O_2O_3, which meets lO_1 in a line of the pencil vertex $(0, 1, 1, 0)$, the intersection of O_2O_3 with O_1X.

To examine a general plane p' of S', say $ax' + by' + cz' - w'$, which $\sim$ a certain homaloid $\phi \equiv lO_1O_2O_3$, we can project the homologue P, of any point P' of p', from O_3 into P_1 in the plane lO_1; then

$$\begin{aligned} w' &= ax' + by' + cz', \quad y_1 = z_1, \\ x_1 : y_1 : w_1 &= x : y : w = x'w' : y'z' : z'w', \\ &= x'(ax' + by' + cz') : y'z' : z'(ax' + by' + cz'), \\ x' : y' : z' &= x_1(w_1 - by_1) : y_1(ax_1 + cw_1) : w_1(w_1 - by_1), \end{aligned}$$

giving a T_2 between the planes p', lO_1, in which the F-points in p' are the traces of l', l_1', l_2', and in lO_1, the traces of the line $w_1 - by_1 = ax_1 + cw_1 = 0$, and of O_2O_3, and the point O_1. The point $(0, 1, 0, b)$ of $l_1', p' \sim$ the same P-line $y - z = w_1 - by_1 = 0$ in this T_2 as in T_{2-3}.

If p' passes through the intersection of l' with l_1' or l_2', then $b = 0$ or $a = 0$, and the plane T_2 has two adjacent F-points.

3. Application to a quartic surface.

By means of T_{2-2}, certain singular quartic surfaces can be reduced to lower degree, in particular, an f_4' with a double line l' and another d.p. O' (p. 198). By means of T_{2-3}, the quartic $f_4' \equiv l'^2$ can be reduced when it has no other d.p.

There are eight pairs of intersecting lines on f_4', all meeting l'; take three skew lines as l_1', l_2', l_3'. A general $f_4' \sim f_8 \equiv (lO_1O_2O_3)^4$; from this we drop the P-plane $O_1O_2O_3$ twice and the three P-planes lO_α; the proper homologue of $f_4' \equiv l'^2$ is $f_3 \equiv lO_1O_2O_3$. Thus f_4' is mapped on the cubic, and all the curves on f_4' can be studied by means of their images on f_3, considered in relation to l, which is any one of the 27 lines, and to the three general points O_1, O_2, O_3 of f_3.

Of the 26 other lines, five intersecting pairs meet l and $\sim$ five of the eight pairs of lines on f_4' meeting l'; the other three pairs lie in the P-planes, and each consists of l_a' and a line meeting l_a' and l', which answers to the tangent plane to f_3 at O_a. The remaining 16 lines of $f_3 \sim 16$ conics on f_4', meeting l', l_1', l_2', l_3'. If f_4' has a line skew to l', we can take l_1', l_2', l_3' to meet it; then it is a P-line, and $\sim$ a point of l double on f_3.

4. Resolution (160).

The general T_{2-3} is compounded of two T_{2-2}'s, in each of which the F-conic breaks up. To the quadric family $(\phi_2) \equiv lO_1O_2O_3$ apply a T_{2-2}, between spaces S, S'', whose F-system in S consists of O_2, l and a line through O_1 meeting l, and in S'' of $O''\,l''\,l_1''$.

The homologue of (ϕ_2) arises from dropping the planes $O''l''$, $l''l_1''$ from $(\phi_4'') \equiv (O''l''l_1'')^2$, and is therefore $(\phi_2'') \equiv O''l_1''$; it also contains O_3'' and the P-line l_2'', meeting l_1'', which $\sim O_1$, and absorbs O'' as an ordinary point on it.

Then O_3'', l_1'', l_2'' are the F-system of another T_{2-2} which converts (ϕ_2'') into planes. The transformation compounded of the two T_{2-2}'s converts (ϕ_2) into planes, and is therefore a general T_{2-3}.

The equations of p. 288 can be replaced by the two sets of quadratic equations:

$$\begin{cases} x : y : z : w = x''(y''-z'') : w''(x''-z'')-z''(x''-y'') : z''(y''-z'') : w''(y''-z'') \\ x'' : y'' : z'' : w'' = x(y-z) : w(x-z)-z(x-y) : z(y-z) : w(y-z) \end{cases}$$

$$\begin{cases} x'' : y'' : z'' : w'' = x'(z'-w') : (x'-y'+w')(z'-w') : z'(x'-y') : z'(z'-w') \\ x' : y' : z' : w' = x''(z''-w'') : -w''(x''-z'')+z''(x''-y'') : w''(z''-w'') : w''(z''-y''). \end{cases}$$

5. F-line of partial contact.

As a particular case, in S' let one of the simple F-lines $l_3' \rightarrow$ the double line l'. In the limit, one sheet of every ϕ_3' touches, at every point of l', the limit of the plane $l'l_3'$, and l' is a double F-line of partial contact, with a contact condition for one set of sheets only.

Let the equations of l_3' now be $\epsilon(x'-y')+z'=z'-w'=0$, those of l', l_1', l_2' being as before. Then as $\epsilon \rightarrow 0$, so $l_3' \rightarrow l'$, moving in the fixed plane $z'-w'$.

The general cubic homaloid is now

$$\phi_3' \equiv (c_1x'w'+c_2y'z'+c_4z'w')(z'-w')+c_3z'w'\{\epsilon(x'-y')+z'\}.$$

At any point $(\xi, \eta, 0, 0)$ of l', the pair of tangent planes to ϕ_3' are

$$t_2' \equiv (c_1\xi w'+c_2\eta z')(z'-w')+c_3\epsilon(\xi-\eta)z'w'.$$

This does not in general fall into factors rational both in z', w' and in ξ, η, and there is no surface with only one sheet through l', touching one sheet of ϕ_3'. But if $\epsilon=0$, then t_2' falls into the two rational factors

$$z'-w', \quad c_1\xi w'+c_2\eta z';$$

one sheet of ϕ_3' touches the fixed plane $z'-w'$, the same for every homaloid and for every point of l'; the other sheet of ϕ_3' touches the quadric

$$c_1x'w'+c_2y'z',$$

and the tangent plane varies for the different homaloids, and for the same one at different points of l'.

Now the P-quadric q', which $\sim l$, breaks up into the two P-planes $l'l_1'$, $l'l_2'$, which $\sim O_1, O_2$; hence O_1 and O_2 lie on l, or rather are adjacent to it, the tangent planes to (ϕ_2) at these points being fixed. The equations of transformation are

$$x' : y' : z' : w' = \quad xz \quad : \ y(z-w) \ : \ zw \ : \ w(z-w)$$

$$x : y : z : w = x'w'(z'-w') : y'z'(z'-w') : z'^2w' : z'w'(z'-w')$$

and (ϕ_2) contains l, $z=w=0$ and $O_3(0, 0, 1, 0)$, and touches z at the point $(1, 0, 0, 0)$ of l, and $z-w$ at $(0, 1, 0, 0)$.

6. **Ruled cubic homaloids** (85, 86, 159, 216, 287).

The transformation just considered is the first of a series in each of which one homaloidal web consists of cubic surfaces with a double line l, and therefore ruled, since any plane through l meets ϕ_3 in l counted twice and one other line.

Now if (ϕ_3) already has l as a double line, the postulation and equivalence of a generator, which meets l, are each 2, as for two simple points. The cubic web of T_{3-2} remains homaloidal if any one of the three fundamental generators is replaced by a pair of points. But since the total degree of the F-lines is reduced by 1 by the change, n' is increased by 1, and the transformations of the series are as follows:

(i) T_{3-2}, considered above; the cubic homaloids have a double F-line and three simple generators.

(ii) T_{3-3}, with a double F-line, two generators and two simple points (13).

(iii) T_{3-4}, with a double F-line, one generator and four simple points.

(iv) T_{3-5}, with a double F-line and six simple points.

All these can be compounded of T_{2-2}'s. To the cubic family (f_3) of (iv) apply an auxiliary T_{2-2} whose F-system consists of one of the six points, A_1 say, and a conic breaking up into l and a line l_1 meeting l and passing through A_2, another of the six. From the homologue there fall away the plane $l'A_1'$ twice and the plane $l'l_1'$ once, and the degree of (f') is $3 . 2 - 2 - 1 = 3$. This cubic family contains l_1' twice; in place of A_2, which is a P-point of T_{2-2}, we have on (f_3') a line l_2' meeting l_1' and absorbing A_1', which is simple; and four simple points arising from the rest of the original six. This is the homaloidal family of (iii).

We can repeat the process, at each step losing two points and gaining one line, and so reduce (iii) to (ii), and (ii) to (i), which we know to be compounded of two T_{2-2}'s; or these families are converted directly into quadrics by a T_{2-2} applied to l, a generator and a simple point, either isolated or on another generator.

7. $\mathbf{T_{2-4}}$; F-point of contact (18, 83, 85, 86, 160, 216, 234).

The only other quadric transformation is that whose reverse is quartic (p. 169). There are no *F*-curves in *H*, which consists of a point of contact O_1 and three simple points O_2, O_3, O_4. The homaloidal curves are $(c_4) \equiv O_1^2 O_2 O_3 O_4$, both branches at O_1 touching the fixed tangent plane of (ϕ_2).

The quartics in S' are a web of Steiner surfaces, and H' consists of three double lines meeting in a point, which is necessarily triple on (ϕ_4'), and a simple conic meeting the three lines.

The simplest forms of the equations are

$$x' : y' : z' : w' = \quad xp \quad : \quad zw \quad : \quad yw \quad : \quad yz, \quad \text{where } p \equiv y + z + w,$$

$$x : y : z : w = x'y'z'w' : z'w'q' : y'w'q' : y'z'q', \quad \text{where } q' \equiv z'w' + y'w' + y'z',$$

and

$$J \equiv 2yzwp, \quad J' \equiv 4y'^2z'^2w'^2q'^3.$$

This is compounded of the two T_{2-2}'s

$$x'' : y'' : z'' : w'' = x'w' : z'w' : y'w' : y'z',$$

and

$$x : y : z : w = x''w'' : y''p'' : z''p'' : w''p''.$$

If a surface f passes through O_1, the factor q' falls away from f'; if f touches p at O_1, then q'^2 falls away.

Consider more in detail the correspondence between the *F*-point of contact O_1 and the *P*-quadric q'. Let $P \rightarrow O_1$ along a general line l, not lying in p, say

$$y : z : w = \eta : \zeta : \theta;$$

then the path of p' is the line

$$y' : z' : w' = \zeta\theta : \eta\theta : \zeta\eta;$$

and as $P \rightarrow O_1$, so $P' \rightarrow A_1'$ and to no other point, giving no indication that O_1 is an *F*-point.

But if l lies in p, then for all positions of P on l, we find that P' is the fixed point Q' $(0, \zeta\theta, \eta\theta, \zeta\theta)$, which lies on the *F*-conic in S'; for now l is a *P*-line and all its points ~ one *F*-point. We cannot make P' approach a general point Q' of q', until we give up lines of approach of P to O_1.

Let $P' \rightarrow Q'$ $(\xi, \eta, \zeta, \theta)$ where q' $(\eta, \zeta, \theta) = 0$, along the general line

$$\frac{\theta x' - \xi w'}{a} = \frac{\theta y' - \eta w'}{b} = \frac{\theta z' - \zeta w'}{c}.$$

The path of P is

$$\frac{\theta xp - \xi yz}{a} = \frac{(\theta w - \eta y)\, z}{b} = \frac{(\theta w - \zeta z)\, y}{c}.$$

This has two branches through O_1, one whose tangent depends on a, b, c, associated with the second intersection of l', q', and one associated with Q', touching the line $\eta y = \zeta z = \theta w$ of p, and osculating the quadric $\theta xp - \xi yz$, which is a member of a certain pencil of quadrics touching p at O_1, which ~ the pencil of planes through $A_2'A_3'$, and are thus in 1, 1 relation to their intersections with any generator of q', such as $A_1'Q'$.

The intersection of (ϕ_2) with p consists of two variable lines of the pencil (O_1). Apply a T_{2-2} with p as P-plane, and otherwise general: $(\phi_2) \sim$ a quartic homaloidal web, which has, at the isolated F-point of T_{2-2}, a binode with a fixed edge arising from O_1, and two variable biplanes (p. 238).

8. $\mathbf{T_{4-4}}$, Steiner quartics (5, 85, 216, 259).

There is another homaloidal web of Steiner quartics, the F-system consisting of the three double lines l_1, l_2, l_3 and three simple points O_1, O_2, O_3; the reverse is of the same nature. The equations can have the form

$$x' : y' : z' : w' = q_2 q_3 : q_3 q_1 : q_1 q_2 : xyzw,$$

where
$$q_1 \equiv \begin{vmatrix} yz, & zx, & xy \\ \eta_2\zeta_2, & \zeta_2\xi_2, & \xi_2\eta_2 \\ \eta_3\zeta_3, & \zeta_3\xi_3, & \xi_3\eta_3 \end{vmatrix},$$

the quadric cone containing l_1, l_2, l_3, O_2, O_3.

Here l_1 is $y = z = 0$, and O_1 is $(\xi_1, \eta_1, \zeta_1, 0)$, etc. The P-surfaces in S are the three planes x, y, z, each containing two F-lines, which $\sim$ the three simple F-points in S', and the three quadric cones q_1, q_2, q_3, each containing l_1, l_2, l_3 and two of the F-points, which $\sim$ the double F-lines in S'. The triple point $(0, 0, 0, 1)$ is not an isolated F-point.

This T_{4-4} is compounded of three T_{2-2}'s or of two of the T_{2-4}'s just described.

If O_1, O_2, O_3 lie by pairs on planes p_1, p_2, p_3 through l_1, l_2, l_3 respectively, then q_1 breaks up into xp_1, and so for q_2, q_3, but q_1', q_2', q_3' are proper. If O_1, O_2, O_3 lie on a quadric cone q through l_1, l_2 and l_3, each of q_1, q_2, q_3 coincides with q. The above equations break down; we can replace $q_2 q_3$ by qyz, etc. The section of q by $w \equiv O_1 O_2 O_3$ meets (ϕ_4) in nine fixed points and is fundamental; the transformation reduces to T_{2-4}.

The involutory case of this T_{4-4} is discussed in 259

9. $\mathbf{T_{4-4}}$, square of $\mathbf{T_{2-2}}$ (86, 97, 160).

The simplest example of composition of Cremona space transformations is afforded by the square of the general T_{2-2}. With tetrahedra of reference in general positions, let V be a T_{2-2} between S, S', given by (p. 201):

$$\begin{aligned} &p_1'(x', y', z', w') : p_2' : p_3' : p_4' \\ &\qquad = p_1(x, y, z, w)\, p_4(x, y, z, w) \qquad : p_2 p_4 : p_3 p_4 : q(p_1, p_2, p_3), \\ &p_1(x, y, z, w) \quad : p_2 : p_3 : p_4 \\ &\qquad = p_1'(x', y', z', w')\, p_4'(x', y', z', w') : p_2' p_4' : p_3' p_4' : q(p_1', p_2', p_3'), \end{aligned}$$

where $p_1, \dots p_4'$ are any eight linear functions; and let W be a T_{2-2} of the same form between S' and a third space S'', given by replacing $x, \dots x', \dots$ by $x', \dots x'', \dots$ in these equations.

A general point P of S $\tilde{V}$ a general point P' of S', and this $\tilde{W}$ a point

P'' of S''. This constitutes a Cremona transformation $T \equiv W.V$, compounded of V and W, between the points of S, S''.

Let the two F-systems of V be O_1, ω_2 in S, and O'_V, ω'_V in S'; and let those of W be O'_W, ω'_W in S' and O'', ω'' in S. If the T_{2-2} is quite general, the two F-systems in S' have no special relations, and no F-element of either is incident with an F-element or lies upon a P-surface of the other T_{2-2}.

A plane of S'' $\tilde{W}$ a quadric in S' through O'_W, ω'_W; as regards V, this is a general quadric, and it $\tilde{V}$ a quartic surface ϕ_4 in S with the first F-system of V as double elements, and also passing once through the point O_2 and curve ω_4 which $\tilde{V}$ O'_W, ω'_W. Here O_2 is a general point of S as regards V, and ω_4 is a quartic curve with two branches through O_1 and four intersections with ω_2; it lies on the quadric $\phi_2 \equiv O_1\omega_2$, which $\tilde{V}$ the plane of ω'_W. Thus the transformation T_{4-4} between S, S'' is quartic, and the first homaloidal web is $(\phi_4) \equiv O_1^2 O_2 \omega_2^2 \omega_4$. The same argument with S, S'' interchanged shows that the reverse of T_{4-4} is of the same nature.

The homaloidal curve c_4 of T_{4-4} $\tilde{V}$ the c_2' which $\tilde{W}$ a general line of S''. Now c_2' passes through O'_W and meets ω'_W twice; it has no incidences with O'_V, ω'_V, but meets the second P-plane of V twice and the P-cone four times. Hence c_4 is a quartic through O_2 meeting ω_4 twice, with two branches through O_1 and meeting ω_2 four times. The residual intersection of ϕ_4, ψ_4 consists of ω_2 counted four times and ω_4 once; the 16 intersections of c_4, χ_4 are 1 at O_2, 2 on ω_4, 4 at O_1, 8 on ω_2 and 1 free.

10. $\mathbf{T_{4-16}}$, square of $\mathbf{T_{2-4}}$.

The square of T_{2-4} is a T_{4-16}, interesting as being the simplest T_{n-n^2} which is not monoidal (p. 308). The quartic homaloids are the homologues in a T_{4-2} of the quadric homaloids of a T_{2-4}. The first F-system of T_{4-16} is therefore made up of the first F-system of one T_{2-4} taken twice over, together with another system of the same nature taken once; it consists of a point where two separate sheets of (ϕ_4) touch the same given plane, that is, a tacnode, three double points, one simple point of contact and three simple points. There is no triple point and the quartics are not monoids.

The explicit singularities of (ϕ_4) are all d.p.'s, and do not present any conditions to any adjoint (p. 166); but ϕ_4 has an infinitesimal double line implicit at the tacnode, which would lie on the first adjoints, of degree 0; this system is therefore self-contradictory, and does not exist.

II. THE BILINEAR T_{3-3}

11. Three bilinear equations (61, 85, 86, 133, 158, 223, 287, 343, 386).

We have considered all the three essentially distinct quadratic transformations; there is just one for each of the possible degrees, 2, 3 and 4, of

T^{-1}. The number of distinct cubic transformations, of degrees $3-2$ to $3-9$, is 75 (Table VI).

There is only one T_{3-2}, the reverse of T_{2-3}. There are four T_{3-3}'s; besides the one non-singular type, there is one with an ordinary double F-point, whose involutory case is the Arguesian transformation (p. 322); the third has a double F-point of contact, and the fourth a double line (p. 291).

The non-singular cubo-cubic transformation was the first space transformation, other than collineation, to be discovered (40). It is in many ways simpler than T_{2-2}, and a more natural extension to space of the plane T_2.

It is defined by three equations linear in each set of variables:

$$x't_1 + y't_2 + z't_3 + w't_4 = 0, \text{ where } t_a \equiv a_{1a}x + a_{2a}y + a_{3a}z + a_{4a}w,$$

$$x'u_1 + y'u_2 + z'u_3 + w'u_4 = 0, \text{ where } u_a \equiv b_{1a}x + b_{2a}y + b_{3a}z + b_{4a}w,$$

$$x'v_1 + y'v_2 + z'v_3 + w'v_4 = 0, \text{ where } v_a \equiv c_{1a}x + c_{2a}y + c_{3a}z + c_{4a}w.$$

These can be rearranged in the equivalent forms

$$xt_1' + yt_2' + zt_3' + wt_4' = 0, \text{ where } t_a' \equiv a_{a1}x' + a_{a2}y' + a_{a3}z' + a_{a4}w', \text{ etc.}$$

If x, y, z, w are given, one of these equations gives a plane in S', and defines a space correlation; our T_{3-3} transforms a point into the intersection of three planes, its homologues in three given correlations (381 and p. 40).

When the equations are solved, x', y', z', w' are given explicitly, proportional to the four determinants, cubic in x, y, z, w,

$$\phi_1, \phi_2, \phi_3, \phi_4 \equiv \left\| \begin{matrix} t_1, & t_2, & t_3, & t_4 \\ u_1, & u_2, & u_3, & u_4 \\ v_1, & v_2, & v_3, & v_4 \end{matrix} \right\|;$$

for example

$$x' : y' = \phi_1 : \phi_2 = t_2 \begin{vmatrix} u_3, & u_4 \\ v_3, & v_4 \end{vmatrix} + u_2 \begin{vmatrix} v_3, & v_4 \\ t_3, & t_4 \end{vmatrix} + v_2 \begin{vmatrix} t_3, & t_4 \\ u_3, & u_4 \end{vmatrix}$$

$$: -t_1 \begin{vmatrix} u_3, & u_4 \\ v_3, & v_4 \end{vmatrix} - u_1 \begin{vmatrix} v_3, & v_4 \\ t_3, & t_4 \end{vmatrix} - v_1 \begin{vmatrix} t_3, & t_4 \\ u_3, & u_4 \end{vmatrix} \text{.......(1)}$$

Similarly, x, y, z, w are proportional to cubic functions of x', y', z', w'.

Thus T is cubic both ways; the degree of the F-curve in each space is $3^2 - 3 = 6$. The total intersection of any two homaloids breaks up into a fixed sextic ω_6 and a variable twisted cubic c_3.

Now ϕ_1, ϕ_2 pass through the twisted cubic

$$t_3/t_4 = u_3/u_4 = v_3/v_4, \text{ ....(2)}$$

which does not lie on ϕ_3, ϕ_4, and is the particular position of c_3 for the pair of homaloids ϕ_1, ϕ_2. The residual is the F-curve ω_6 of genus 3, with seven apparent d.p.'s and in general no actual d.p.'s; it meets c_3 eight times and absorbs all but one of the intersections of ϕ, ψ, χ. There is no other F-element of any kind, H consists of ω only. In S' there is an F-sextic ω' of the same nature, which constitutes H'.

A point Q' of ω' is simple on (ϕ'), and $\sim$ a P-line κ in S. This meets (ϕ) in three points which are all F-points, and κ is a trisecant of ω. The Jacobian J_8 is the locus of these trisecants; on it ω is triple, for the quintic cone projecting ω from a point O of itself is of genus 3 and has three double edges, which are trisecants of ω through O. The six intersections of ω' with a variable plane $\sim$ six variable P-lines lying on one ϕ, which with ω^3 make up the intersection of ϕ, J. The eight intersections of ω', c' $\sim$ those of J with a general line.

Thus a general line $l \sim$ a cubic c_3' meeting ω' in eight points; if l meets ω, a trisecant κ' is separated from c_3', and $l \sim$ a conic meeting ω' in five points; while a chord of $\omega \sim$ a chord of ω'.

We know (p. 170) that there is only one type of non-singular T_{3-3}, and its F-system consists of one curve ω_6. If such a transformation T is given, we can form bilinear equations to represent it. For two of its homaloids ϕ_1, ϕ_2 have a residual intersection c_3, which can be represented, in many ways, in the form (2). Since ϕ_1, ϕ_2 contain c_3, we can express them, again in many ways, in the form (1). The twelve functions $t_1 \ldots v_4$ then determine a bilinear transformation, with ϕ_1, ϕ_2 as two homaloids and ω_6 as F-sextic. But T is determined by its F-system, that is, by ω_6 alone, and is therefore identical with the bilinear T_{3-3} which we have constructed. For a geometrical proof, see 316.

In its general form, T_{3-3} is as simple as a transformation can be. There are no multiple F-points or contact conditions; the F-system consists of one curve and the P-system of one surface; J coincides with G, each having one sheet counted once; and T^{-1} is exactly like T.

It is the only $T_{n-n'}$ with only one F-element in each space. This must be an F-curve of first species; let H consist of ω^i_m and H' of $\omega'^{i'}_{m'}$. Then the first P-surface is counted once in J and i times in G:

$$nn' - 1 = 4i\,(n-1),$$

whence $$n\,(n'-1) = (4i-1)\,(n-1),$$

and similarly $$n'\,(n-1) = (4i'-1)\,(n'-1).$$

Thus each of $n-1$, $n'-1$ is a factor of the other. But n, $n' > 1$, since there is an F-system; hence $i = i'$, $n = n' = 4i - 1$. Then from the degree of the F-curve,

$$i^2 m = n^2 - n' = (4i-1)\,(4i-2) \equiv 2 \pmod{i},$$

$$i = 1, \quad n = n' = 3, \quad m = 6,$$

and we have the T_{3-3} under discussion.

12. Singularities of ω_6.

In practice, the general T_{3-3} is not so useful as some of the many particular cases where ω is singular or breaks up. Then there may be different types of trisecants, according as their intersections with ω fall on the different components, or at the multiple points of ω; if so, J breaks up. Then ω', J' may behave in the same way as ω, J, or differently. We consider various examples.

If ω acquires an actual d.p. O, it still has seven apparent d.p.'s, and its genus is reduced by 1. Then O is a point of contact of (ϕ), which touch the plane p containing the two tangents to ω; it is not an isolated F-point, since c does not pass here; it $\sim$ a P-line κ' double on J'. Any line through O in p is a chord of ω, and $\sim$ a chord of ω' meeting κ'. Hence $p \sim$ a ruled cubic ϕ' with κ' as double line; this meets any other ϕ' in a $k_9' \equiv \omega_6' \,.\, c_3'$ with three double points on κ', which cannot all lie on c' as this is not plane, and one is a d.p. of ω'. Hence ω' acquires a d.p. whenever ω does, and they are of the same genus.

If ω acquires a triple point O, there are two distinct cases according as the three branches of ω do or do not all touch a plane p. In the first case, ω still has seven apparent d.p.'s, and is rational; O is a point of osculation of (ϕ), and ∞^2 ϕ's have a binode at O with p as one biplane. There is one proper trisecant $\kappa \equiv OQ_1Q_2$ through O, and $O \sim$ a trisecant $\kappa' \equiv O'Q_1'Q_2'$ of ω', triple on J', where $O' \sim \kappa$, and Q_1', $Q_2' \sim$ the other two trisecants κ_1, κ_2 through O, which are generators of the osculating quadric of (ϕ). Now κ is a double line on one homaloid ϕ, the cubic cone projecting ω from O; this does not contain κ_1 or κ_2, and meets any plane through O in three chords of ω through O. Hence $\phi \sim$ a plane p' containing O' but not Q_1', Q_2' or κ', meeting any ϕ' through κ' in three chords of ω', which meet κ' and therefore pass through O' the trace of κ' on p'. Thus each of these ∞^2 homaloids ϕ' contains four lines through the F-point O', three lying in p', and has a binode at O' with p' as a biplane; and O' is a point of osculation of (ϕ') and a triple point of ω'. Thus again ω' has the same singularity as ω.

In the second case, when the three branches of ω do not touch a plane, O is a d.p. of (ϕ) and absorbs one apparent d.p. of ω, which is of genus 1. Now c passes once through O, which is an isolated F-point, and must be reckoned as a distinct second F-element of H, although it is a necessary consequence of the first F-element ω; and $O \sim$ a P-surface of degree 1, counted twice in J', meeting (ϕ') in F-curves only, that is, a P-plane j_1', meeting ϕ' in an F-cubic ω_3', of genus 1. Hence ω_6' breaks up into ω_3' and a rational twisted cubic ω_3'', meeting ω_3' in the three points where ω_3'' meets j_1'. Now J' breaks up into j_1' counted twice and the locus j_6' of chords of ω_3'' meeting ω_3'. Any chord of ω through O meets (ϕ) in three F-points, and is a P-line, and J breaks up into the cubic cone $j_3 \equiv O^3\omega$, and the locus $j_5 \equiv \omega^2$ of proper trisecants.

Conversely, if ω' breaks up and part is a plane cubic, its plane $\sim$ an F-point O through which passes one branch of c; this is therefore double on (ϕ) and triple on ω. In this case, ω, ω' behave quite differently. In each space we have two F-elements, one being irrational of genus 1; but the second element in H is an isolated double F-point lying on ω_6, and in H' it is a rational curve.

If ω has four branches through a d.p. O of (ϕ), it is not an isolated F-point. Now ω_6 must break up into ω_5 and a line ω_1 through O; for if it all lay on the quadric cone j_2 projecting it from O, then ϕ would consist of j_2 and a general plane. Now the generators of $j_2 \equiv \omega_5$ are P-lines, and so are the chords of ω_5 meeting ω_1, whose locus is $j_4 \equiv \omega_1^3\omega_5$. The rest of $J_8 \equiv \omega_1^3\omega_5^3$ is a quadric through ω_5, that is, j_2 over again. Now j_2 meets ϕ in ω_5 and one line, hence $j_2 \sim$ a simple F-line ω_1' of H', which is a line of contact of (ϕ') because j_2 is counted twice in J. And j_4 meets ϕ in $\omega_1^3\omega_5$ and four lines, and $\sim$ a simple ω_4'. Hence ω_6' breaks up into two components, a line of contact and a quartic.

All the F-elements are rational.

Since a triple point of ω double on (ϕ) absorbs one of the seven chords of ω drawn from any point of space, and two of the eight intersections of c_3, ω, it is convenient to say that any curve through O meets two branches of ω and misses the third (p. 223); with this and similar conventions, we can count up the seven apparent d.p.'s in all cases when ω is singular or degenerate.

A further convention is needed if ω has a repeated part, that is, a curve of contact. The definition of the contact condition must give a fixed surface touched by (ϕ), preferably the surface of lowest degree. For a line of contact ω_1, this degree is 1, 2 or 3, according as there are 2, 1 or 0 d.p.'s of (ϕ) on ω_1. If (ϕ) touches a plane, the two lines that coincide in ω_1 are the limit of two intersecting lines, and have one actual and no apparent intersection; if (ϕ) touches a quadric, they are the limit of two generators of the same system, and have no actual and one apparent intersection. But if (ϕ) does not touch any surface of degree < 3, the case cannot arise as a limit, and we can count the seven apparent d.p.'s of ω only if we admit the paradox, that the two ω_1's meet in -1 points and have two apparent intersections (p. 259).

If ω_1 is counted four times in ω_6, it may be an F-line of four-point contact (63); or it may be a double F-line, giving a particular case of the ruled cubic transformation of p. 291, when the two isolated F-points lie on ω_1, becoming F-points of partial contact. Then ω_1 is met by c_3 in these points only, and is of second species; there is an F-line of the same nature in S', and this is the simplest example when the multiplicity of a curve of second species is not equal to the degree of its homologue. A particular case is worked out on p. 269 (158, 309), and another is given on p. 160.

13. Projective pencils of planes (287).

Let T_{3-3} convert a pencil of planes $u_1 - av_1$ into a pencil $u_1' - av_1'$. Then one of the bilinear equations is

$$u_1 v_1' - v_1 u_1' = 0,$$

which is not general, being *two-membered.* Conversely, if one of the equations has this form, there is a pair of projective pencils of corresponding planes.

If a plane contains l, the axis of the pencil, its homologue is a plane, and a P-quadric is dropped; hence l is an F-line that meets c_3 twice. But it meets its total residual four times, and is therefore a chord of the other F-curve ω_5. Then ω_5 has three apparent intersections with l, and four apparent d.p.'s, and is of genus 2.

If there are two such pairs of pencils of corresponding planes, then in each space the F-curve is a quartic of genus 1 and two chords.

Now let there be three pairs of projective pencils, with axes $l_1 \ldots l_3'$. This is another natural extension to space of a construction for the general plane T_2 (p. 39, 9, 116, 201, 378, 391).

Let
$$u_1 - a_1 v_1 \sim u_1' - a_1 v_1',$$
$$u_2 - a_2 v_2 \sim u_2' - a_2 v_2',$$
$$u_3 - a_3 v_3 \sim u_3' - a_3 v_3'.$$

The three two-membered bilinear equations are
$$u_1 v_1' - v_1 u_1' = u_2 v_2' - v_2 u_2' = u_3 v_3' - v_3 u_3' = 0.$$

If P lies on l_1, then $u_1 = v_1 = 0$, and a_2, a_3 are determinate but not a_1. The locus of P' is the P-line
$$u_2' - a_2 v_2' = u_3' - a_3 v_3' = 0,$$
where
$$a_2 = u_2/v_2, \quad a_3 = u_3/v_3,$$
which describes a P-quadric through l_2', l_3' but not l_1'.

A line l' meeting l_1', l_2', l_3' meets ϕ' in three F-points and is a P-line; its locus is a fourth P-quadric j_2', which completes J_8'. Now $l' \sim$ a point Q, intersection of three planes of the pencils which are projective with the generators of j_2', and therefore with each other; and Q describes a twisted cubic ω_3 meeting each of the axes twice.

Conversely, if ω_6 breaks up into a cubic and three chords, the transformation is of this type, with three pairs of projective pencils.

Any cubic curve with l_1, l_2, l_3 as chords $\sim$ another such cubic. If the transformation is repeated, in one space, the axes being fixed in coincident pairs and the F-cubic varied, we obtain a group of transformations which permute this family of cubics (414).

Let two axes intersect in a point O, lying in a plane p which belongs to both pencils. The last P-quadric breaks up into p and Ol_3, and ω_3' breaks up, part being the line ω_1', intersection of the two planes of the pencils (l_1'), (l_2') each associated with p, and the other part a conic ω_2', meeting ω_1', l_1', l_2' once and l_3' twice. Now ω_3 does not break up; it meets l_1, l_2 each twice, but p only three times: hence it passes through O without touching p there; O is triple on ω_6 and a d.p. of (ϕ), and $\sim$ the P-plane meeting (ϕ') in a plane F-cubic ω_2', l_3'. Now O is an isolated F-point, and there are five F-elements in each space.

If the three axes form a triangle, this is a fundamental plane section of (ϕ); and ω_3 passes through the corners, which are double F-points. Then (ϕ') has three fundamental plane sections and one double F-point O', and ω_3' breaks up into three lines through O' (362).

If in each space the three axes form a triangle, each F-cubic breaks up into three concurrent lines through the corners, and ω_6, ω_6' consist of the six edges of a tetrahedron (p. 301).

If there are four pairs of projective pencils with skew axes, then ω_6 breaks up into the axes l_1, l_2, l_3, l_4 and their two transversals l_5, l_6 (14, 158). There are four P-quadrics, each containing three axes and l_5, l_6, and c_3 meets each of $l_1 \ldots l_4$ twice and does not meet l_5, l_6, which are of second species, and $\sim l_5'$, l_6', F-lines of second species in S'. We consider the correspondence between these in greater detail.

The transformation can be expressed as follows; a more symmetrical notation is given in 14.

F-lines:

$$\begin{array}{lrcrl lrcrl}
l_1, & x & = & z & = 0, & l_1', & x' + cy' & = & az' + w' & = 0,\\
l_2, & y & = & w & = 0, & l_2', & x' + dy' & = & bz' + w' & = 0,\\
l_3, & ax + by & = & z + w & = 0, & l_3', & x' & = & z' & = 0,\\
l_4, & x + y & = & cz + dw & = 0, & l_4', & y' & = & w' & = 0,\\
l_5, & x & = & y & = 0, & l_5', & x' & = & y' & = 0,\\
l_6, & z & = & w & = 0, & l_6', & z' & = & w' & = 0.
\end{array}$$

Two-membered bilinear equations: any three of

$$\begin{array}{ll}
x(az' + w') = z(x' + cy'), & (ax + by)z' = (z + w)x',\\
y(bz' + w') = w(x' + dy'), & (x + y)w' = (cz + dw)y'.
\end{array}$$

Equations of transformation:

$$\begin{aligned}
x' : y' : z' : w' &= (ax + by)(dxw - cyz) : -(x + y)(axw - byz)\\
&\quad : (z + w)(dxw - cyz) : -(cz + dw)(axw - byz),\\
x : y : z : w &= (x' + cy')(x'w' - bdy'z') : -(x' + dy')(x'w' - acy'z')\\
&\quad : (az' + w')(x'w' - bdy'z') : -(bz' + w')(x'w' - acy'z').
\end{aligned}$$

Let $P \rightarrow$ a general point $O(0, 0, \zeta, \theta)$ of l_5, by a path in the plane $p \equiv \eta x - \xi y$. Then x, y are small compared with z, w, and are in the fixed ratio $\xi : \eta$; and $P' \rightarrow$ the definite point O' of l_5', whose coordinates

$$0,\ 0,\ (\zeta + \theta)(d\xi\theta - c\eta\zeta),\ -(c\zeta + d\theta)(a\xi\theta - b\eta\zeta),$$

are linear functions of ξ, η and quadratic functions of ζ, θ.

If O, and therefore ζ/θ, is constant, and p rotates about l_5, so ξ/η takes every value once and O' describes l_5' once. We have a homography between the pencil of planes through l_5 and the points of l_5'. There is a different homography between the same elements for each point O of l_5. But if p, and therefore ξ/η, is constant, and O describes l_5, then ζ/θ takes every value once, and O' describes l_5' twice; for $p \sim$ a ϕ' having l_5' as a double line, and there are two distinct patches of ϕ' adjacent to every point O', which $\sim$ two patches of p adjacent to distinct points of l_5.

14. T_{tet} (3, 9, 20, 55, 69, 120, 191, 192, 223, 287, 297, 402).

If ω_6 consists of the six edges of a tetrahedron, we have a very important transformation, which we denote by T_{tet}. Any three faces of the tetrahedron form a degenerate homaloid, and we can take

$$x' : y' : z' : w' = yzw : zxw : xyw : xyz = \frac{1}{x} : \frac{1}{y} : \frac{1}{z} : \frac{1}{w},$$

and the bilinear equations are

$$xx' = yy' = zz' = ww',$$

another extension of the general plane T_2.

Now each vertex is a d.p. of (ϕ) and c_3 passes through each and meets χ twice there; it does not meet any F-line again, and they are all of second species, and are a necessary consequence of the four d.p.'s; this is a chief property of the punctual transformations, of which T_{tet} is the simplest (p. 318).

There are ten F-elements in H, in two sets: six lines and four d.p.'s; and either set is a necessary consequence of the passage of (ϕ) through the other set. The only P-surfaces are the four faces of each tetrahedron, corresponding to the vertices of the other, reckoned twice in J.

For one d.p., the postulation and equivalence are $P = 4$, $E = 8$. If n were high enough, the four d.p.'s would have $P = 16$, $E = 32$; the appearance of the F-lines when $n = 3$ does not affect P, since they present no fresh conditions; but it completely alters E, which falls to 26, owing to the effect on the degree of c.

If the coordinates are proportional to the distances of P, P' from the same planes of reference, then the equations

$$xx' = yy' = zz' = ww'$$

express that the products of perpendiculars from P, P' to these four planes are equal, and P, P' are the foci of an inscribed quadric of revolution. These foci are therefore related by I_{tet}, the involutory case of T_{tet}. If one describes a line, the other describes a cubic curve through the vertices; if one describes a plane, the other describes a cubic surface through the edges. In particular, if one describes the plane at ∞, the quadric is a paraboloid of revolution touching four given planes, and the locus of its focus is a cubic (133).

More generally, let q_1 be any quadric. With a proper system of tangential coordinates $(\xi, \eta, \zeta, \theta)$, we can take

$$q_1 \equiv \xi^2 + \eta^2 + \zeta^2 + \theta^2 + a_1\eta\zeta + b_1\zeta\xi + c_1\xi\eta + d_1\xi\theta + e_1\eta\theta + f_1\zeta\theta.$$

Let $$q_2 \equiv a_2\eta\zeta + b_2\zeta\xi + c_2\xi\eta + d_2\xi\theta + e_2\eta\theta + f_2\zeta\theta$$

be any quadric inscribed in the tetrahedron of reference. If the family

$q_1 + aq_2$ contains a pair of points $P(x, y, z, w)$ and $P'(x', y', z', w')$, then for this member

$$q_1 + aq_2 \equiv (x\xi + y\eta + z\zeta + w\theta)(x'\xi + y'\eta + z'\zeta + w'\theta),$$

we have $$xx' = yy' = zz' = ww',$$

and P, P' are related by I_{tet}.

In particular, if the family are confocal quadrics of revolution, with q_1 as the absolute quadric, then P, P' are the foci of q_2 (235, 236).

I_{tet} can also be constructed by means of a fixed quadric q and two of its points O_1, O_2. Let PO_1, PO_2 meet q again in X_1, X_2; then P' is the intersection of O_1X_2, O_2X_1. Thus P' lies in the polar plane of P, and PP' meets O_1O_2 and also the conjugate chord O_3O_4 of q. If P describes a line, the locus of P' is the product of three projective pencils of planes, and is a cubic curve through the four fixed points O_1, O_2, O_3, O_4 (69).

In another construction of I_{tet}, P' is the point of concurrence of the six planes, reflexions, in the bisector planes of the dihedral angles of a tetrahedron, of the six planes joining P to the edges (169, 348).

15. $\mathbf{T}_{3-3}$ in one space; invariant system.

If the spaces are superposed, we can extend the notation of p. 43, and separate each bilinear expression into its symmetric and skew parts S, Σ, where

$$\begin{aligned} S_a \equiv a_{11}xx' + a_{22}yy' + a_{33}zz' + a_{44}ww' \\ + a_{23}(yz' + zy') + a_{31}(zx' + xz') + a_{12}(xy' + yx') \\ + a_{14}(xw' + wx') + a_{24}(yw' + wy') + a_{34}(zw' + wz'). \end{aligned}$$

$$\begin{aligned} \Sigma_a \equiv A_{23}(yz' - zy') + A_{31}(zx' - xz') + A_{12}(xy' - yx') \\ + A_{14}(xw' - wx') + A_{24}(yw' - wy') + A_{34}(zw' - wz'). \end{aligned}$$

At an invariant point, $x = x'$, etc., $\Sigma \equiv 0$; the bilinear equation $S_a + \Sigma_a = 0$ reduces to $q_a = 0$, where

$$\begin{aligned} q_a \equiv a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{44}w^2 + 2a_{23}yz + 2a_{31}zx + 2a_{12}xy \\ + 2a_{14}xw + 2a_{24}yw + 2a_{34}zw, \end{aligned}$$

and the invariant system is that common to the three quadrics q_a, q_b, q_c, which can have any relative positions. One or two of them can vanish identically, but if all three vanish, T_{3-3} reduces to identity.

The invariant system is therefore in general a set of eight points, but may have any of the forms:

eight points,
a line and four points,
a conic and two points,
a twisted cubic,
a quartic of genus 1,
a plane and a point,
a plane and a line,
a quadric.

16. Bilinear involutions I_{3-3} (55, 99, 126, 316).

If T_{3-3} is involutory, the two F-systems coincide and also the two P-systems, and ω_6 breaks up in the same way in each space. Exactly as for the plane I_2 (p. 50), the equations hold with P, P' interchanged, and by adding and subtracting we get six equations, three symmetric and three skew, of which three are independent. Each symmetric equation $S_a = 0$ represents reciprocation with regard to a quadric q_a, and correlates P to its polar plane, on which P' lies; each skew equation $\Sigma_a = 0$ represents a null-system, and correlates P to a plane through P, on which P' lies.

Three null-systems lead only to $P \equiv P'$, the identical transformation; and there are three main types of bilinear involutions, given by

three reciprocations, or
two reciprocations and one null-system, or
one reciprocation and two null-systems.

17. Reciprocation with regard to a net of quadrics (40, 131, 153, 223, 373, 378, 379).

If the three independent equations are all symmetric, P, P' are conjugate with regard to q_a, q_b, q_c, and therefore with regard to any quadric of the net which they determine.

Since an F-point O does not determine a unique homologue, its polar planes are collinear, meeting in the P-line which $\sim O$, and form a pencil instead of a star; one is therefore indeterminate, and one quadric of the net is a cone vertex O. Hence ω_6 is the locus of vertices of cones of the net.

A point of any base curve ω of the net is an F-point, for the polar planes are tangent planes and meet in the tangent line to ω; thus $O \sim$ the tangent to ω at O, and ω is an F-curve, part of ω_6.

If a line l contains two pairs of conjugates, it meets its homologue c_3 in four points, and c_3 breaks up into l and two P-lines; hence l is self-corresponding, and contains two F-points and two invariant points. Now an invariant point lies on each of its polar planes, and therefore on each quadric, and is a base point of the net. Conversely, a line joining two base points is self-corresponding, and contains two invariant points.

If the quadrics (q) have a pair of skew base lines, PP' meets both and the transformation is linear. If they have a common conic, and therefore two other common points, we have seen that the transformation is I_{2-2} (p. 207); the plane of conic drops away from (ϕ_3). If (q) have a common plane, this drops away twice and the involution is linear. Thus there are three cases of this type of I_{3-3}, with invariant systems

(i) eight points, (ii) a line and four points, (iii) a twisted cubic.

(i) With eight invariant points, the simplest example is I_{tet}, where

$$q_a \equiv x^2 - w^2, \quad q_b \equiv y^2 - w^2, \quad q_c \equiv z^2 - w^2,$$

and the invariant points are $(\pm 1, \pm 1, \pm 1, 1)$, lying by pairs on sixteen self-conjugate lines through the double F-points, and twelve meeting pairs of opposite edges of the tetrahedron.

(ii) With a base line ω_1 and four isolated points D_1, D_2, D_3, D_4, every point of $\omega_1 \sim \omega_1$, and it is an F-line of second species. Four other F-lines are the intersections of pairs of planes such as $\omega_1 D_1$, $D_2 D_3 D_4$. No other point can be vertex of a cone of the net, and ω_1 must count twice in ω_6. We find that it is an F-line of contact, along which (ϕ) touch a quadric, having a variable double point on ω_1. There are four pencils of self-conjugate lines, each through one D, meeting ω_1; and six isolated self-conjugate lines joining two D's.

(iii) If the base of the net of quadrics is a twisted cubic ω_3, then P, P' lie on the chord of ω_3 through P, and divide it harmonically. The associated complex reduces to the congruence of chords of ω_3, each of which is self-conjugate (48, 104, 133, 152, 333, 349).

Any quadric cone through ω_3 has its vertex on the curve, and ω_6 consists of ω_3 taken twice, being an F-curve of contact. The P-lines are the tangents to ω_3 and J_8 is their locus j_4 taken twice.

Let ω_3 be given in terms of a parameter τ by

$$x : y : z : w = 1 : \tau : \tau^2 : \tau^3.$$

Three independent quadrics of the net are

$$q_a \equiv xz - y^2, \quad q_b \equiv xw - yz, \quad q_c \equiv yw - z^2.$$

The bilinear equations are

$$xz' + zx' - 2yy' = xw' + wx' - yz' - zy' = yw' + wy' - 2zz' = 0,$$

and the web of homaloids is

$$\left\| \begin{array}{cccc} z, & -2y, & x, & 0 \\ w, & -z, & -y, & x \\ 0, & w, & -2z, & y \end{array} \right\|,$$

that is,

$$\phi_1 : \phi_2 : \phi_3 : \phi_4 = 2yq_a - xq_b : zq_a - xq_c : wq_a - yq_c : wq_b - 2zq_c.$$

The tangent plane at the point τ to the general homaloid

$$c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4$$

is

$$(c_1 + \tau c_2 + \tau^2 c_3 + \tau^3 c_4)(\tau^3 x - 3\tau^2 y + 3\tau z - w).$$

The first factor gives three values of τ at which the tangent plane is indeterminate, and ϕ has a d.p.; these are not double F-points, but vary on ω_3 with the parameters $c_1 \dots c_4$ of ϕ (48). The second factor is independent of the parameters, and is the fixed tangent plane to (ϕ) at τ. It is also the osculating plane of ω:

$$\left| \begin{array}{cccc} x, & y, & z, & w \\ 1, & \tau, & \tau^2, & \tau^3 \\ 0, & 1, & 2\tau, & 3\tau^2 \\ 0, & 0, & 2, & 6\tau \end{array} \right| \equiv 2(\tau^3 x - 3\tau^2 y + 3\tau z - w),$$

and its envelope is the locus of the tangent line to ω_3, common to the three tangent planes to q_a, q_b, q_c;

$$t_a \equiv \tau^2 x - 2\tau y + z, \quad t_b \equiv \tau^3 x - \tau^2 y - \tau z + w, \quad \tau t_c \equiv \tau^3 y - 2\tau^2 z + \tau w.$$
$$\equiv \tau t_a + t_c.$$

By eliminating τ between any two of these, we find the P-surface

$$j_4 \equiv 4 q_a q_c - q_b^2.$$

18. Pencil of quadrics and one null-system.

If two of the bilinear equations are symmetric, and one is skew, P, P' are conjugate with regard to the quadrics q_a, q_b, and to the pencil which they determine, and the invariant system is the base of this. If q_a, q_b have a common plane, the involution is of lower degree; hence the invariant system is a proper or degenerate quartic δ_4. A general chord of δ_4 contains no conjugate pairs other than the two whose points coincide at the invariant points. There are ∞^1 chords that are rays of the associated complex, each containing one conjugate pair, and a finite number of common chords of δ_4, ω_6, which are self-conjugate lines and bear ∞^1 conjugate pairs. An example is given by (158 J 6)

$$xy' + yx' - ww' = (x + y)(x' + y') - zz' = xy' - yx' = 0.$$

Here the null-system is special, and PP' meets the axis $x = y = 0$, which is an F-line; δ_4 breaks up into two conics

$$x + y = \pm z, \quad 2xy - w^2 = 0,$$

and ω_6 consists of a line of contact and four other lines.

If the two quadrics touch the same plane j_1 at O, and also j_1 is related to O in the null-system, then O is an F-point and $\sim j_1$ in I_{3-3}. It follows that O is a d.p. on (ϕ) and triple on ω_6, and also that j_1 meets (ϕ) in a plane F-cubic ω_3. The residual F-cubic k_3 cannot be plane, for ω_6 cannot be wholly in two planes; hence k_3 has one branch through O and ω_3 has two and is rational. Now a general ray of the congruence of chords of k_3 contains two F-points and one general point of j_1; hence it $\sim$ a ray of the star (O) (246, 316 and p. 309).

19. One quadric and two null-systems.

With one symmetric and two skew equations, the one quadric with regard to which P, P' are conjugate is an invariant surface Δ_2. The two null-systems determine a pencil of null-systems relating to P, P', and two of these are special; hence PP' meets the fixed axes l_1, l_2, which are F-lines, and the involution is skew (p. 179). The associated complex reduces to a bilinear congruence, and any ray contains two F-points of l_1, l_2, two invariant points of Δ_2, and ∞^1 conjugate pairs. If l_1, l_2 met in O, then PP' would pass through O, and the transformation would be central, and reduce to quadric inversion.

Altogether there are five kinds of bilinear I_{3-3}, distinguished by their invariant systems:

eight points,
a line and four points,
a cubic,
a quartic,
a quadric.

III. MONOIDAL TRANSFORMATIONS (37, 100, 287, 369)

20. Definitions.

A *monoid* is a surface ϕ_n of degree n with an $(n-1)$-fold point O, called the *vertex*. The type includes all planes, quadrics, singular cubics, Steiner quartics, etc. For a plane, any point of space can be considered as the vertex, and for a quadric, any point of the surface. A line through O will be called a *ray*; it meets ϕ again in a single point, and projection from O gives a 1, 1 representation of ϕ upon a plane: every monoid is rational.

The word *monoid* has also been used in an extended sense, which we do not adopt, of a surface meeting all the rays of any congruence in only one point not singular for the congruence (249).

A transformation T is *monoidal* if the first homaloidal web consists of monoids with a common vertex. Then T^{-1} need not be monoidal; if it is, T is *bimonoidal* (396). If the vertex is variable, we do not call T monoidal, though every ϕ is a monoid (325).

If $O\,(0, 0, 0, 1)$ is the vertex, the general monoid is

$$\phi \equiv w u_{n-1} + u_n,$$

where u_{n-1} is the tangent cone at O, and u_n the cone vertex O standing on the section of ϕ by the general plane w. On ϕ there lie the $n\,(n-1)$ rays given by $u_n = u_{n-1} = 0$.

Let $\psi \equiv w v_{n-1} + v_n$ be another general monoid vertex O, the tangent cone v_{n-1} at O having no sheet in common with that of ϕ. The intersection of ϕ, ψ is projected from O on to w into the plane curve k_{2n-1} given by

$$u_n v_{n-1} - u_{n-1} v_n$$

of degree $2n-1$, passing through the $n\,(n-1)$ traces ΣA_α of the rays on ϕ, which are the base points of the plane representation. The net of homaloidal curves $(c_{n'})$ on ϕ projects into a plane homaloidal net, say $(k_\mu) \equiv \Sigma A_\alpha{}^{\gamma_\alpha} \Sigma X_\beta{}^{\gamma_\beta}$, where the X's are the projections of the isolated F-points on ϕ, and ω projects into a curve through all the A's that are not base points of (k_μ).

If ϕ has no other $(n-1)$-fold point, the rays on ϕ are on a different footing from any other lines that may lie on ϕ, and different plane transformations always lead to different space transformations. But if ϕ has two

$(n-1)$-fold points O, O_1, any type of space transformation, in which O, O_1 are differently related to H, arises twice, the plane family being altered if the parts played by O, O_1 are exchanged.

Since each ϕ is at most linear in w, so is any common factor of a set of ϕ's; hence if there is an F-point, or F-line of first species, in S', the corresponding P-surface, being part of three or two ϕ's, is a monoid or cone. In particular, if T is bimonoidal, each vertex $\sim$ a P-monoid.

21. Limits for n'.

If $c_{n'}$ has β branches through O, it projects into k_μ of degree

$$\mu = n' - \beta.$$

Each of these branches projects into a part of k_μ meeting the trace of the tangent cone u_{n-1} in a point other than ΣA. Let the ray OA_a be τ_a-fold on u_{n-1}, then

$$\beta = \mu(n-1) - \Sigma\tau_a\gamma_a.$$

If ϕ is a general monoid, no ray on ϕ is a multiple edge of u_{n-1}, and every $\tau_a = 1$; then

$$n' = \mu + \beta = \mu n - \Sigma\gamma_a.$$

But since $\Sigma A_a{}^{\gamma_a}$ are some or all of the F-points of (k_μ),

$$\Sigma\gamma_a \leqq 3(\mu-1)$$
$$n' - n \geqq (\mu-1)(n-3)$$
$$n' \geqq n \text{ if } n \geqq 3;$$

and if $n > 3$, we have $n' = n$ only if $\mu = 1$, and k_μ is a general line of w, and $c_{n'}$ a section of ϕ by a general plane through O. The fixed plane curve, image of ω, is $k_{2n-2} \equiv \Sigma A$, and the F-curve of T is

$$\omega_{n(n-1)} \equiv O^{(n-1)(n-2)},$$

meeting $c_{n'}$ in $2(n-1)$ free points; O and ω constitute H, and T^{-1} is of the same nature.

Thus, provided there is no contact of (ϕ) at O, and no multiple edge of the tangent cone to ϕ at O lies on ϕ, we have $n' \geqq n$, and if T is bimonoidal, $n \geqq n'$ also, whence $n = n'$. But if any of the rays on ϕ are multiple on u_{n-1}, we can have $T_{n,n'}$ bimonoidal with $n' < n$, for example T_{4-2}, where the Steiner quartics have three double rays. There are three base points A and k_μ is a conic through them, image of a conic.

If $n = 3$, the condition $n = n'$ is not sufficient to make T bimonoidal; the cubic web of the bilinear T_{3-3} can acquire a d.p. without the second web acquiring one also (p. 297).

All values between the limits n, n^2 are possible for n'. For example, there is a monoidal $T_{n-(2n-3)}$, where ϕ'_{2n-3} is a de Vries' surface, with an $(n-2)$-fold F-cubic (398).

22. Transformations with no F-curves (18, 396).

If n' has its maximum value n^2, there is no ω; any fixed part $k_{2n-1-\mu}$ of the plane k_{2n-1} is an exceptional curve of the projection, which does not represent a curve on ϕ. The cone $t \equiv O\,(k_{2n-1-\mu})$ meets ϕ in rays only, and any other generator of t meets ϕ at O only; t is a sheet of the tangent cone to ϕ, common to ϕ, ψ, and O is an F-point of partial or total contact.

If there is no such contact, there is no fixed curve in the plane, $\mu = 2n-1$, all the A's are base points of (k_μ), and are simple since there are no F-rays. But the plane homaloidal net has at most $2\mu - 2$ simple base points:

$$n(n-1) \leqq 2(2n-2),$$

$$n \leqq 4.$$

We can have $n = 2$, 3 or 4, and there is one monoidal T_{n-n^2} of each of these degrees, with no contact at the vertex; in each case there is contact at a simple F-point elsewhere.

For all values of n, there exist several kinds of T_{n-n^2} with contact at O (161 and p. 314).

Conversely, we shall now prove, by the method of adjoints, that any transformation with no F-curves is monoidal. In other words, if any $T_{n-n'}$ is not monoidal, it has an F-curve, either finite or infinitesimal. The former reduces n' below its maximum n^2, the latter does not; hence if a T_{n-n^2} is not monoidal, it has an infinitesimal F-curve, and therefore an F-point of self-contact, at least a tacnode: for example, the T_{4-16} of p. 294.

Let $n = 4\alpha + \beta$, $\beta = 0, 1, 2$ or 3; then since no adjoints exist, some of the F-points ΣO^i survive on the α^{th} adjoint g_β and prevent it existing. We review the various minimum sets of points impossible on a surface of degree $\leqq 3$, and exclude sets that would prevent (ϕ) existing, and also sets with $i_1 + i_2 > n$, since then O_1O_2 would be an F-line. In all the remaining cases there is a $(\beta+1)$-fold point O_1 that destroys g_β, and $i_1 \geqq 2\alpha + \beta + 1$.

Let $i_1 = n - 2\gamma - \epsilon$ where $\epsilon = 0$ or 1 according as $n - i_1$ is even or odd; then

$$n - 1 \geqq i_1 \geqq n - 2\alpha + 1,$$

$$0 \leqq \gamma \leqq \alpha - 1.$$

If $\gamma = 0$, then $i_1 = n-1$, and T is monoidal. Assume $\gamma > 0$ if possible. Then γ is a possible order for an adjoint g of degree $n - 4\gamma$; on this, O_1 is of multiplicity $i_1 - 2\gamma = n - 4\gamma - \epsilon$, and does not by itself offer any contradiction. Hence the next highest F-point O_2 survives on this adjoint. But

$$i_2 \leqq n - i_1 \text{ which } = 2\gamma + \epsilon,$$

and its multiplicity on g is at most $\epsilon \leqq 1$. Hence $\epsilon = 1$, $i_2 = 2\gamma + 1$, and the only multiple F-point on g is O_1.

To prevent g existing, there survive on it simple F-points in number

$$\binom{n-4\gamma+3}{3} - \binom{n-4\gamma+1}{3} = (n-4\gamma+1)^2.$$

These are all $(2\gamma+1)$-fold on (ϕ), and together with O_1 reduce its freedom to

$$\binom{n+3}{3}-\binom{n-2\gamma+1}{3}-(n-4\gamma+1)^2\binom{2\gamma+3}{3}-1.$$

This can be expressed as

$$-\tfrac{2}{3}\gamma(\gamma+1)\{(n-4\gamma-4)^2(2\gamma+4)+(n-4\gamma-4)(20\gamma+31)+36\gamma+54\}-1.$$

On the assumption $\gamma>0$, we have $\alpha \geqq \gamma+1$, $n-4\gamma \geqq 4$, and this freedom would be negative, (ϕ) would not exist. Therefore $\gamma=0$, and T is monoidal.

It follows that if H has no F-curves, there is at least one F-point of contact. For if there is no contact at O, the intersection is a proper rational $c_{n^2} \equiv O^{(n-1)^2}$. If $n>1$, the rational cones of degree $2n-1$ projecting (c) from O are singular, and have a fixed multiple edge l as part of their base. Since l is not an F-line, it meets ϕ and c in one point only other than O, which is therefore multiple on every c lying on ϕ, and is an F-point of contact.

23. The star (O) and its corresponding congruence (246).

A general ray meets ϕ in $n-1$ points at O and one free point, and the star $(O) \sim$ a congruence C' of first degree of lines meeting (ϕ') in $n'-1$ F-points. If C' is a star, its vertex is $(n'-1)$-fold on (ϕ'), and T is bimonoidal.

A monoid has no multiple curves except rays, hence if T is bimonoidal, C' is either

(i) lines through a fixed $(n-1)$-fold point, or

(ii) lines meeting an $(n-1)$-fold line, or

(iii) lines meeting an $(n-2)$-fold line ω_1' and a simple F-curve ω'_m meeting ω_1' in $m-1$ points, or

(iv) if $n=3$, lines meeting two branches of simple F-curves.

Case (iv) does not arise for the T_{3-3} with a nodal F-line (no. 5 of Table VI) which belongs to (ii); nor for the other two types of singular T_{3-3}'s (nos. 3, 4 of Table VI): for in the general case of each of these, the congruence of chords of ω_6 is not of first degree, and therefore C' is the star (O'), and T belongs to (i), both in general and in any special case, however ω_6 breaks up. Hence (iv) can only arise for the bilinear T_{3-3}, when this becomes bimonoidal. Now if (ϕ) is monoidal, with a d.p. O triple on ω_6, the septimic congruence of chords of ω_6 breaks up into the star (O) and a sextic congruence of proper chords. In S', in general ϕ' has no d.p., ω_6' consists of a plane cubic ω_3' and a twisted cubic ω_3'' (p. 297); there is a congruence of first degree of chords of ω_3'' and a sextic congruence of lines meeting ω_3' and ω_3''; the former is C'. This relation survives when (ϕ') also acquires a d.p. Thus in case (iv), C' is the set of chords of a twisted cubic (316 and p. 305).

When we know C' and the correspondence between the rays of the two congruences, any correlation determines a T in which P' is the intersection of the ray which $\sim OP$ with the plane correlated to P (249).

In the involutory case, where $n=n'$, and the two homaloidal webs coincide, all four cases are possible, and there are two varieties of the first according as the corresponding stars coincide or not. If not, each vertex is an $(n-1)$-fold F-point of (ϕ), and their join is an $(n-2)$-fold F-line (p. 327).

An involutory case of (ii) is given by the equations of p. 271:

$$x' : y' : z' : w' = xw^2 : yz^2 : zw^2 : z^2w.$$

A ray of the star whose vertex is the $(n-1)$-fold point $(\xi, \eta, 0, 0)$ can be represented by

$$\eta x = \xi y + aw, \qquad bz = w,$$

where a, b are parameters of the ray. The homologue is

$$\eta x = b^2\xi y + ab^2 w, \quad z = bw,$$

through the variable $(n-1)$-fold point $(b^2\xi, \eta, 0, 0)$, which describes the linear congruence of tangent lines to the surface $\eta xw^2 - \xi yz^2$ at the points of $z = w = 0$.

The reduction of monoidal space involutions can be made to depend on that of the two-dimensional transformation between the star (O) and C' (369 and p. 96).

24. $\mathbf{T_M}$.

We denote by T_M or $T_{Mn-n'}$ the first bimonoidal case, when the rays of $(O) \sim$ those of (O') in a two-dimensional Cremona transformation T_μ. Some examples of T_{M3-3} and T_{M5-5} are described in 309 and applied to the resolution of a singular point of a surface.

In particular, if T_M changes the planes of (O) into those of (O'), then T_μ is a homography (37, 112, 229).

In the general T_M, let T_μ be given by

$$x' : y' : z' = \phi_\mu(x, y, z) : \psi_\mu : \chi_\mu,$$
$$x : y : z = \phi_\mu'(x', y', z') : \psi_\mu' : \chi_\mu'.$$

These can be taken as two of the equations of T_M; since the three are rationally reversible, the third is linear in each of w, w', and T_M can be expressed by

$$x' : y' : z' : w' = \phi_\mu j : \psi_\mu j : \chi_\mu j : \phi,$$

where

$$\phi \equiv wu_{n-1} + u_n,$$
$$j \equiv wu_{n-\mu-1} + u_{n-\mu},$$

and j is the P-monoid which $\sim O'$. It meets ϕ in an F-curve ω of degree $n(n-\mu)$; and H consists of O, ω and any F-rays of T_μ that lie on ϕ.

The solution T_M^{-1} is

$$x : y : z : w = \phi_\mu' j' : \psi_\mu' j' : \chi_\mu' j' : \phi',$$

where

$$j' : \phi' = w'\chi_\mu u_{n-\mu-1} - z'u_{n-1} : -w'\chi_\mu u_{n-\mu} + z'u_n,$$

when on the right we substitute ϕ_μ', ψ_μ', χ_μ' for x, y, z.

Now from the properties of T_μ, if any γ_α-fold F-ray $l_\alpha \sim$ a P-cone g_α, and $G' \equiv \Pi g_\alpha{}^{\gamma_\alpha}$,

$$\chi_\mu(\phi_\mu', \psi_\mu', \chi_\mu') \equiv z'G'.$$

Thus z' can be cancelled from the expression for j'/ϕ'.

Also $u_n(\phi_\mu', \psi_\mu', \chi_\mu')$ is the product of the homologue of u_n and the set of P-cones which $\sim$ the F-rays of T_μ lying on u_n. Thus if u_n, u_{n-1} have any l_α in common, g_α can be cancelled also. After making all these reductions,

$$\phi' \equiv w'u'_{n'-1} + u'_{n'},$$
$$j' \equiv w'u'_{n'-\mu-1} + u'_{n'-\mu},$$

where

$$\frac{G'u_{n-\mu}(\phi_\mu', \psi_\mu', \chi_\mu')}{u'_{n'-1}} \equiv \frac{-u_n(\phi_\mu', \psi_\mu', \chi_\mu')}{u'_{n'}} \equiv \frac{-G'u_{n-\mu-1}(\phi_\mu', \psi_\mu', \chi_\mu')}{u'_{n'-\mu-1}}$$
$$\equiv \frac{u_{n-1}(\phi_\mu', \psi_\mu', \chi_\mu')}{u'_{n'-\mu}} \equiv G_1', \text{ say,}$$

where G_1' is a set of P-cones of T_μ, and apart from sheets that are P-cones, the cones

$$u'_{n'-1}, \quad u'_{n'}, \quad u'_{n'-\mu-1}, \quad u'_{n'-\mu}$$

are the respective homologues of

$$u_{n-\mu}, \quad u_n, \quad u_{n-\mu-1}, \quad u_{n-1}.$$

Of these, u_n, $u_{n-\mu}$ stand on the sections of ϕ, j by the general plane w, and u_{n-1}, $u_{n-\mu-1}$ are the tangent cones of ϕ, j at O.

Another pair of corresponding cones is

$$u_{2n-\mu-1} \equiv u_n u_{n-\mu-1} - u_{n-1}u_{n-\mu},$$

standing on ω and the cone of the same nature in S'.

Unless G_1' contains the whole of G', there is a common factor of $u'_{n'-1}$, $u'_{n'-\mu-1}$, and a fixed sheet of the tangent cone to (ϕ') at O'.

In the involutory case, $n' = n$, and $u'_{n'} \dots u'_{n'-\mu-1}$ are the same functions as $u_n \dots u_{n-\mu-1}$; the cones u_n, $u_{n-\mu-1}$ are self-conjugate, and u_{n-1}, $u_{n-\mu}$ are conjugate, as can be seen directly. By varying the plane w, we obtain ∞^2 self-conjugate cones such as u_n, standing on the sections of the homaloids by their corresponding planes, and ∞^2 pairs of conjugate cones such as u_{n-1}, $u_{n-\mu}$, being the tangent cones at O to the homaloids, and the cones standing on the sections of j by the corresponding planes.

Now the degree of G' is $\mu^2 - 1$ and that of G_1' is $n(\mu - 1)$; we see that $u_{n-\mu}$, $u_{n-\mu-1}$ contain the same set of F-rays $\Sigma l_\alpha{}^{\beta_\alpha}$ say, and each of u_n, u_{n-1} contains $\Sigma l_\alpha{}^{\beta_\alpha+\gamma_\alpha}$.

In one space, if the vertices coincide, the invariant system of T_M is given by

$$\frac{\phi_\mu}{x} = \frac{\psi_\mu}{y} = \frac{\chi_\mu}{z} = \frac{\phi}{wj},$$

and consists of the intersections, other than O, of $x\phi - wj\phi_\mu$ with the system of invariant rays of T_μ. In general there are $\mu + 2$ such rays, and two intersections on each, altogether $2(\mu+2)$ isolated invariant points of T_M. If T_μ has a cone of invariant rays, then T_M has an invariant curve; and if T_μ is identity, there is an invariant surface $\Delta_n \equiv \phi - wj$, on which O is $(n-2)$-fold (112).

25. F- and P-systems of $\mathbf{T_M}$.

On corresponding rays l, l' of the stars, T_M sets up a homography, in which the one free intersection of l, $\phi \sim$ that of l' with the plane which $\sim \phi$. If O is regarded as a point of l, it $\sim$ the free intersection of l', j', and the intersection of $l, j \sim O'$.

If l meets ω, in Q say, but is not an F-ray of T_M, it is a P-ray and a generator of $u_{2n-\mu-1}$ and l' of $u'_{2n'-\mu-1}$, meeting ω' in Q' say. The homography is improper; a general point of $l \sim Q'$, and is associated with a definite plane through the tangent line to ω' at Q'; and $\omega \sim$ the cone standing on ω'.

Thus, apart from possible F-rays, there is a 1, 1 relation between the points of the two F-curves, and they are of the same genus; if one breaks up, so does the other, and the components are associated in pairs of the same genus, such that the P-surface which $\sim$ each of the pair is the cone standing on the other.

Besides these F-curves, there may be F-rays of T_M of two kinds.

(i) A ray common to j, $\phi \sim$ a definite ray l', and both are of second species; l meets the residual F-curve ω in two points Q_1, Q_2 say, and gives an apparent d.p. of ω as seen from O. Each of Q_1, Q_2 is an F-point of contact of (ϕ), the fixed tangent plane containing OQ; l' is of the same nature, and each of its two intersections with ω' is associated with a definite one of Q_1, Q_2. There are the same number of such F-rays in H, H'; however ω, ω' break up, the components are associated in pairs of the same genus; but the association of points breaks down for homologous rays.

If ω has an actual d.p. Q, the fixed tangent plane to (ϕ) does not in general contain O, and OQ is not an F-ray.

(ii) The second kind of F-ray of T_M is an F-ray l of T_μ that lies on ϕ but not on j; this is of first species, and $\sim$ a P-cone g'. An s-fold F-ray of T_μ that is not an F-ray of T_M meets (ϕ) in an F-point of s-point contact.

If T_μ is a homography, $n = n'$; between homologous planes p, p' of (O), (O') there is set up by T_M a plane T_{Jn}, whose F-points in p are O and the $2n-2$ other intersections of p, ω; its P-lines join these to O (112).

If p contains an F-ray l, which is of second species since T_μ has no F-rays, then p' contains l'; the T_J reduces to degree $n-1$, with $2n-4$ simple F-points in p, the two intersections Q_1, Q_2 of ω, l being ordinary points for T_J. This T_{Jn-1} has l, l' as ordinary corresponding lines, and sets up a proper homography between their points; and the homography varies in a pencil as p rotates about l.

If p touches ω at Q_2, this and the point Q_2', associated with Q_2 by T_M, are F-points of T_J, and Q_1 is a general P-point, and $\sim Q_2'$; if p touches ω at Q_1, then Q_1 is an F-point and $\sim$ every point of l', including Q_2'. Thus $Q_1 \sim Q_2'$ in two transformations of the pencil, and therefore in them all; and similarly $Q_2 \sim Q_1'$. These are the only points of l that have fixed homologues in T_J as p rotates.

If p contains an actual d.p. Q of ω, not lying on an F-ray, another F-point of T_{Jn} is adjacent to Q on the trace on p of the tangent plane π to (ϕ) at Q. A general point of $l' \sim$ a point adjacent to Q on π, and $Q' \sim$ all other points of l.

26. Useful kinds of $\mathbf{T_M}$.

(i) Let there be a pair of corresponding planes not passing through the vertices. Take them as w, w'; then $u_n \equiv 0$, and the T_M

$$x' : y' : z' : w' = \phi_\mu j : \psi_\mu j : \chi_\mu j : w u_{n-1}$$

transforms w into w' and effects between them the arbitrary plane transformation T_μ. The characteristic numbers of T_μ reappear in the numbers and orders of contact of F-points of T_M on w (188).

In particular, if the spaces are superposed, with $w' \equiv w$, and T_μ is identity, we have a T_M leaving the plane w invariant.

(ii) If k is any given curve, we can always find a monoid ϕ_n through it, if n is great enough, and construct a T_M that transforms ϕ into a plane and k into a plane curve k'; we must take j not to pass through k, so that k is not an F-curve.

By the known theory of transformation of plane singularities, there is a T_μ in which $k' \sim$ another plane curve k'' with ordinary singularities only (p. 129), and this can be effected by a second T_M. Then k'' can be regarded as the projection of a twisted curve k''' free from all singularities, and this projection can be effected as part of a third T_M such that T_M^{-1} transforms k'' into k'''. By compounding the three, we have a Cremona space transformation that converts any given proper curve k into one free from singularities (p. 143).

(iii) We can also construct a T_M having any given curve k_m as F-curve. Choose for ϕ, j any two monoids through k with a common vertex O, a general point of space. There is always a lower limit for $n - \mu$ for which this is possible (96).

If k has h apparent d.p.'s, the h chords Σl of k through O are necessarily F-lines of second species, lying on both ϕ and j. For simplicity, we take T_μ to be identity; then there are no F-rays of first species, and the only other F-element is ω the residual intersection of ϕ, j, of degree $n(n-1) - m - h$, on which O is $\{(n-1)(n-2) - h\}$-fold. The P-cone u_{2n-1} falls into two proper components, j_k, say, standing on k and j_ω on ω; the former contains Σl^2.

If n is great enough, we can assume that ω is proper, and free from multiple points other than O. Then ϕ, j have no common rays other than Σl, and no ray meets ω twice or ω, k each once. We can also assume that the intersections of ω with k and with Σl are all distinct and at general points of all these curves, regarded both as curves in space, or as curves on any particular surface f_ν through k: unless the intersections fall at multiple points of k which are necessarily singular for any surface through k. Further, we can assume that the sections of ϕ, j, by any plane p, are general plane curves, subject only to the condition of passing through the m intersections of p, k.

Now consider the effect of this T_M on the singularity of a general point Q of k, regarded as a point of the section k_ν of f by a general plane p. Since O is in general position with regard to f, we can take p to pass through O; then we have a T_{Jn} between p and the corresponding plane p', with Q as a simple F-point; and Q, regarded as a point of k_ν, is spread out into the set of intersections $\Sigma Q'$ of a P-line of T_J with the section k'_ν of f' by p'.

This has just the same effect on the nature of the plane singularity as a T_2 applied to Q; a point Q' is of definitely simpler nature on k'_ν than Q is on k_ν. The plane p' is not in general position with regard to f', since it passes through the multiple point O'; but it is general as regards the neighbourhood of $\Sigma Q'$; in any case, the section of f' by a general plane cannot have a higher singularity at Q' than that by a special plane. Thus after a finite number of steps, the multiplicity of Q on k_ν is reduced.

But this resolution applies only to the general points of k. The same transformations may introduce all kinds of other singularities, either accidental, due to the transformations themselves, or arising from special points of k (Chapter XVI).

(iv) We can secure that all the F-elements except k are rational. For example, let ϕ consist of the cone $u_m \equiv O(k)$ and a set of planes all passing through O except one; and let j meet that one in $n-1$ lines:

$$\phi \equiv w u_m \prod_{(n-m-1)} u_1, \quad j \equiv w v_{n-2} + \prod_{(n-1)} v_1,$$

whence

$$\phi' \equiv w' \Pi v_1', \qquad j' \equiv - w' v'_{n-2} + u'_m \Pi u_1'.$$

The F-curves are k, the $n-1$ lines $w = v_1 = 0$, the $n-m-1$ plane monoids, sections of j by Πu_1, and the intersection of j, u_m residual to k, consisting of a curve of degree $(n-1)m-m$ with $(n-2)m$ branches through O, and therefore a set of rays.

Similarly, H' consists of a set of rational curves and the plane curve $w' = u'_m = 0$, of the same genus as k.

Thus H, H' have each only one F-genus, that of k, and it is explicit in both spaces. This is a type that may possibly be useful in some theory of classification or decomposition of space transformations.

(v) Let O be a point of partial or total contact; then u_{n-1}, $u_{n-\mu-1}$ have a common factor t of degree $n-1-\nu$ say:

$$\phi \equiv w t_{n-1-\nu} u_\nu + t_n,$$

$$\psi \equiv w t_{n-1-\nu} v_\nu + v_n.$$

The intersection projects into

$$k_{n+\nu} \equiv t_n v_\nu - u_\nu v_n$$

with $n\nu$ simple base points. If these are all distinct, and if $n' = n^2$, there is no fixed part of $k_{n+\nu}$ and as on p. 308

$$n\nu \leqq 2(n+\nu-1),$$

$$\nu \leqq \frac{2(n-1)}{n-2} < 3 \text{ if } n > 4.$$

Hence if $n > 4$, there are three kinds of T_{Mn-n^2} (161).

(1) $\nu = 0$: tangent cone entirely fixed; k_n subject to no conditions. For each value of n there is such a transformation answering to each plane T_n (393).

(2) $\nu = 1$: tangent cone consists of a fixed t_{n-1} and a variable plane; (k_{n+1}) has n simple base points, and is a plane monoidal net. The plane transformation is T_J, and there is one T_M of this kind for each n (396).

(3) $\nu = 2$: tangent cone consists of a fixed t_{n-2} and a variable quadric cone; (k_{n+2}) has $2n$ simple base points, and again the plane transformation is T_J, leading to one T_M for each n.

The simplest of these transformations, with total contact at O, is when $\mu = 1$, $\nu = 0$, and $v_{n-2} \equiv u_n \equiv 0$,

$$x' : y' : z' : w' = xv_{n-1} : yv_{n-1} : zv_{n-1} : wu_{n-1},$$

$$x : y : z : w = x'u' : y'u' : z'u' : w'v'.$$

The F-system consists of the $(n-1)$-fold point O, the fixed tangent cone u, which may be regarded as an infinitesimal simple curve adjacent to O, the plane curve $w = v = 0$, and the set of $(n-1)^2$ F-rays $u = v = 0$.

The $(n-1)$-fold point O is a necessary consequence of the F-rays, for the rays do not lie on any cone of degree $< n-1$; but the contact at O is an independent condition.

There are only two F-genera, one explicit in v and one implicit in u; both can be arbitrary. The reverse transformation is of the same nature, and the F-genera exchange positions. In particular, if we take either u or v to be rational, say a set of planes, the T_M has only one F-genus other than 0, explicit in one space and implicit in the other.

(vi) There is osculation at O if $\mu = 1$ and

$$(\phi) \equiv wu_{n-1} + u_n + u_{n-1}v_1,$$

the coefficients in v only being parametric; this family is homaloidal without further condition (399); there are $n(n-1)$ F-rays $u_{n-1} = u_n = 0$.

A monoidal family cannot have total contact at the vertex of order higher than this. If t_ν is a fixed sheet of the tangent cone to (ϕ), there is 4-point contact of the sheets of (ϕ) touching t, with a fixed monoid $j_{n-\mu}$, also having one sheet touching t, if $\nu \leqq \frac{1}{2}n$ and

$$(\phi) \equiv jv_\mu + t^2 v_{n-2\nu}.$$

If t_ν is a plane, z say, we have the extreme case of $(n+2)$-point contact, with

$$(\phi) \equiv jv_\mu + z^n \quad \text{(p. 247).}$$

27. An $(n-1)$-fold F-line (139, 216, 247, 287, 352).

(vii) If there is an $(n-1)$-fold F-line l, the web is monoidal, and any point of l may be regarded as the common vertex. This is the general transformation of genus 0 (p. 172).

Now l counts $(n-1)^2$ times in the intersection of ϕ, ψ. The residual c_{2n-1} meets l in $2n-2$ points; hence if it breaks up, one part is a line meeting l. The F-system consists of l and α rays meeting l and β simple points, where

$$2\alpha+\beta=3(n-1), \quad 2\beta+\alpha=3(n'-1), \quad n'=\alpha+\beta-n+2,$$

and T^{-1} is of the same nature, with an $(n'-1)$-fold line, β F-rays and α simple F-points. Since

$$\alpha=2n-n'-1, \quad \beta=2n'-n-1,$$

we can form such a $T_{n-n'}$ provided $2n'>n$ and $2n>n'$.

For $\alpha=0$, see 284, 397, 413. Generalizations are given in 263.

In particular, the homaloids can be cones with vertices varying on l, which is a line of contact and counts $n(n-1)$ times in the intersection of ϕ, ψ; hence $n' \leqq n$. If the second homaloidal web also consists of cones, $n'=n$; then $n-1$ F-rays coincide with each of l, l'; there are no others, and therefore $n-1$ simple F-points in each space (97, 324).

Now c_n meets l in $n-1$ points, and $l \sim j'_{n-1}$; a line meeting l or plane through $l \sim$ a line meeting l' or plane through l', and there is a homography between the two pencils of planes. An involutory case of even degree $n=2\mu+2$ can be constructed by assuming the involution I_1 in the pencil of planes (l), a rational curve k_μ and a 1, 1 relation between the points X of k and the pairs of planes p, p' of I_1. Then P, P' are conjugate points of I_{n-n} if they lie on planes p, p' conjugate in I_1 and are collinear with the point X related to p, p'.

Other involutory cases, when the complex of lines meeting $l \sim$ itself, are studied in 139, 263.

28. Transformation of polars (66).

Like T_{2-2}, T_M has a useful polar property. Let f_μ be any surface and

$$f_X \equiv \left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}+\zeta \frac{\partial}{\partial z}+\theta \frac{\partial}{\partial w}\right) f$$

be the first polar surface of any point X $(\xi, \eta, \zeta, \theta)$ with regard to f. The intersection of f, f_X is called the *total polar* curve of X; parts of this are all the multiple curves of f, each reckoned a certain number of times, according to the nature of the singularity for f of its general points; the residual, variable with X, is called the *pure polar* curve of X with regard to f (p. 372).

If a curve k is i-fold on f, it is $(i-1)$-fold on f_X. Any implicit multiple curve of f adjacent to k also lies on f_X; if f has a cuspidal sheet, there is in some neighbourhood a simple curve adjacent to k, an essential part of its singularity, lying on f_X. A simple curve adjacent to k on an ordinary sheet of f does not lie on every f_X.

To resolve the singularity of k, we apply a simple T_{Mn-n} with k as an F-curve, and otherwise general (p. 313):

$$x = x'j', \quad y = y'j', \quad z = z'j', \quad w = \phi',$$

where $$\phi' \equiv -w'v'_{n-1} + u'_n, \quad j' \equiv w'v'_{n-2} - u'_{n-1};$$

let square brackets denote that this substitution is effected.

This is the reverse of

$$x' : y' : z' : w' = xj : yj : zj : \phi,$$

where $$\phi \equiv wu_{n-1} + u_n, \quad j \equiv wv_{n-2} + v_{n-1}.$$

The F-curves in S form the total intersection of ϕ, j, and consist of k, the chords Σl of k through O, and a proper residual ω. The l are of second species, and the P-surfaces in S' are the monoid j' which $\sim O$, and two cones j_k', j_ω', where

$$j_k' j_\omega' \equiv u'_{2n-2} \equiv u_n' v'_{n-2} - u'_{n-1} v'_{n-1}.$$

Since k is i-fold on f and $(i-1)$-fold on $f_O \equiv \dfrac{\partial f}{\partial w}$, we drop $j'^{\,i}_{k}$, $j'^{\,i-1}_{k}$ from the homologues:

$$f' \equiv j'^{\,-i}_{k}[f], \quad (f_O)' \equiv j'^{\,-(i-1)}_{k}\left[\frac{\partial f}{\partial w}\right].$$

The first polar of O' with regard to f' is

$$\begin{aligned} f'_{O'} \equiv \frac{\partial f'}{\partial w'} &= j'^{\,-i}_{k}\left[\left(\frac{\partial x}{\partial w'}\frac{\partial}{\partial x} + \frac{\partial y}{\partial w'}\frac{\partial}{\partial y} + \frac{\partial z}{\partial w'}\frac{\partial}{\partial z} + \frac{\partial w}{\partial w'}\frac{\partial}{\partial w}\right) f\right] \\ &= j'^{\,-i}_{k}\left\{\frac{1}{j'}\frac{\partial j'}{\partial w'}\left[\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) f\right] + \frac{\partial \phi'}{\partial w'}\left[\frac{\partial f}{\partial w}\right]\right\} \\ &= j'^{\,-i}_{k}\left\{\frac{v'_{n-2}}{j'}\left[\mu f - \phi'\frac{\partial f}{\partial w}\right] - v'_{n-1}\left[\frac{\partial f}{\partial w}\right]\right\} \\ &= j'^{\,-i}_{k} j'^{\,-1}\left\{\mu v'_{n-2}[f] - u'_{2n-2}\left[\frac{\partial f}{\partial w}\right]\right\}, \end{aligned}$$

$$j' f'_{O'} \equiv \mu v'_{n-2} f' - j_\omega' (f_O)'.$$

Thus $f'_{O'}$ is not identical with $(f_O)'$, but their intersections with f' are related.

Since O does not lie on f, the intersection of f'. j' consists of F-curves. That of f', $f'_{O'}$ is the total polar of O', and consists of the pure polar and the multiple curves of f', which are the F-curves and the homologues of the multiple curves of f, other than k, either explicit on f or implicit in the first neighbourhood of k. That of f', j_ω' consists of F-curves and the P-rays of j_ω' that $\sim$ the intersections of f, ω other than those lying on k; on account of the generality of T_M, these are ordinary points of f, and the P-rays are simple on f' and are therefore part of the pure polar of O'.

Lastly, the intersection of f', $(f_O)'$ consists of F-curves and the homologue of the total polar of O, that is, the homologues of the pure polar and of the multiple curves of f explicit or adjacent to k, and also any simple curves of f' that arise from cuspidal sheets of f along k.

Thus the pure polar of O' consists of the homologue of the pure polar of O, and in addition, certain simple P-rays lying on j_ω' and possibly certain simple curves lying on j_k', arising from the neighbourhood of k.

The relation is simpler if $v'_{n-2} \equiv 0$, when j, j' are cones, and j_ω' is a sheet of j'; the vertices are F-points of total contact. Then $(f_O)', f'_{O'}$ are identical, save possibly for certain P-cones; and save for certain branches lying on the latter, the pure polar curves of O, O' are homologues. If we choose $j \equiv j_k$, then $j' \equiv j_\omega'$, and there are no supplementary sheets in $f'_{O'}$.

IV. OTHER SPECIAL TYPES

29. Punctual transformations (74, 75, 120, 191, 192).

For a general space transformation, there is nothing to answer either to a plane characteristic or to Noether's theorem. There is however a group of transformations, called *punctual* because each is determined by a set of points, which are defined to satisfy an analogue of Noether's theorem, and possess characteristics, and for which we can set up parallels to a good deal of the plane theory.

The elementary transformation, which takes the place of the plane T_2, is T_{tet}, and we define a punctual space transformation as one compounded of T_{tet}'s. In **192** it is suggested that all transformations with no F-curves of first species are compounded of T_{tet}'s; but the so-called proof is admittedly "gewagt," and merits a stronger adjective. Punctual transformations are called *regular* in **74**, but this word has a different accepted meaning in connection with transformations (p. 58).

The simplest type of punctual transformation is T_{tet} itself, and its properties that concern us now are:

(i) it is completely determined by its degree and the number and multiplicity of its isolated F-points; that is, it has a characteristic, $n=4$; 4^2;

(ii) all the F-curves are of second species, and all the P-surfaces $\sim$ isolated F-points. Hence the nature of the homologue of any surface depends only on its multiplicities at ΣO; in particular, each P-surface is completely defined by these multiplicities.

Let a transformation T be compounded of any number of T_{tet}'s. We assume that the four F-points of each new component T_{tet} are distinct, and that each either coincides with an F-point of the transformation V already constructed, or is in general position. Then no F-line of T_{tet} can meet an F-curve of V, unless the two coincide, being determined by the same two F-points. The F-curves of T all arise as homologues of F-curves of the T_{tet}'s, are all of second species, and are a necessary consequence of the passages of (ϕ) through the F-points; (ϕ) is completely determined by its degree and isolated F-points. Thus T has the same properties (i) and (ii) as T_{tet}, and has a characteristic

$$n;\ i_1 \ldots i_\sigma.$$

30. Incidence relations.

These characteristic numbers are connected by certain relations, and satisfy an inequality answering to Noether's in the plane. We cannot imitate the plane proofs, for these start with the two equations of condition, involving the total postulation, equivalence and genus. But in space (p. 225) these involve explicitly the constants of the F-curves of second species entailed by the F-points, and there is no simple way of expressing these in terms of n and Σi. We therefore use induction.

In a punctual $T'_{n-n'}$, let the systems of isolated F-points be

$$\sum_{\alpha=1}^{\sigma} O_\alpha^{i_\alpha}, \quad \sum_{\beta=1}^{\sigma'} O'_\beta{}^{i'_\beta},$$

where the i_α, i_β' are in descending order, and let

$$O_\beta' \sim j_\beta \equiv \sum_\alpha O_\alpha^{i_{\alpha\beta}}, \text{ of degree } \mu_\beta, \quad O_\alpha \sim j_\alpha' \equiv \sum_\beta O'_\beta{}^{i'_{\beta\alpha}}, \text{ of degree } \mu_\alpha'.$$

These satisfy the relations

$$\begin{array}{llll} n = n', & \sigma = \sigma', & i_\alpha = 2\mu_\alpha', & i_{\alpha\beta} = i'_{\beta\alpha}. \\ \sum_\alpha i_\alpha \ = 4(n-1), & & \sum_\alpha i_\alpha^2 \ = 2(n^2-1), & \\ \sum_\alpha i_{\alpha\beta} = 4\mu_\beta' - 1, & & \sum_\alpha i_\alpha i_{\alpha\beta} = i_\beta' n, & \\ \sum_\alpha i^2_{\alpha\beta} = 2\mu'^2_\beta + 1, & & \sum_\alpha i_{\alpha\beta} i_{\alpha\gamma} = 2\mu_\beta' \mu_\gamma' & (\beta \mp \gamma), \end{array}$$

and the similar equations with the spaces reversed. These all hold for T_{tet}, when

$$n = n' = 3, \quad \sigma = \sigma' = 4, \quad i = i' = 2, \quad \mu = \mu' = 1, \quad i_{\alpha\beta} = 0 \text{ or } 1.$$

Now if any surface $f_\mu \equiv \sum_{\alpha=1}^{\sigma} O_\alpha^{\gamma_\alpha}$ is transformed by a T_{tet} with O_1, O_2, O_3, O_4 as F-points in S, and O_1', O_2', O_3', O_4' in S', and if $O_5, \ldots O_\sigma \sim O_5' \ldots O_\sigma'$, we have for f'

$$\mu' = 3\mu - \sum_{\alpha=1}^{4} \gamma_\alpha; \quad \gamma_1' = 2\mu - \sum_{\alpha=1}^{4} \gamma_\alpha + \gamma_1, \text{ and so for } \gamma_2', \gamma_3', \gamma_4',$$

$$\gamma_5' = \gamma_5, \qquad \text{and so for } \gamma_6' \ldots \gamma_\sigma'.$$

Let T be a given punctual transformation between S, S'. By definition, this has the property analogous to Noether's theorem, and is compounded of a series of T_{tet}'s, between S, S''; ...; S''', S' say. If we isolate in turn the first and last of these, we can in two ways regard T as compounded of a T_{tet} and a punctual V which has one component fewer than T.

The two homaloidal webs of T are respectively the homologues by T_{tet} of the first homaloidal web of V in S'', and the homologues by V of cubics in S'' with four d.p.'s. Similarly, the P-surfaces of T are found from those of T_{tet} and V by transformation. From these considerations we can write down a large number of relations between the characteristic numbers of T, V, sufficient to reduce the required relations for T to depend on those for two punctual V's, each of which has one component fewer than T, and so finally on those for T_{tet} itself.

In order to keep the same number of terms in the various sums, when F-points drop out, it is convenient to define, for a point O_α that is not an F-point,

$$i_\alpha = \mu_\alpha = i_{\alpha\beta} = 0, \quad i_{\alpha\alpha} = -1 \text{ (192 and p. 26).}$$

The whole calculation is given in 74.

Some of the relations have simple direct proofs. Thus $\sigma = \sigma'$ holds, since when the characteristic is given, T is completely determined by $3\sigma + 15$ parameters, of which 3σ are the coordinates of the F-points, and 15 determine the base homaloids of the web; and T is equally determined by $3\sigma' + 15$ parameters in S'.

This is also proved by induction as follows. Let $T \equiv V.W$ where V, W are punctual. In the intermediate space S'', let V, W have α common F-points O'' and β common P-surfaces j''. Then the two P-surfaces of V, W that $\sim O''$ are ordinary homologues for T, and so are the two F-points of V, W that $\sim j''$. All other F- and P-elements of V, W give F- and P-elements of T; hence if we assume $\sigma = \sigma'$ for V, W,

$$\sigma_T = \sigma_T' = \sigma_V + \sigma_W - \alpha - \beta.$$

Another relation comes by superposing the spaces and considering the two isologic surfaces of a general line (p. 174),

$$\Omega_{n+1} \equiv \Sigma O_\alpha^{i_\alpha}, \quad \Omega'_{n'+1} \equiv \Sigma O'^{\,i'_\beta}_\beta,$$

which are homologues; hence, since $n = n'$,

$$n + 1 = n(n+1) - \sum_\alpha i_\alpha \mu_\alpha',$$

and since $i_\alpha = 2\mu_\alpha'$,

$$2(n^2 - 1) = \sum_\alpha i_\alpha^2 = \sum_\beta i'^2_\beta.$$

The analogue of Noether's inequality is

$$i_1 + i_2 + i_3 + i_4 \geqq 2(n+1),$$

for from the equations for Σi, Σi^2 we deduce

$$(n-1)\{i_1 + i_2 + i_3 + i_4 - 2(n+1)\} = \sum_{\alpha=1}^{3} (i_\alpha - i_4)(n - 1 - i_\alpha) + \sum_{\alpha=5}^{\sigma} i_\alpha (i_4 - i_\alpha) \geqq 0.$$

The inequality expresses that a T_{tet} applied to the four highest F-points reduces the degree of T by at least 2. On this rests a theory of normal resolution, as in the plane.

We can have $i_1 + i_2 + i_3 + i_4 = 2(n+1)$ only if either

$$i_1 = n - 1 \quad \text{or} \quad i_\alpha = i_4 \, (\alpha = 1 \ldots \sigma),$$

that is, if T is either monoidal or symmetric.

Since the P-surfaces in S all $\sim$ isolated F-points in S', and are exactly determined by their multiplicities at the isolated F-points in S, the whole plane argument, about the partitioning of groups of F-points of equal multiplicity by P-elements of equal degree (p. 23), holds of punctual space transformations; Clebsch's theorem can be carried over, and also, since $\sigma = \sigma'$, the complete 1, 1 relation between the two sets of F-points.

31. Special punctual transformations.

Since $i = 2\mu'$, every i is even, $\Sigma i^2 \equiv M(4)$, and n is odd. The only punctual cubic characteristic is that of T_{tet}, and the only quintic is $n = 5;\ 2^4 4^2$. For $n \geqq 7$, there are several for each n.

The only surfaces completely determined by seven or fewer points are

$$f_1 \equiv 3^1; \quad f_2 \equiv 1^2 5^1; \quad f_3 \equiv 4^2 3^1; \quad f_4 \equiv 1^3 6^2.$$

Hence if $\sigma \leqq 7$, every P-surface is of degree $\mu \leqq 4$, and i can only have the values 2, 4, 6, 8. There are nine characteristics with seven or fewer F-points,

$$n = 3;\ 4^2, \qquad n = 5;\ 2^4 4^2, \qquad n = 7;\ 1^6 3^4 3^2, \quad n = 7;\ 6^4, \qquad n = 9;\ 1^8 6^4;$$

$$n = 9;\ 3^6 3^4 1^2, \quad n = 11;\ 1^8 4^6 2^4, \quad n = 13;\ 3^8 4^6, \quad n = 15;\ 7^8,$$

and $\sigma \geqq 8$ if $n \geqq 16$.

For the symmetric punctual characteristics, we find

$$i = \frac{n+1}{2}, \quad \sigma = \frac{8(n-1)}{n+1},$$

and besides T_{tet} there are only $n = 7;\ 6^4$ and $n = 15;\ 7^8$. The involutory cases of these are considered below, pp. 323, 326.

We can find any number of general characteristics by assigning any law of coincidences of the F-points of the component T_{tet}'s. If all are in general positions, we have

$$n = 3^\mu; \quad 4^{2\cdot 3^{\mu-1}} 4^{2\cdot 3^{\mu-2}} \ldots 4^2.$$

If two F-points of each new T_{tet} are the highest F-points of the characteristic already formed, and all the others are general, we have

$$n = 2\mu + 1; \quad 2^{2\mu} (2\mu)^2;$$

the reverse is similar, and this is a bimonoidal T_{n-n} with an $(n-2)$-fold F-line joining the two vertices in each space (120).

More generally, let one F-point of each T_{tet} be the highest F-point of the last characteristic, and let the law of coincidences of the other three F-points in the successive spaces be the same as in the normal composition of any given plane characteristic $\mu;\ \mu_2' \ldots \mu_\sigma'$. The space characteristic is

$$n = 2\mu - 1; \quad 2\mu - 2,\ 2\mu_2', \ldots\ 2\mu_\sigma',$$

and is called the *dilation* of T_μ. The example just given is the dilation of T_J (75).

Conversely, any monoidal punctual characteristic is a dilation; for if

$$n = 2\mu - 1, \quad i_1 = 2\mu - 2,$$

we have

$$\sum_{\alpha=2}^{\sigma} \mu_\alpha' = 3(\mu - 1), \quad \sum_{\alpha=2}^{\sigma} \mu'^2_\alpha = \mu^2 - 1,$$

and μ; $\mu_2' \dots \mu_\sigma'$ is a plane characteristic.

32. Arguesian $\mathbf{I}_{3-3}$ (85, 106, 159, 287, 339, 408 and p. 170).

We shall now consider a few well-known involutions.

The simplest singular cubic transformation is no. 3 of Table VI, when the homaloidal curves project from the d.p. O into the lines of a plane, and are general plane sections of ϕ_3, with two branches through O. The F-curve is $\omega_6 \equiv O^2$; it projects into a non-singular plane k_4, and is therefore of genus 3.

The Arguesian space transformation is a special involutory case of this, when P, P' are conjugate in the involution cut out on OP by a given pencil of quadrics. The quartic curve ω_4 of genus 1, base of the pencil, is an F-curve, for any point Q of $\omega_4 \sim$ the whole of OQ; and $\omega_4 \sim$ the cone $j_4 \equiv O^4\omega_4$. The other F-lines are the two chords of ω_4 which pass through O; every point of one of these $\sim$ the whole line, and they are of second species.

This is a T_{M3-3}, and $O \sim$ a quadric $j_2 \equiv O\omega_4$ and $J_8 \equiv j_2^2 j_4$. The transformation is central, and in any plane through O it sets up a plane Arguesian transformation (p. 116).

33. A skew $\mathbf{I}_{5-5}$ (135, 318).

An involution of degree 5, of which some incorrect details have been published, is cut out on a bilinear congruence by a pencil of quadrics.

Let l_1, l_2 be the axes of the congruence, and ω_4 the quartic of genus 1, base of the pencil; these are F-curves. If P lies on l_1, the ray is any line through P in the plane Pl_2, and $P \sim$ the conic, section by this plane of the quadric through P; hence l_1, l_2 are double F-lines. If P lies on ω_4, the quadric is indeterminate, and $P \sim$ the whole ray through P; hence ω_4 is a simple F-curve.

There are eight rays which are chords of ω_4; each is a generator of the quadric through any point of itself, which $\sim$ the whole ray; these are eight simple F-lines of second species.

There are no other F-points, the total degree of F-curves is

$$2 . 4 + 4 + 8 = 20 = n^2 - n;$$

hence $n = 5$. The homaloids are quintics with two double lines.

34. Geiser's I_{7-7} (119, 132, 365, 380).

The earliest known space transformation, after T_{2-2} and T_{3-3}, is the involution connecting the seventh and eighth base points of nets of quadrics through six given points $O_1, \ldots O_6$.

This is the only web with two free intersections determined by simple base points only; for if μ is the degree and α the number of base points, $\alpha = \binom{\mu+3}{3} - 4 = \mu^3 - 2$, whence $\mu = 2$, $\alpha = 6$ (347).

The same involution has been approached from four dimensions (10).

It follows from the definition that:

The line $O_1O_2 \sim$ itself; there are fifteen such lines, and they are F-lines of second species.

The twisted cubic $\omega_3 \equiv O_1 \ldots O_6 \sim$ itself, and is also an F-curve of second species.

The point $O_1 \sim$ the quadric cone $q_1 \equiv O_1^2 O_2 \ldots O_6$; there are six such P-cones.

The plane $O_1O_2O_3 \sim$ the plane $O_4O_5O_6$; for if P lies on $O_1O_2O_3$, one quadric of the net through P is this pair of planes, and P' lies on one or other of them. If it lay on $O_1O_2O_3$, all the quadrics through P would contain the conic $O_1O_2O_3PP'$, and would not form a net; hence P' lies on $O_4O_5O_6$. The intersection of the two planes is a self-corresponding line. There are twenty such planes and ten such lines.

The vertex of a quadric cone of the web is an invariant point of I_{7-7}, which has therefore a quartic invariant surface Δ_4, the Jacobian of the quadrics. It has $O_1 \ldots O_6$ as d.p.'s, and contains all the F-curves.

A net of the quadrics contain ω_3, and a pencil of these pass through any point P and contain the chord of ω_3 through P. Hence P' lies on the same chord, and the associated complex of I reduces to the congruence of chords of ω_3. The involution on any chord is cut out by any pencil of the quadrics which has no base point on the chord, and the two intersections with ω_3 are a conjugate pair.

If P moves on a line l, then P' lies on the $f_4 \equiv \omega_3^2 l$, locus of chords of ω_3 meeting l, and also on the one quadric that contains l. These two surfaces meet in l and a c_7 which $\sim l$; this meets l four times, passes twice through $O_1 \ldots O_6$, and does not meet any of the F-curves again. The degree of I is that of c, and $n = 7$.

We can also find this by considering degenerate homaloids. The ϕ_n which $\sim O_1O_2O_3$ breaks up into:

q_1 containing $O_1^2O_2O_3O_4O_5O_6\omega_3$ and the five F-lines $O_1O_2, \ldots O_1O_6$,
q_2 „ $O_1O_2^2O_3O_4O_5O_6\omega_3$ „ „ „ „ $O_2O_1, \ldots O_2O_6$,
q_3 „ $O_1O_2O_3^2O_4O_5O_6\omega_3$ „ „ „ „ $O_3O_1, \ldots O_3O_6$,
p „ $O_4O_5O_6$ and the three F-lines O_5O_6, O_6O_4, O_4O_5,

and is a $\phi_7 \equiv (O_1 \ldots O_6)^4 \omega_3^3$. The twenty such degenerate ϕ's do not form a net, and any other ϕ is a linear function of them; hence $n = 7$, and H consists of $(O_1 \ldots O_6)^4$, ω_3^3, and the fifteen simple lines $O_\alpha O_\beta$.

These F-curves are of total degree $3 \times 3^2 + 15 = 42 = 7^2 - 7$; hence there are no others. J_{24} consists of the six P-cones each taken twice, and there are no other isolated F-points. The F-curves have no free intersections with c_7, and are all of second species, necessary consequences of the six 4-fold points. The I_{7-7} is a symmetric punctual transformation, and is normally resolved into three T_{tet}'s. The first, applied to four of the F-points, reduces the characteristic to $n = 5$; $2^4 4^2$; another T_{tet} reduces this to $n = 3$; 4^2, that is, to a third T_{tet}.

It can also be resolved into two particular cases of the bilinear T_{3-3} (p. 300 and B 1 of 158). The F-sextics of the first consist, in S, of ω_3, O_2O_3, O_3O_1, O_1O_2, and in the intermediate space S'', of three skew lines (λ'') and three lines (l'') concurrent at O'' meeting pairs of (λ''). This converts (ϕ_7) into (ϕ_3'') through three generators of the opposite system of the quadric through (λ''), which $\sim O_4$, O_5, O_6, and three lines through O'', meeting pairs of these, which $\sim O_5O_6$, O_6O_4, O_4O_5. This is the homaloidal family of a T_{3-3} of the same nature as the reverse of the first, and I_{7-7} is resolved into the two.

This is the simplest example of the argument of p. 273, which shows that if a plane p' meets ω_3 in Q_1, Q_2, Q_3, then the homologue ϕ has three separate sheets through ω_3, touching the cones which project ω_3 from Q_1, Q_2, Q_3 respectively; these vary with p' and ϕ, and ω_3 is not a curve of contact. This also follows from the fact that ϕ has six 4-fold points on ω_3, and conversely.

Let ω_3 be represented as on p. 304. The cone projecting ω_3 from the point τ_1 is

$$q_1 \equiv (\tau_1 x - y)(\tau_1 z - w) - (\tau_1 y - z)^2 \equiv \tau_1^2 q_a - \tau_1 q_b + q_c,$$

and touches
$$\tau_1^2 t_a - \tau_1 t_b + \tau t_c \equiv (\tau_1 - \tau)\, t_1,$$

where
$$t_1 \equiv \tau_1 t_a - t_c.$$

The web $(\phi_7) \equiv \omega^3$ has the form $(q_a, q_b, q_c \overset{3}{)(} x, y, z, w)^1$, and its three tangent planes are

$$t_\phi \equiv (t_a, t_b, \tau t_c \overset{3}{)(} 1, \tau, \tau^2, \tau^3)^1 \equiv (t_a, t_c \overset{3}{)(} 1, \tau)^6.$$

Now ϕ_7 has six fixed 4-fold points τ_a; at each of these, $t_\phi \equiv 0$, and t_ϕ has $\prod\limits_{(6)} (\tau - \tau_a)$ as a fixed factor. The remaining factor is $(t_a, t_c)^3$ with constant coefficients; it falls into three factors $t_1 t_2 t_3$ where $t_1 \equiv \tau_1 t_a - t_c$, the coefficients τ_1, τ_2, τ_3 depending on the parameters of ϕ. Thus

$$t_\phi \equiv t_1 t_2 t_3 \prod_{(6)} (\tau - \tau_a),$$

and ϕ has three sheets touching q_1, q_2, q_3 respectively, which are three cones whose vertices vary on ω_3 with the parameters of ϕ.

Conversely, if ϕ_7 touches these cones, t_ϕ has the factor

$$(\tau_1 t_a - t_c)(\tau_2 t_a - t_c)(\tau_3 t_a - t_c),$$

whose four coefficients $\tau_1\tau_2\tau_3$, $\Sigma\tau_1\tau_2$, $\Sigma\tau_1$, 1 give all the parameters of (ϕ). The remaining factor $(1, \tau)^6$ is fixed for all of (ϕ), and gives six fixed values of τ, and six points of ω_3, for each of which t_ϕ is indeterminate, and ϕ has a 4-fold point.

This can also be proved by applying an auxiliary transformation which converts ω_3 into a line, for example, a bilinear T_{3-3} whose F-sextic meets ω_3 in eight points.

35. $\mathbf{I_{9-9}}$ on the chords of a cubic (360).

Now let the involution on each chord of ω_3 be determined by two pairs, its intersections O, O' with ω_3 and its other intersections Q, Q' with a given quartic surface $f_4 \equiv \omega_3$. As before, ω_3 is a triple F-curve of second species, and ϕ_n has three sheets touching three variable quadric cones along ω_3. But we shall find that ϕ has a fourth sheet through ω_3, which touches a fixed surface, and that with respect to this sheet, ω_3 is of first species. This means that there are branches of c' meeting ω_3, and they all lie on the sheets in contact, not on the first three sheets.

For the F-points, where I fails, either the chord l is indeterminate, or the involution on it is improper or indeterminate. Now l is indeterminate only if P lies on ω_3, and we have seen that from this cause every point O of $\omega_3 \sim \omega_3$.

If l is determinate, the involution on it is improper if $O \equiv Q$, and O', Q' are distinct; then $O \sim$ both O' and Q', and therefore $O \sim$ every point of l. Now l touches f_4 at O, and is the one chord of ω_3 in the tangent plane to f_4 at O. Thus $O \sim$ the variable P-line l as well as the fixed P-curve ω_3, and ω_3 is at the same time a simple F-curve of first species and a triple F-curve of second species, and is 4-fold on (ϕ).

If $P_1 \rightarrow$ a point P of l, the chord $P_1O_1O_1'Q_1Q_1'$ is near the tangent plane to f_4 at O, and O_1, Q_1 are points of f_4 near O; the plane $OO_1Q_1 \rightarrow$ the tangent plane to f_4 at O. But P_1' is near O_1 and Q_1, and therefore near O, and lies in the plane OO_1Q_1, and $\rightarrow O$ in a direction touching f_4. Hence if P_1 describes a general plane p meeting l in P, the homologue ϕ of p touches f_4 at O and similarly at every point of ω_3. Hence the fourth sheet of ϕ touches the fixed surface f_4 along ω_3.

Since ω_3 is 4-fold on (ϕ_n), and any chord of ω_3 is self-corresponding, and meets ϕ_n in one free point, $n = 2 . 4 + 1 = 9$, and the transformation is I_{9-9}.

The involution on l is indeterminate if $O \equiv Q$ and $O' \equiv Q'$, for then we are given only one pair of conjugates. Any point of $l \sim$ every other point, and l is an F-line of second species; it is a chord of ω_3 bitangent to f_4.

Let there be α such lines of multiplicity i. Now ω counts $4^2 + 1 = 17$ times in the aggregate of F-curves, whose total degree is

$$9^2 - 9 = 17 . 3 + \Sigma i^2 \alpha,$$

whence

$$i = 1, \quad \alpha = 21,$$

as can also be proved directly.

Now c_9' does not meet any F-curve of second species, and where it meets ω_3 it touches f_4 and meets a homaloid χ in 5 points. Since 80 of its 81 intersections with χ are thus absorbed, it meets ω_3 16 times.

Now $f_4 \equiv (q_a, q_b, q_c \between x, y, z, w)^2$, and touches $t_f \equiv (t_a, t_c \between 1, \tau)^7$,

$\phi_9 \equiv (q_a, q_b, q_c \overset{4}{\between} x, y, z, w)$, " " $t_\phi \equiv (t_a, t_c \overset{4}{\between} 1, \tau)^7$,

and t_ϕ has the four factors $t_f t_1 t_2 t_3$ and no other. Hence (ϕ_9) has no 5-fold points on ω_3 unless f_4 is singular.

36. $\mathbf{I}_{11-11}$, inversion of $\mathbf{I}_{\text{tet}}$ (348).

There is an I_{11-11}, an extension to space of the metrical plane I_5 of p. 122.

Let O_1, O_2, O_3, O_4 be the feet of the four perpendiculars on the faces of a tetrahedron drawn from the circumcentre O_5 say. Any sphere q through $O_1O_2O_5$ determines another q' such that the plane $O_1O_2O_5$ bisects the angle between q, q', and also that between the spheres $O_1O_2O_3O_5$ and $O_1O_2O_4O_5$. A point P determines six spheres such as q, and six such as q'; the last set concur at P'. By inversion with regard to O_5, this is reduced to the construction of I_{tet} by symmetric planes (p. 302).

If $P \equiv O_1$, three spheres q' are indeterminate, and three coincide in $O_2O_3O_4O_5$, which is the P-surface which $\sim O_1$; while $O_5 \sim$ a sextic P-surface, as is most easily seen by inversion. The plane $O_1O_2O_5 \sim$ itself, after dropping P-surfaces of degree $2+2+6$; hence $n = 11$.

If, instead, we invert the construction of I_{tet} with regard to O_4, we obtain an I_{6-6}, where the spheres $PO_1O_2O_4$, $P'O_1O_2O_4$ are equally inclined to the sphere $O_1O_2O_3O_4$ and the plane $O_1O_2O_4$ respectively; and also the pairs of planes PO_1O_4, $P'O_1O_4$ are symmetric with regard to the bisectors of the dihedral angles $O_1O_2O_4$, $O_1O_3O_4$.

37. The symmetric punctual $\mathbf{I}_{15-15}$ (75, 77, 191).

The last symmetric punctual transformation has seven 8-fold F-points $O_1 \dots O_7$. It is generated by the ∞^6 family of quartics $(f_4) \equiv (O_1 \dots O_7)^2$. If f_P is any proper quartic of the family through P, and q_1, q_2 two independent quadrics through P of the net (q_1, q_2, q_3) determined by $O_1 \dots O_7$ and the associated eighth point X, then any other of the quartics through P is

$$f_P + (q_1, q_2 \between q_1, q_2, q_3),$$

and contains P' the other free intersection of f_P and the $k_4 \equiv O_1 \dots O_7 P$, intersection of q_1, q_2.

One f_4 has a triple point at O_1; this is the P-surface which $\sim O_1$, and (c_n) have four branches through $O_1 \dots O_7$, and being determined by these seven 4-fold points, are of degree 15. Hence $n = 15$, $i = 8$.

The quadrics q are self-corresponding, and the two-dimensional transformation set up on any one of them is projected from X into Bertini's plane I_{17} (p. 127).

To reduce I_{15-15} normally, apply a T_{tet} to the four highest F-points, giving in turn

$$n=13;\ 3^8 4^6,\quad n=9;\ 3^6 3^4 1^2,\quad n=5;\ 2^4 4^2,\quad n=3;\ 4^2;$$

hence I_{15-15} is compounded of five T_{tet}'s.

38. Types of general degree.

(i) Let there be an $(n-2)$-fold line l, and let the remaining F-curve ω be simple and of degree m, meeting l in d points; then $n'=4n-4-m$ **(216, 225·1, 251, 263)**.

Lines meeting l, ω contain $n-1$ F-points, and $\sim$ another congruence of lines. If $d=m-1$, these congruences are of first degree (p. 309).

Now l meets $c_{n'}$ in $4n-8-d$ points, and $\sim j'_{4n-8-d}$; a plane p through $l \sim$ a surface of degree $n'-(4n-8-d)=d-m+4$. If $d=m-3$, this is a plane p', and T changes the pencil of planes (l) into another pencil of planes, whose axis l' is an F-line, of multiplicity $i'_{l'}$ say.

The plane pencil of rays in p meeting l, whose vertex is any point O of $\omega \sim \infty^1$ lines in p', and therefore also a plane pencil, whose vertex O' is a point of an F-curve ω' of multiplicity $i'_{\omega'}$ say, where $i'_{l'}+i'_{\omega}=n'-1$. This sets up a 1, 1 relation between the points of ω, ω', and these curves are of the same genus; if ω is proper, so is ω', and any other F-curve of H' lies in a plane through l'. Any p contains $m-d=3$ such pencils of rays; hence ω' meets p' in three points O' not on l', and meets l' in $d'=m'-3$ points.

An example is the T_{3-4}, no. 6 of Table VI, when ω_5 breaks up into a line l and an ω_4 of genus 1 meeting l once; there is also a simple F-point O. The P-elements are: chords of ω_4 meeting l, whose locus is a $j_5 \equiv l^2\omega_4{}^2$, conics on the quadric $O\omega_4$ and conics in the plane Ol. Then (ϕ_4') have a simple ω_5', a double chord l' of ω_5' and a double trisecant of ω_5' meeting l'.

Another example is the T_{3-5}, no. 12 of Table VI, where l is simple and ω_3 does not meet it; there are also two simple F-points. Then (ϕ_5') have a rational ω_5', a triple chord l' and two double trisecants meeting l' (p. 373 of **385**). For $n=5$ see **21**.

If $n=n'$, then $m=3n-4$, $d=3n-7$. An example is the bilinear T_{3-3}, where ω_6 breaks up into a quintic of genus 2 and a chord **(86)**. It can be involutory, the rays of the congruence conjugate in pairs **(251)**.

(ii) The only other involutions that permute the rays of a congruence are I_M and an $I_{(2n+1)-(2n+1)}$ with an n-fold F-quartic and $2n$ simple F-chords **(191, 216, 225·1, 256, 287, 320, 368)**.

(iii) There have also been studied transformations with an i-fold and an $(n-i-1)$-fold F-line, which convert the congruence of lines meeting both into another such. For example, the I_{5-5} of p. 322 **(216, 263)**.

A similar type has an $(n-3)$-fold line and a certain number of triple points **(397)**.

Other special transformations are described in the following memoirs:

I_{4-4}, 7.
T_{3-5}, no. 23 of Table VI, 49.
T_{3-6}, no. 41 of Table VI, 41.
T_{4-5}, 138, 254.
T_{4-6}, 88.
T_{5-6}, 89.
I_{6-6}, 245.
I_{7-7}, 260.
I_{11-11}, 320, 368.
T_{8-13}, 252.
T_{15-15}, I_{27-27}, 47.
I_{19-19}, 250, 412.

Large numbers of examples are given in:

6
10
21
23
24
85
86
158
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160
217
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269
283
316
318
361
368
370
384
385.

CHAPTER XV

A CUBO-QUARTIC TRANSFORMATION

1. Equations of the transformation.

Consider the transformation, no. 10 of Table VI, given by the cubic equations

$$\begin{aligned} x' : y' : z' : w' &= \phi_1 : \phi_2 : \phi_3 : \phi_4 \\ &= xt + dyzw + ezw^2 : yw(z-w) : z^2(z-w) : zw(z-w), \end{aligned}$$

where $$t \equiv yw + z^2 + azw + bw^2.$$

We first show that this is a Cremona transformation, by actually solving for $x : y : z : w$. From the three last ratios,

$$y' : w' = y : z, \qquad z' : w' = z : w,$$

whence $$y : z : w = y'z' : z'w' : w'^2;$$

and from the two first ratios,

$$\begin{aligned} \frac{x'}{y'} &= \frac{xt + dyzw + ezw^2}{yw(z-w)} \\ &= \frac{x}{y} \cdot \frac{y'z'w'^2 + z'^2w'^2 + az'w'^3 + bw'^4}{w'^2(z'w' - w'^2)} + \frac{z'w'(dy'z' + ew'^2)}{y'z'(z'w' - w'^2)}, \\ \frac{x}{y} &= \frac{w'\{x'(z'-w') - dy'z' - ew'^2\}}{y'u_2'}, \end{aligned}$$

where $$u_2' \equiv y'z' + z'^2 + az'w' + bw'^2.$$

Hence

$$\begin{aligned} x : y : z : w &= \phi_1' : \phi_2' : \phi_3' : \phi_4' \\ &= z'w'\{x'(z'-w') - dy'z' - ew'^2\} : y'z'u_2' : z'w'u_2' : w'^2u_2'. \end{aligned}$$

These are the equations of the reverse transformation, which is therefore of degree 4.

2. First F-system H.

We shall first find the different points and curves of H, and then determine their multiplicity, and order of contact, if any.

The conditions of intersection or contact satisfied by the general ϕ are also satisfied by the three degenerate base homaloids ϕ_2, ϕ_3, ϕ_4. The simplest of these is $\phi_3 \equiv z^2(z-w)$; hence all the F-elements lie on one or other of the planes z, $z-w$.

First consider z; it is part of ϕ_4, and meets ϕ_2 in the two lines $y = z = 0$, $z = w = 0$, the latter counted twice; any F-element lying on z lies therefore on one or other of these two lines. The first meets ϕ_1 where

$x \,.\, bw^2 = 0$, that is, in the two points $A_4\,(0, 0, 0, 1)$ and $A_1\,(1, 0, 0, 0)$, which are therefore F-points. The second line lies entirely on ϕ_1, and is therefore an F-line l; it passes through A_1 but not through A_4.

The second sheet $z - w$ of ϕ_3 is also part of ϕ_2 and ϕ_4; it meets ϕ_1 in the line $z = w = 0$, which is the F-line l already obtained, and also in the conic

$$z - w = 0, \quad x\,\{y + (1 + a + b)\,z\} + dyz + ez^2 = 0.$$

This is therefore an F-conic ω_2; it passes through A_1 but not through A_4.

Thus there are no F-points other than A_4 and the points of l and ω_2. Now the total degree of curves in H is $n^2 - n' = 3^2 - 4 = 5$; hence one or other of the F-curves must be taken more than once.

F-line l of osculation.

While we are considering the line l, $z = w = 0$, it is best to arrange the general homaloid

$$\phi \equiv c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4$$

in the first instance according to degree in z, w:

$$\begin{aligned} \phi \equiv \;& c_1 xyw && \text{of degree 1 in } z, w \\ &+ c_1\,\{x\,(z^2 + azw + bw^2) + dyzw\} + c_2 yw\,(z - w) && \text{of degree 2 in } z, w \\ &+ c_1 ezw^2 + c_3 z^2\,(z - w) + c_4 zw\,(z - w) && \text{of degree 3 in } z, w. \end{aligned}$$

The tangent plane to ϕ at the point $(x, y, 0, 0)$ of l is given by the linear terms, namely $c_1 xyw$. Hence every ϕ touches the fixed plane w along l.

At points of ϕ near l therefore, x, y are finite, z is small of first order and w of second. Rearrange the terms in order of smallness:

$$\begin{aligned} \phi \equiv \;& c_1 x\,(yw + z^2) && \text{of order of smallness 2} \\ &+ c_1\,(axzw + dyzw) + c_2 yzw + c_3 z^3 && \text{of order of smallness 3} \\ &+ \text{terms of order} \geqq 4\,; && \end{aligned}$$

hence ϕ osculates the quadric

$$q \equiv yw + z^2,$$

which is independent of the parameters $c_1, \ldots c_4$, and fixed for the whole family.

At points of ϕ near l therefore, q is small of third order; the other terms of third order depend on the parameters c_1, c_2, c_3, and there is no contact of order > 2 along l, which is an F-line of osculation.

We next verify that l counts just three times in the intersection of two general homaloids, ϕ and $\psi \equiv d_1\phi_1 + d_2\phi_2 + d_3\phi_3 + d_4\phi_4$. First eliminate q by forming $c_1\psi - d_1\phi$, of which the terms of lowest degree z, w are

$$(c_1 d_2 - c_2 d_1)\, yw\,(z - w)\,;$$

on this l is double and one sheet touches ϕ. Then eliminate the factor w from the lowest terms, by forming

$$\Lambda \equiv c_1 x\,(c_1\psi - d_1\phi) - (c_1 d_2 - c_2 d_1)\,(z - w)\,\phi,$$

whose lowest terms are cubic in z, w, not having w as a factor; on this, l is triple, no sheet touching w along l. Hence l counts just three times in the intersection of ϕ, Λ, that is of ϕ with $x\psi$, of which the sheet x does not contain l. Thus l counts just three times in the intersection of ϕ, ψ.

The co-factor of z^3 in Λ is a multiple of x; that of w in ϕ is c_1xy. These do not vanish except when $x=0$, at A_2, or when $y=0$ at A_1; hence at no other point of l does ϕ touch a sheet of Λ, or l meet its total residual in the intersection of ϕ, ψ. It follows that l is an F-line of second species.

In particular, if we examine the intersection of ϕ_1 with the other three base homaloids, we find:

a single sheet of ϕ_1 through l, touching w and osculating q;

two sheets of ϕ_2 through l, one intersecting ϕ_1 without touching it, and one touching w and ϕ_1 but not osculating q, so that l counts three times in the intersection of ϕ_1, ϕ_2;

two distinct sheets of ϕ_3, one repeated, neither touching w, so that l counts three times in the intersection of ϕ_1, ϕ_3;

three sheets of ϕ_4, two intersecting ϕ_1 without touching it, and one touching w but not osculating q, so that l counts four times in the intersection of ϕ_1, ϕ_4.

Simple F-conic ω_2.

Along ω_2, which is simple on every ϕ, each of ϕ_2, ϕ_3, ϕ_4 touches the plane $z-w$, and ϕ_1 does not. Hence ω_2 is not a curve of contact, and counts once only in the intersection of ϕ, ψ.

But ψ touches ϕ at one point of ω_2 given by

$$c_1\{d_2y+(d_3+d_4)z\}=d_1\{c_2y+(c_3+c_4)z\},$$

which is the one variable intersection of ω_2 with c the variable intersection of ϕ, ψ.

Thus the F-curves of H consist of l counted three times and ω_2 counted once; the total degree is $3\times 1+2=5$, as it must be.

Simple F-point A_4.

We have seen that the only F-point which does not lie on l or ω_2 is $A_4(0, 0, 0, 1)$. Here the tangent plane to ϕ is $c_1(bx+ez)-c_2y-c_4z$, and is variable with the parameters; A_4 is a simple F-point without contact, and c passes through it.

It remains to be seen whether there are any special points on the F-curves, which are either of higher multiplicity or of higher order of contact than the general points, and which must be considered as distinct F-elements. Our attention has already been directed to A_1, which is one of the points of intersection of l, ω_2.

Double F-point of contact A_1.

In considering A_1 (1, 0, 0, 0), we arrange ϕ according to powers of x; the lowest terms are c_1xt. This shows that (ϕ) has a d.p. at A_1 with the fixed tangent cone t, which contains l as a generator, and along it touches the same plane w as ϕ, and osculates the same quadric q; it also contains as a generator the tangent line to ω_2,

$$z-w=y+(1+a+b)z=0.$$

The terms of ϕ cubic in y, z, w are all of third order of smallness and effectively involve the parameters; there is no higher contact at A_1, which is a double F-point of simple contact.

Through A_1 there pass six branches of the total intersection of ϕ, ψ (p. 235), that is, l counted three times, ω_2, and two branches of c, each touching t and meeting a third homaloid χ in three points at A_1. We can find their tangents as follows.

By eliminating t we form

$$\Lambda \equiv c_1\psi - d_1\phi \equiv \{(c_1d_2-c_2d_1)\,yw+(c_1d_3-c_3d_1)\,z^2+(c_1d_4-c_4d_1)\,zw\}\,(z-w),$$

a degenerate cubic cone, not having t as a sheet. The six branches of the intersection of ϕ, ψ through A_1 touch the six common generators of t, Λ. The first factor of Λ gives a quadric cone, which touches t along l and meets it in two other variable generators; these are the two tangents to c. The second factor of Λ is the fixed plane $z-w$, meeting t in l and the tangent to ω_2.

F-binode A_2.

To examine the other points of l, we consider that the tangent plane to ϕ is given by c_1xyw; this becomes indeterminate not only at A_1, where $y=0$, but also where $x=0$, at A_2 (0, 1, 0, 0), the other intersection of l, ω_2. Hence A_2 is a common d.p. of the family. The tangent cone

$$w\,\{c_1(x+dz)+c_2(z-w)\}$$

breaks up into the fixed plane w and a variable plane passing through the fixed line $x+dz=z-w=0$, which is the tangent to ω_2 at A_2.

A_2 is therefore an F-binode with one fixed and one variable biplane. The point is a B_3, and there is no higher contact; through it there pass five branches of the total intersection, four touching the fixed plane, and one the variable plane. The former consist of l counted three times and one branch of c, the latter is ω_2.

There could be a point of higher contact on l only if, for some value of x/y, the terms of ϕ of third order of smallness reduced to a form depending on the parameter c_1 only. Since these terms include c_3z^3, this does not occur. Also there are no special points on ω_2 other than A_1 and A_2; for ϕ_1 never touches the same plane as ϕ_2, ϕ_3 and ϕ_4; and there is no d.p. of ϕ_3 on ω_2 except where $z=0$.

We have now obtained all the elements of H, and give a list for reference.

(i) Double point of simple contact, $A_1(1, 0, 0, 0)$, fixed tangent cone

$$t \equiv yw + z^2 + azw + bw^2.$$

(ii) Binode of partial simple contact, $A_2(0, 1, 0, 0)$, fixed biplane w, variable biplane through fixed line $x + dz = z - w = 0$.

(iii) Simple point $A_4(0, 0, 0, 1)$.

(iv) Simple line of osculation l, $z = w = 0$ of second species, fixed tangent plane w, fixed osculating quadric $q \equiv yw + z^2$; joins A_1A_2, lies on the fixed cone at A_1 and on the fixed plane at A_2.

(v) Simple conic ω_2,

$$z - w = 0, \quad x\{y + (1 + a + b)z\} + dyz + ez^2 = 0;$$

passes through A_1, A_2, touches the fixed cone at A_1 and the variable plane at A_2.

3. Postulation and equivalence of H.

A general cubic surface ϕ contains $\binom{6}{3} = 20$ terms and therefore depends on 19 parameters; the intersection of two such surfaces ϕ, ψ is a c_9 meeting a third χ in 27 points. We shall make ϕ acquire the five F-elements one after the other in the above order, and find the changes in P and E due to each fresh element.

(i) If ϕ has a d.p. at A_1 with the assigned tangent cone t, it has the form

$$\phi \equiv c_1xt + (y, z, w)^3;$$

and for this $P = 9$, $E = 18$; there are six branches of c_9 through A_1, each meeting χ in three points.

(ii) If ϕ were general, we should have for the binode A_2, $P = 7$, $E = 14$ and c_9 would be a proper curve (p. 239); but the presence of A_1 changes all this. With ϕ in its present form, in order that A_2 may be a d.p., the terms in y^3, y^2z, y^2w in $(y, z, w)^3$ must vanish; and in order that the tangent cone may break up into w and another plane, the term in yz^2 must also vanish. The additional postulation is only 4, and ϕ is reduced to

$$\phi \equiv c_1xt + c_2yzw + c_3yw^2 + (z, w)^3.$$

This form shows that ϕ necessarily contains l, touches w and osculates q along l, so that c_9 breaks up into l counted three times and c_6, as a consequence of the conditions at A_1 and A_2.

Of the six branches of c_9 at A_1, three are accounted for by l, and the other three are branches of c_6, each meeting χ in three points. Of the five branches of c_9 at A_2, four touch the fixed biplane, meeting χ in three points, and one touches the variable biplane, meeting χ in two points; l accounts for three of the four; the fourth and also the last are branches of c_6. Hence of the

18 intersections of c_6, χ, the number falling at A_1 and A_2 is $3 \times 3 + 3 + 2 = 14$; since c_6 does not meet l again, the other 4 are free, instead of 9 at the last stage: the additional equivalence is 5.

(iii) To make ϕ also contain the simple point A_4, the term in w^3 must vanish. The additional postulation is 1 and equivalence 1; and ϕ has now the form

$$\phi \equiv c_1 xt + c_2 yzw + c_3 yw^2 + z\,(z,\ w)^2.$$

(iv) All the conditions relative to l are automatically satisfied by virtue of the conditions at A_1, A_2; there is no further postulation or equivalence.

(v) To make ϕ also contain the conic ω_2, at points of which

$$z = w, \quad xt = -\,z^2\,(dy + ez),$$

we must have two conditions, which give $c_2 = dc_1 - c_3$, and the sum of the last three coefficients $= ec_1$. The additional postulation is 2, and ϕ is reduced to its final form:

$$\begin{aligned}\phi &\equiv c_1\,(xt + dyzw + ezw^2) - c_3\,(yzw - yw^2) + c_4\,(z^3 - zw^2) + c_5\,(z^2w - zw^2)\\ &\equiv c_1\phi_1 - c_3\phi_2 + c_4\phi_3 + (c_4 + c_5)\,\phi_4,\end{aligned}$$

which only differs by a change of constants from the form with which we started the chapter.

Now c_6 breaks up into ω_2 and c of degree 4. Of the three branches of c_6 at A_1, one is ω_2 and two are branches of c; at A_2, the branch of c_6 which touches the variable plane is ω_2, and that which touches the fixed plane is c; and A_4 lies on c, not on ω_2. Also c meets ω_2 once again at a variable point. Of the 12 intersections of c, χ, there are 6 at A_1, 3 at A_2, 1 at A_4, 1 on ω_2, and 1 is the free intersection of ϕ, ψ, χ. The additional equivalence of ω_2 is 2.

The total P and E of H are made up thus:

Element (i) (ii) (iii) (iv) (v)

$$P = 9 + 4 + 1 + 0 + 2 = 16 = \binom{6}{3} - 4;$$

$$E = 18 + 5 + 1 + 0 + 2 = 26 = 3^3 - 1.$$

In this calculation, the additional postulation and equivalence of each F-element depend on what elements have already been taken into consideration. This has been shown analytically, and can also be seen by pure geometry.

Suppose, for example, that we first make ϕ satisfy the conditions relative to the two F-curves. Then each of its sections, by a series of planes p normal to l, touches the intersection of p, w, and has the same curvature as the section of q by p. As $p \rightarrow A_1$ the vertex of q, the successive parallel sections of q are similar conics of diminishing scale. The curvature of q, and therefore of the section of ϕ, increases without limit and becomes infinite at A_1; this is therefore a singular point, and (ϕ) has a d.p. at A_1, as a necessary consequence of osculating the fixed cone q along l. This does not fix the tangent cone t at

A_1; from considerations of continuity, we see that t contains l, touches w and osculates q; also since ω_2 passes through A_1, the tangent to ω_2 is another fixed generator of t, which is therefore subject to four conditions and has one degree of freedom. To fix it requires one more condition. Hence if (ϕ) already satisfies the conditions relative to l and ω_2, the additional postulation of A_1 is 1.

When p passes through A_2, the section of q has a definite tangent and finite curvature; near A_2 therefore, ϕ has a series of sections with finite curvatures tending to a finite limit, and cannot have an ordinary conical point; if it had a simple point, the tangent plane would be w the tangent plane to q. But ω_2 passes through A_2, and its tangent line does not lie in w; hence ϕ has neither a simple point at A_2 nor a conic node, but at least a binode; one sheet touches w and osculates q not only at adjacent points of l but also by continuity at A_2 itself. Hence if (ϕ) already satisfies the conditions relative to l and ω_2, those relative to A_2 are necessarily satisfied, and the additional postulation is 0.

4. Second F-system H′.

In S', the whole of H' lies on ϕ_4', that is on the plane w' or on the quadric cone u_2'. Since u_2' is a common sheet of ϕ_2', ϕ_3', ϕ_4', its intersection with each sheet of ϕ_1' gives an F-curve. The first sheet z' of ϕ_1' gives the line l', $z' = w' = 0$ counted twice; the second sheet w' gives l' again and another line l'', $w' = y' + z' = 0$; the quadric sheet gives l' again and a twisted cubic ω_3'.

The F-points which do not lie on u_2' must lie on w', and ϕ_2' shows that they lie on either y' or z'. The latter gives l' again, and the former gives l''', $y' = w' = 0$, which also lies on ϕ_1' and is a third F-line. There are no F-points not lying on one or other of these four curves.

The three F-lines all meet at A_1' $(1, 0, 0, 0)$ through which ω_3' also passes; l', l'' are generators of u_2', and meet ω_3' again in the points A_2' $(0, 1, 0, 0)$ and $(d, 1, -1, 0)$ respectively; l''' does not lie on u_2' or meet ω_3' again.

Tacnodal F-line l′.

In considering l', we rearrange ϕ' according to degree in z', w';

$$\begin{aligned}\phi' \equiv \quad & c_2 z'^2 y'^2 \\ & + z' \left[c_1 w' \{x'(z'-w') - dy'z'\} + c_2 y'(z'^2 + az'w' + bw'^2) + (c_3 z' + c_4 w') y'w'\right] \\ & - c_1 e z' w'^3 + (c_3 z' + c_4 w') w' (z'^2 + az'w' + bw'^2).\end{aligned}$$

Hence l' is a double line on ϕ', and at every point both tangent planes coincide with the fixed plane z'. Also z' is a factor of the cubic terms, which shows that the section by a general plane has a tacnode.

If we assume w' small of first order, we find z' is of second order, say $y'z' = \Lambda' w'^2$, where Λ' is given to a first approximation by

$$c_2 y' \Lambda'^2 + (-c_1 x' + c_2 b y' + c_4 y') \Lambda' + c_4 b y' = 0.$$

The two roots Λ_1', Λ_2' of this equation are irrational functions of x'/y'. If they are regarded as constants, then $y'z' - \Lambda_1' w'^2$, $y'z' - \Lambda_2' w'^2$ are two quadrics, which osculate the separate sheets of ϕ' at the definite point $(x', y', 0, 0)$. These quadrics depend upon c_1, c_2, c_4, so that there is no higher contact of (ϕ') along l'.

Now ψ' has two sheets through l', each having simple contact with each sheet of ϕ'; and l' counts eight times in the intersection of ϕ', ψ'.

The only point of higher multiplicity on l' is where the tangent planes $c_2 y'^2 z'^2$ are indeterminate, that is, where $y' = 0$ at A_1', the isolated F-point already mentioned.

At a point of l' where one sheet of ψ' osculates one sheet of ϕ', one of the values of Λ' is the same for both surfaces. This leads to a quartic equation in x', y' from which we drop the factor $y'^2 x'$, corresponding to two fixed points, the F-point A_1', and A_2' the other intersection of l', ω_3', which is not otherwise remarkable. There remains a linear equation in x'/y', giving one point of osculation, which is a variable intersection of l', c'.

Simple F-line l''.

In considering l'', we arrange ϕ' in the form

$$\phi' \equiv w' \,.\, z'^2 \{c_1 (x' - dy') + c_2 a y'\} + (y' + z') \,.\, c_2 y' z'^2 + w' (w', y' + z' \big)\!\big(x', y', z', w')^2.$$

The tangent plane is variable at all general points of l''; it is indeterminate only where $z' = 0$, at A_1'; it is determinate and independent of c_1/c_2, giving a point of contact of (ϕ'), where $x' - dy' = 0$, the other intersection of l'', ω_3' and not otherwise remarkable. There is no other point of contact of ϕ', ψ'. Hence l'' is a simple F-line without contact, and with no variable intersection with c'; it is therefore of second species.

Simple F-line l'''.

The tangent plane at any point of l''' is

$$w' \,.\, z'^2 (c_1 x' + c_3 z') + y' \,.\, c_2 z'^3,$$

indeterminate at A_1', but not fixed for any value of x'/z'. Hence there is no contact of (ϕ') along l''', or at any point of it; there is no other intersection with ω_3'. But there is one point of contact of ϕ', ψ', and one variable intersection of l''', c'.

Simple F-cubic ω_3'.

This passes through A_1' and touches the line

$$z' - w' = y' + (1 + a + b) z' = 0.$$

We have already found the intersections of ω_3' with the F-lines, and shall show that it has two variable intersections with c'.

Now ω_3', l' are the intersections of the two quadrics

$$x' (z' - w') - dy'z' - ew'^2 \text{ and } u_2'.$$

Let p', p'' be their tangent planes; these are definite, except p'' at A_1'. The tangent plane to ϕ' at any point of ω_3' is

$$[c_1 z'w']\, p' + [c_2 y'z' + c_3 z'w' + c_4 w'^2]\, p'',$$

where in the square brackets we substitute the coordinates of the point of contact. This agrees with the corresponding plane deduced from ψ' at the points of ω_3' lying on

$$d_1 (c_2 y'z' + c_3 z'w' + c_4 w'^2) - c_1 (d_2 y'z' + d_3 z'w' + d_4 w'^2).$$

This is a quadric cone, vertex A_1', which meets u_2' in l' counted twice and two variable generators joining A_1' to the points of contact of ϕ', ψ' lying on ω_3'. Hence there are two such points, and two variable intersections of ω_3', c'.

The tangent plane is independent of the parameters only when it coincides with p'', and $z'w' = 0$. Now z' meets ω_3' at A_1', and touches it at A_2'; and w' meets ω_3' at A_1', A_2' and the other intersection of l'', ω_3'. These are all points already discussed and do not give fresh F-elements.

We have now considered all the F-curves of H', and their total degree is

$$8 + 1 + 1 + 3 = 13 = 4^2 - 3,$$

as it must.

Triple F-point A_1'.

At A_1' (1, 0, 0, 0), we write

$$\phi' \equiv x' \,.\, c_1 z'w' (z' - w') + (y', z', w')^4.$$

There are three sheets touching three distinct fixed planes z', w', $z' - w'$. At points of the first sheet, z' is small compared with y' or w', and the second approximation is given by

$$- c_1 x'z' + c_4 b w'^2 = 0.$$

On the second sheet, w' is small of second order and approximately

$$c_1 x'w' + c_2 y' (y' + z') = 0.$$

On the third sheet, $z' - w'$ is the small quantity:

$$c_1 x' (z' - w') - c_1 z' (dy' + ez') + \{c_2 y' + (c_3 + c_4) z'\} \{y' + (1 + a + b) z'\} = 0.$$

All these depend upon the parameters, and there is no osculation of (ϕ').

The three sheets do not all play the same part in the transformation. They all pass through l', the second contains l'' and l''' and the third the tangent to ω_3'. The total intersection of ϕ', ψ' has twelve branches through A_1', four touching each plane, all lying on the degenerate quartic cone

$$c_1\psi' - d_1\phi' \equiv \{(c_1 d_2 - c_2 d_1) y'z' + (c_1 d_3 - c_3 d_1) z'w' + (c_1 d_4 - c_4 d_1) w'^2\}\, u_2'.$$

The four branches touching z' all coincide with l'; of the four touching w', two are l' and the others are l'' and l'''; of the four touching $z' - w'$, two are l', one is ω_3' and the last is c'.

We have now obtained all the elements of H', which we rearrange as follows.

(i) Triplanar point of contact A_1' $(1, 0, 0, 0)$, fixed triplanes z', w', $z'-w'$.

(ii) Tacnodal line l', $z'=w'=0$, through A_1', fixed double tangent plane z'.

(iii) Simple line l'', $w'=y'+z'=0$ of second species, through A_1'.

(iv) Simple line l''', $w'=y'=0$, through A_1'.

(v) Simple twisted cubic ω_3' given (with l') by

$$x'(z'-w')-dy'z'-ew'^2=0, \quad u_2'=0,$$

through A_1', meeting each of l', l'' once again.

5. Postulation and equivalence of H′.

A general quartic surface ϕ' contains $\binom{7}{3}=35$ terms, and therefore depends on 34 parameters; the intersection of two such surfaces ϕ', ψ' is a c_{16}' meeting a third χ' in 64 points. We shall make ϕ' acquire the five F-elements in the above order.

(i) If ϕ' has the triplanar point A_1' with the given triplanes,

$$\phi' \equiv c_1x'z'w'(z'-w')+(y', z', w')^4,$$

and we know from p. 235 that $P=19$, $E=48$. There are twelve branches of c_{16}' through A_1', each meeting χ' in four points; and there are 16 free intersections of c_{16}', χ'.

(ii) To make ϕ' have l' as a tacnodal line with z' as fixed tangent plane, the terms of degree 0 and 1 in z', w' vanish, those of degree 2 are a multiple of z'^2 and those of degree 3 of z'. The additional postulation is 6, and now

$$\phi' \equiv c_1x'z'w'(z'-w')+c_2y'^2z'^2+y'z'(z', w')^2+(z', w')^4.$$

Since l' counts eight times in the total intersection of ϕ', ψ', the residual is now c_8' with four branches through A_1', two touching each of the two triplanes which are not the tangent plane along l', and each branch meeting χ' in four points.

As we showed above, l' meets its residual in two points besides A_1', at each of which c_8' touches each sheet of χ', and four intersections are absorbed. The number of free intersections of c_8', χ' is therefore $8\times4-4\times4-2\times4=8$, and the additional equivalence of l' is 8.

(iii) Now l'' already meets (ϕ') in four points at A_1', and one more condition makes it lie wholly on (ϕ'); the additional postulation of l'' is 1, and we have

$$\phi' \equiv c_1x'z'w'(z'-w')+c_2y'z'^2(y'+z')+c_3z'^3(y'+z')+y'z'w'(z', w')+w'(z', w')^3.$$

Now c_8' breaks up into l'' and c_7', with three branches through A_1', two touching $z'-w'$ and one touching w', meeting l' in two points as before, and meeting l'' once at a simple intersection of c_7', χ'. The number of free intersections of c_7', χ' is $7 \times 4 - 3 \times 4 - 2 \times 4 - 1 = 7$, and the additional equivalence of l'' is 1.

(iv) Similarly, the additional postulation of l''' is 1, and now

$$\phi' \equiv c_1 x'z'w'(z'-w') + c_2 y'z'^2(y'+z') + y'z'w'(z', w') + w'(z', w')^3.$$

Also c_7' breaks up into l''' and c_6' with two branches through A_1', both touching $z'-w'$, meeting l', l'' as before and meeting l''' once. The number of free intersections is $6 \times 4 - 2 \times 4 - 2 \times 4 - 1 - 1 = 6$, and the additional equivalence of l''' is 1.

(v) Finally, we make ϕ' contain ω_3'; it now contains the total intersection of the two quadrics $x'(z'-w') - dy'z' - ew'^2$ and u_2', and ϕ' becomes a sum of multiples of these two expressions.

By comparison with the last form, the only admissible terms are

$$z'w'\{x'(z'-w') - dy'z' - ew'^2\}, \quad u_2'.(y'z', z'w', w'^2),$$

which gives the original form of the general homaloid ϕ'. The additional postulation of ω_3' is 4.

Now ω_3' passes once through A_1', where it touches $z'-w'$, and meets l' and l'' again, but not l'''; hence c_6' breaks up into ω_3' and c' of degree 3, passing once through A_1', where it touches $z'-w'$, and meeting l' and l''', but not l''; also, as we have seen, c' meets ω_3' twice. The number of free intersections of c', χ' is $3 \times 4 - 4 - 4 - 1 - 2 = 1$, and the additional equivalence of ω_3' is 5.

The total P and E of H' are made up thus:

$$\begin{array}{l} \text{Element (i)} \quad \text{(ii)} \;\; \text{(iii)} \;\; \text{(iv)} \;\; \text{(v)} \\ P = 19 + 6 + 1 + 1 + 4 = 31 = \binom{7}{3} - 4, \\ E = 48 + 8 + 1 + 1 + 5 = 63 = 4^3 - 1. \end{array}$$

As before, the additional postulation and equivalence of the separate F-elements, as just calculated, depend on the order in which the elements are taken. For example, let ϕ' satisfy all the conditions relative to the F-curves; we shall show geometrically that all the conditions relative to A_1' are necessarily satisfied.

First, A_1' is at least a d.p. since it lies on the double line l'. If it were not of higher multiplicity, the tangent cone would be the tacnodal plane z' taken twice. But the tangent cone at A_1' must contain the tangents to the three other F-curves through A_1', none of which lie on z'; hence A_1' is triple at least. Then the tacnodal conditions require that the cubic tangent cone has two sheets touching z'; it therefore breaks up into z' and the quadric cone determined by the five conditions of touching z' and containing the tangents to the three other F-curves. Since two of these lie on a plane through l', this cone breaks up into the plane w' containing l', l'' and l''', and the plane $z'-w'$ containing l' and the tangent to ω_3'. Hence all the conditions relative to A_1' are satisfied.

6. First P-system.

To investigate the first P-system, we consider the Jacobian

$$J \equiv \begin{vmatrix} t, & \dfrac{\partial\phi_1}{\partial y}, & \dfrac{\partial\phi_1}{\partial z}, & \dfrac{\partial\phi_1}{\partial w} \\ 0, & w(z-w), & \dfrac{\partial\phi_2}{\partial z}, & \dfrac{\partial\phi_2}{\partial w} \\ 0, & 0, & 3z^2-2zw, & -z^2 \\ 0, & 0, & 2zw-w^2, & z^2-2zw \end{vmatrix} \equiv 3z^2w(z-w)^3t.$$

We also calculate

$$G \equiv (c_1\phi_1' + c_2\phi_2' + c_3\phi_3' + c_4\phi_4')/(c_1x + c_2y + c_3z + c_4w) \equiv z^4w(z-w)^4t.$$

Both J and G consist of the three planes z, w, $z-w$ and the quadric cone t, taken with different multiplicities. These are therefore the four P-surfaces, and each meets ϕ in P-curves and F-curves only; so by determining these intersections we shall find some at least of the typical P-elements.

The P-plane z meets ϕ in l and a variable conic κ_2,

$$z = c_1x(y+bw) - c_2yw = 0,$$

which does not break up in general; it meets l in the F-points A_1, A_2 and also passes through A_4. Now c meets the plane of κ_2 in four F-points, two coinciding with A_1, and A_2, A_4; it does not meet κ_2 in a variable point. Also κ_2 meets ψ in six F-points only: three at A_1, where it passes through the d.p. and touches the fixed tangent cone, two at A_2, where it touches the variable biplane of ϕ and does not touch ψ, and one at A_4.

Thus the homologue of κ_2 does not meet a general plane in S', and is not a curve but an F-point. Since κ_2 is incident twice with every plane in S, the F-point is double on every ϕ'; since every ϕ contains one conic such as κ_2, the P-plane z, locus of κ_2, corresponds to a locus of double F-points in S' incident once with every plane, that is, a double F-line, which can only be l'.

The P-plane w touches ϕ along l and meets it in a variable line κ_1,

$$w = c_1x + c_3z = 0$$

through A_2, which meets ψ in three points at A_2 only, where it passes through the d.p. and lies in the fixed biplane. Since κ_1 meets every plane once, and ϕ contains one such P-line, the P-plane $w \sim$ a simple F-line of first species, which is l'''.

The P-plane $z-w$ meets ϕ in the F-curves l, ω_2 only; hence it $\sim$ a single F-point, say O'. It meets any line in S once, hence c' has one branch through O'. If any ϕ contains a general point of $z-w$, it contains the whole of it, and breaks up into the P-plane and a quadric $c_2yw + c_3z^2 + c_4zw$, touching w along l and also containing the fixed line $y = z = 0$; two such degenerate homaloids ϕ, ψ have for their variable intersection a line λ only. Now ϕ, ψ, $\lambda \sim$ two planes and a line λ' through O'; since λ meets any plane once, λ' meets any ϕ' in one free point and three points at O'; which is therefore a triple F-point, namely A_1'.

The quadric P-cone t, which is the fixed tangent cone to (ϕ) at A_1, meets ϕ in l counted three times and in three variable P-lines, each meeting ψ in three points at A_1 and in no other point. This P-surface, meeting every line twice, and with three P-elements in common with every ϕ, each incident once with every plane, corresponds to the simple F-cubic ω_3', meeting c' in two variable points.

Thus the sheets of J lead to four out of the five F-elements of H'. One simple F-line l'' has not appeared in this connection, and does not $\sim$ a P-surface, being of second species. Thus, though J gives a short cut to all the first P-surfaces, and hence to most of the elements of H', it does not exhaust H', either by finding all its elements or by disclosing the full nature of the elements which it does find.

We shall therefore now attack the first P-system indirectly, from the other space, and find exactly what happens in S when a surface in S' satisfies in turn the various conditions relative to the elements of H', in the order given on p. 338.

7. Correspondence between A_1' and $z - w$.

If a general surface f_μ' in S' is made to pass through the F-point A_1', each of its terms contains y', z' or w' as a factor, and after transformation the factor $z - w$ is dropped. This P-plane therefore, as we have seen, corresponds to A_1'. If f' had an i-fold point at A_1', by the same argument $(z - w)^i$ would be dropped.

The transformed expression, $(z - w) f_{3\mu-1}$ say, besides possessing the factor $z - w$, also differs from the function $f_{3\mu}$ arising from a general f_μ, in having no term in $(xyw)^\mu$, and being a sum of multiples of yw, z^2, zw. Hence l is $(\mu + 1)$-fold on $(z - w) f_{3\mu-1}$, and is still μ-fold on $f_{3\mu-1}$, which also contains the fixed line $y = z = 0$. This is not an F-line, and together with l forms a particular case of the P-conic κ_2 which $\sim A_1'$ regarded as a point of the double F-line l'.

Now A_1' is also a point of l''', and, as we shall see, $l''' \sim$ the plane w, locus of P-lines κ_1; in the present case, κ_1 coincides with l. If A_1' is regarded as a point of ω_3', which $\sim t$, similarly $A_1' \sim$ a generator of t which again coincides with l.

If a point $P' \rightarrow A_1'$, moving along a general line, we can take the coordinates of P' to be $(1, \eta\epsilon, \zeta\epsilon, \theta\epsilon)$, and let $\epsilon \rightarrow 0$. To a first approximation, the homologue P is

$$(\zeta\theta(\zeta - \theta)\epsilon^3, \eta\zeta u_2'(\eta, \zeta, \theta)\epsilon^4, \zeta\theta u_2'\epsilon^4, \theta^2 u_2'\epsilon^4),$$

and $P \rightarrow A_1 (1, 0, 0, 0)$ along a definite line, unless $\zeta\theta(\zeta - \theta)$ is small, of the same order as ϵ. We find the same limit for P if $P' \rightarrow A_1'$ along any path touching the line.

The case of exception is when either ζ, θ or $\zeta - \theta$ is a finite multiple of ϵ, and the path is replaced by one touching one of the triplanes. If $\zeta \equiv \zeta_1\epsilon$, then as $\epsilon \rightarrow 0$, the point $P'(1, \eta\epsilon, \zeta_1\epsilon^2, \theta\epsilon) \rightarrow A_1'$ along a conic touching z'. The coordinates of P are approximately $(-\zeta_1\theta^2\epsilon^4, b\eta\zeta_1\theta^2\epsilon^5, b\zeta_1\theta^3\epsilon^5, b\theta^4\epsilon^4)$ and $P \rightarrow Q(-\zeta_1, 0, 0, b\theta^2)$, a definite point of the P-line $y = z = 0$. Also $P \rightarrow$ the same point Q if $P' \rightarrow A_1'$ along any curve osculating the conic.

If, instead, we put $\theta = \theta_1\epsilon$, then $P'(1, \eta\epsilon, \zeta\epsilon, \theta_1\epsilon^2) \rightarrow A_1'$ along a conic touching w'; in the same way we find $P \rightarrow Q(\theta_1, \eta(\eta + \zeta), 0, 0)$, a definite general point of l, $z = w = 0$, which is the F-line of second species, and therefore also to be considered as a P-line.

Finally, if $\zeta - \theta$ is small, we put $\theta \equiv \zeta - \theta_2\epsilon$; then $P'(1, \eta\epsilon, \zeta\epsilon, \zeta\epsilon - \theta_2\epsilon^2)$ $\rightarrow A_1'$ along a conic touching $z' - w'$, which is an approximation to any curve touching $z' - w'$ and in particular to a general c'. We find that Q is $(\zeta(\theta_2 - d\eta\zeta - e\zeta^2), \eta u_2'(\eta, \zeta, \zeta), \zeta u_2', \zeta u_2')$, which is a general point of the P-plane j_1, $z - w$.

Thus the neighbourhood of A_1' corresponds to the aggregate neighbourhoods of four distinct elements of different natures:

(1) the single point A_1, corresponding to all the neighbourhood of A_1' except the parts near the three triplanes,

(2) the ordinary P-line $y = z = 0$, corresponding to the part near z',

(3) the P-line of second species l, corresponding to the part near w',

(4) the P-surface j_1, corresponding to the part near $z' - w'$,

only the last containing points of a general line in S or of a general c' in S'.

Not only is A_1' a triple F-point, it is also an F-point of contact, with three separate fixed tangent planes. Suppose next that f' passes through A_1' and also touches z':

$$f_\mu' \equiv x'^{\mu-1}z' + (y', z', w' \overset{2}{)(} x', y', z', w')^{\mu-2}.$$

The result of substitution is

$$(z - w)\{\phi_1^{\mu-1}z^2 + (z - w)(yw, z^2, zw \overset{2}{)(} \phi_1, \phi_2, \phi_3, \phi_4)^{\mu-2}\};$$

the factor dropped is still $z - w$, which ~ the passage through A_1', and no P-surface ~ the contact condition. The same is true of the second fixed tangent plane w'. But if f' touches the third plane $z' - w'$, we obtain

$$(z - w)^2\{\phi_1^{\mu-1}z + (yw, z^2, zw \overset{2}{)(} \phi_1, \phi_2, \phi_3, \phi_4)^{\mu-2}\}$$

and in this case an additional factor $z - w$ is dropped. This tangent plane is on a different footing from the other two, due to the fact that c' touches $z' - w'$ at A_1', but has no branch touching z' or w'.

More generally, let f' have an i-fold point at A_1', the tangent cone being $u_i'(y', z', w')$. We drop $(z-w)^i$ and the homologue is

$$f_{3\mu-i} \equiv \phi_1^{\mu-i} u_i'(yw, z^2, zw) + (z-w)(yw, z^2, zw \overset{i+1}{\between} \phi_1, \phi_2, \phi_3, \phi_4)^{\mu-i-1};$$

answering to u_i' we have the section of f by the plane $z-w$, residual to the F-curves, l counted μ times and ω_2 counted $\mu-i$ times; this consists of i variable lines through A_1, in the P-plane $z-w$, given by $u_i'(y, z, z)$. We have l a μ-fold line on f; all the sheets touch w.

If the tangent cone to f' has z' or w' as a sheet, one of the set of i lines coincides with l, which becomes $(\mu+1)$-fold on f. But if $z'-w'$ is a sheet, then $u_i'(y, z, z)$ vanishes identically, and the nature of the intersection of f with $z-w$ is changed, and no longer depends only on the coefficients of u_i'. It consists of l counted μ times, ω_2 counted $\mu-i-1$ times and a proper residual of degree $i+1$, having A_1 as an i-fold point.

8. Correspondence between l' and z.

Let f_μ' contain the point $O'(\xi', \eta', 0, 0)$ of l'. Then

$$f_\mu' \equiv (\eta' x' - \xi' y')(x', y')^{\mu-1} + (z', w' \between x', y', z', w')^{\mu-1},$$

$$f_{3\mu} \equiv (\eta'\phi_1 - \xi'\phi_2)(\phi_1, \phi_2)^{\mu-1} + z(z-w)(z, w \between \phi_1, \phi_2, \phi_3, \phi_4)^{\mu-1},$$

and contains the P-conic which $\sim O'$,

$$z = \eta' x(y+bw) + \xi' yw = 0.$$

Next, let f' contain the whole of l', and therefore pass through A_1'; every term of f' contains z' or w', the factor dropped is $z(z-w)$, of which, as we have seen, $z-w \sim A_1'$, and therefore $z \sim$ the rest of l'. The homologue is

$$f_{3\mu-2} \equiv (z, w \between \phi_1, \phi_2, \phi_3, \phi_4)^{\mu-1},$$

on which l is μ-fold; $\mu-1$ of the sheets touch w, and the last sheet touches, at all points of l, a fixed plane depending on the coefficients of f'.

If f' also satisfies the contact condition relative to l', and touches z', every term of f' has either z' or w'^2 as a factor. We drop $z^2(z-w)$, so that not only does $z \sim l'$, but also $z \sim$ the contact condition along l'. If f' has l' as an i-fold line, we drop $z^i(z-w)^i$, and if one sheet touches z', we drop an additional z. If two tangent planes coincide with z', so that l' is cuspidal, no additional factor is dropped unless the terms $w'^{i+1}(x', y')^{\mu-i-1}$ are absent from f', that is, when z' is a factor of the terms of f' of degree $i+1$ in z', w' and the pair of sheets touching z' are tacnodal.

9. Correspondence between l'' and l.

If f' contains l'', it is the sum of multiples of w', $y'+z'$. The only factor dropped is $z-w$, which $\sim A_1'$, and there is no P-surface corresponding to the rest of l'', which is of second species. The homologue of f' is

$$f_{3\mu-1} \equiv (zw, yw+z^2 \between \phi_1, \phi_2, \phi_3, \phi_4)^{\mu-1}.$$

Since l'' is simple on ϕ' and meets any plane once, its homologue meets any plane once and is simple on ϕ, and is itself a simple F-line of second species in S; this can only be l. Yet l is not of higher multiplicity on $f_{3\mu-1}$ than on the homologue of a general f_μ' passing through A_1' but not through l''.

So far, we have not found anything to correspond to a particular point of l'', for l corresponds to them all equally. We therefore assume that f' does not pass through l'', but contains the point O' $(1, \eta', -\eta', 0)$ of l'', and we look for a property of f which effectively depends on η'. We expect this property to relate to l; and since, for all values of η', we know that f has μ sheets through l all touching w and osculating q, we expect it to relate to contact of next higher order. We shall show that there exists a cubic surface f_3, determined by O', having four-point contact with one sheet only of f along l. Hence there are μ such cubics, associated with the μ intersections of f', l''. With due caution, we may say that these μ F-points $\sim$ μ P-lines on f all adjacent to l, and that each sheet of f contains

the line l itself,
an adjacent line l_1 in the fixed plane w,
another adjacent generator l_2 of the fixed quadric q,
another adjacent line l_3 lying on the variable cubic f_3;

the first three being F-lines, the same for each sheet of f and for every O', and the last a P-line, different for the different sheets of f and for different points O'. Since f_3 is found to have two d.p.'s A_1, A_2 on l, four of its 27 lines do coincide in l, which makes this language the more justifiable. As O' moves on l'', we may think of l_3 as moving, but always adjacent to l, so that its locus is a P-surface of degree 0, reducing to l only.

This is elaborate, but we cannot, as in simpler cases, associate the different points O' with different directions of approach to a general point O of l. A general line through O $\sim$ a curve meeting l'' in A_1' only, and not passing through any general point of l''. It is only when the curve of approach to O touches w and osculates q, that the corresponding curve can pass through O'. Thus the neighbourhood of A_1' $\sim$ the neighbourhood of every general point O of l except the set of points whose distances from w and q are small of second and third orders respectively; and this set of points $\sim$ the neighbourhood of every general point O' of l''.

We shall also show that if one of the μ intersections of f', l'' comes to A_1', one sheet of f ceases to osculate q, and has a different osculating quadric f_2, determined by the tangent plane to f' at A_1'. Now f drops the factor $z-w$ corresponding to the passage of f' through A_1', and it is the dropped plane $z-w$ and the exceptional sheet of f that together replace one sheet osculating q. It is still true that f has μ sheets through l, each having four-point contact along l with a separate surface f_3 on which l is simple.

If a second intersection of f', l'' also comes to A_1', no further factor is dropped, for A_1' is not double on f'; the tangent plane to f' at A_1' now contains l''; and the last osculating quadric f_2 coincides with q. The μ sheets of f all touch w and osculate q; they fall into two groups: $\mu - 2$ having four-point contact with separate surfaces f_3 as before, and a pair which are not so separable, either all along l or at any point of it, but form a cycle. The simplest surface having more than three-point contact with either of these sheets is a quintic with l as a double line, which has contact of the same order with both sheets of the pair.

If more intersections of f', l'' come to A_1', more sheets detach themselves from the first group, which have separate surfaces of four-point contact, and are merged in the second group which form one cycle. When all the μ intersections are at A_1', all the sheets touch w and osculate q; but the next approximation to any one has μ sheets through l, and there is no surface having one sheet through l with four-point contact with one sheet of f, either all along l or at any general point of it.

When we go one step further and make f' contain the whole of l'', we shall find that the only change is in the very last clause of this description. There now exist separate surfaces having four-point contact with f at an isolated general point of l, but not all along it.

We have now to prove all this algebraically.

10. Analysis of four-point contact along l.

If f' contains O' $(1, \eta', -\eta', 0)$, it is a sum of multiples of

$$y' - \eta' x', \quad y' + z', \quad w',$$

and f is a sum of multiples of

$$\phi_2 - \eta'\phi_1, \quad \phi_2 + \phi_3, \quad \phi_4.$$

Now f has μ sheets through l touching w and osculating q; the proper assumptions are:

z	small of order 1,
z^2, w	small of order 2,
z^3, zw, q, ϕ_1, ϕ_2, ϕ_3	small of order 3,
z^4, z^2w, w^2, zq, $\phi_2 + \phi_3$, ϕ_4	small of order 4.

If we neglect terms of order $\geqq 3\mu + 3$, we can replace f by a multiple of

$$f_3 \equiv \phi_2 - \eta'\phi_1,$$

which is the cubic of four-point contact with one sheet of f, associated with the point O'.

If f' meets l'' in the μ points O_α' $(1, \eta_\alpha', -\eta_\alpha', 0)$, $\alpha = 1, 2, \ldots, \mu$, then

$$f' \equiv \prod_{\alpha=1}^{\mu}{}' \,(y' - \eta_\alpha' x') + (y' + z', w')\!\!(\, x', y', z', w')^{\mu-1},$$

$$f \equiv \Pi\,(\phi_2 - \eta_\alpha'\phi_1) + (z - w)\,(q, zw)\!\!(\, \phi_1, \phi_2, \phi_3, \phi_4)^{\mu-1}.$$

The terms of lowest degree in z, w are $(xyw)^\mu\,\Pi\,(-\eta_\alpha')$, giving μ sheets touching w. To distinguish between these sheets, which all osculate q, we make the same assumptions as above. The first group of terms in f is small of order 3μ, and the second is of order $3\mu+1$ and is negligible. The third approximation to f is $\Pi(\phi_2-\eta_\alpha'\phi_1)$, falling into μ separate cubic surfaces such as f_3, associated with the μ points O'.

Now let $O_\mu' \rightarrow A_1'$; then $\eta_\mu' \rightarrow 0$, and the last cubic $\rightarrow \phi_2$, which has l as a double line. Now Π' has y' as a factor, and f' after substitution breaks up into $(z-w)f_{3\mu-1}$, where

$$f_{3\mu-1} \equiv yw \prod_{\alpha=1}^{\mu-1} (\phi_2-\eta_\alpha'\phi_1) + (q,\, zw)(\phi_1,\, \phi_2,\, \phi_3,\, \phi_4)^{\mu-1}.$$

The terms of lowest degree in z, w are multiples of $yw(xyw)^{\mu-1}$ arising from both groups, and still give μ sheets touching w. To find the osculating quadrics, we first assume only that w is small of second order; then both groups are of the same order 2μ, and the lowest terms are

$$yw(xq)^{\mu-1}\Pi(-\eta_\alpha'), \quad q(xq)^{\mu-1}$$

arising from the terms in $x'^{\mu-1}y'$, $x'^{\mu-1}(y'+z')$ in f'.

Thus $\mu-1$ sheets osculate q and have four-point contact with cubics f_3 as before. The last sheet osculates a quadric cone vertex A_1, of the form

$$f_2 \equiv gyw + hz^2,$$

determined by the intersection $gy'+hz'$, say, of w' with the tangent plane to f' at A_1', and different from q if this line is different from l''. For four-point contact with this sheet, f_2 is small of third order and q of second; we retain the terms of order $2\mu+1$, which are

$$(xq)^{\mu-1}f_2, \quad yw(xq)^{\mu-2}\{yzw,\, (ax+dy)zw\}, \quad zw(xq)^{\mu-1}, \quad q(xq)^{\mu-2}(yzw,\, z^3);$$

since yw is a sum of multiples of q, f_2, we can drop the factor $x^{\mu-2}q^{\mu-1}$, and obtain a cubic surface.

More generally, let $\beta \leqq \mu$ of the points O' lie at A_1'. Then Π' is replaced by $y'^{\beta}\prod_{\alpha=1}^{\mu-\beta}{}'(y'-\eta_\alpha'x')$, and

$$f_{3\mu-1} \equiv (yw)^{\beta}(z-w)^{\beta-1}\prod_{\alpha=1}^{\mu-\beta}(\phi_2-\eta_\alpha'\phi_1) + (q,\, zw)(\phi_1,\, \phi_2,\, \phi_3,\, \phi_4)^{\mu-1}.$$

There are still μ sheets touching w. If we first assume only that w is small of second order, the two groups are of orders $2\mu+\beta-1$ and 2μ; for $\beta>1$, the first is negligible, we have q repeated μ times, showing that all the sheets osculate q. With q of third order, the groups become of orders $3\mu-1$, 3μ and the second is negligible. This leads to the $\mu-\beta$ cubics f_3, but fails to give an approximation to the β other sheets. For these, we must assume the two groups small of the same order, that is $(yw)^{\beta}z^{\beta-1}(xq)^{\mu-\beta}$ of the same order as $q(xq)^{\mu-1}$, and q of order $(3\beta-1)/\beta$. The simplest surface of four-point contact is of degree $3\beta-1$ having β sheets through l. The expression for q in terms of z is irrational in z, w as well as in x, y; there is no surface having one sheet through l with four-point contact with one of these sheets of f, even at a single general point of l.

If all the μ intersections of f', l'' lie at A_1', all the sheets pass into the second group, and the simplest surface of four-point contact has l as a μ-fold line, and along it does not behave more simply than f itself.

Finally, let f' contain the whole of l''; then Π' vanishes identically, and

$$f_{3\mu-1} \equiv (q, zw \between \phi_1, \phi_2, \phi_3, \phi_4)^{\mu-1}$$

giving w, q as the first and second approximations to all the μ sheets, and q small of third order. For the next approximations, we retain terms of order 3μ:

$$(q, zw \between xq + (ax+dy)\,zw, yzw, -yzw, 0)^{\mu-1} \equiv (q, zw \overset{\mu}{\between} x, y)^{\mu-1}.$$

This falls into μ factors rational and linear in q, zw but irrational in x, y. Hence at any general point of l, with a given value of x/y, there exists a quadric having four-point contact with one sheet of f, at that point only, and not all along l. This constitutes the difference in behaviour along l, of the homologue $f_{3\mu-1}$ of an f_μ' containing l'', compared with that of an f_μ' meeting l'' at A_1' only.

11. Correspondence between l''', w and between ω_3', t.

If f' contains l''', every term has y' or w' as a factor; the factor dropped from f is $w(z-w)$, where, as before, $z-w \sim A_1'$, and $w \sim$ the rest of l'''. If f' contains the particular point $(\xi', 0, \zeta', 0)$ of l''', then f contains the P-line κ_1,

$$w = \zeta' x - \xi' z = 0.$$

In order to find the P-surface corresponding to ω_3', we make f' contain ω_3' and also l', forming the total intersection of two quadrics. Then f' is the sum of multiples of u_2' and $x'(z'-w') - dy'z' - ew'^2$, and after substitution, of $z^2(z-w)^2t$ and $xz(z-w)^2t$. The dropped factor is therefore $z(z-w)^2t$, of which z corresponds to l' and $z-w$ to A_1'; the tangent lines to l' and ω_3' constrain f' to touch the third tangent plane at A_1', which accounts for $z-w$ being dropped twice. Hence $t \sim \omega_3'$. If we had associated a different line with ω_3' to make it a complete intersection, f' would have had a different tangent plane at A_1', and $z-w$ would only have been dropped once.

12. Multiplicities of the sheets of G and J.

We can now account for the multiplicities of the various sheets of

$$G \equiv z^4 w (z-w)^4 t$$

dropped from the general homaloid ϕ' after substitution. We have

$(z-w)^3$ because A_1' is triple on ϕ',

$(z-w)$ again because one sheet touches $z'-w'$ at A_1',

z^2 because l' is double,

z^2 again because two distinct sheets touch z' along l',

w, t because ϕ' passes simply through l''', ω_3', and nothing because ϕ' passes through l'' and touches the other two fixed planes at A_1', which are elements of second species.

The multiplicity σ of any P-surface j in G is the number of intersections of each branch of c' with χ', absorbed by the corresponding F-element; the degree of j is the number of branches incident with the F-element. On the other hand, the multiplicity τ of j in J is the number of conditions presented to the branch of c' by the F-element. We have the following table:

F-element	No. of branches of c'	Contact	σ	τ	j
A_1'	1	Simple	4	3	$z-w$
l'	1	Simple	4	2	z
l''	0	—	—	—	—
l'''	1	None	1	1	w
ω_3'	2	None	1	1	t

The columns headed σ, τ give the index of the factor j in the last column in G, J respectively, and we read off

$$G = (z-w)^4 z^4 wt, \quad J = (z-w)^3 z^2 wt,$$

save for a numerical factor.

13. Second P-system.

The second Jacobian is

$$J' \equiv \begin{vmatrix} z'w'(z'-w'), & \dfrac{\partial \phi_1'}{\partial y'}, & \dfrac{\partial \phi_1'}{\partial z'}, & \dfrac{\partial \phi_1'}{\partial w'} \\ 0, & z'u_2' + y'z'\dfrac{\partial u_2'}{\partial y'}, & y'u_2' + y'z'\dfrac{\partial u_2'}{\partial z'}, & y'z'\dfrac{\partial u_2'}{\partial w'} \\ 0, & z'w'\dfrac{\partial u_2'}{\partial y'}, & w'u_2' + z'w'\dfrac{\partial u_2'}{\partial z'}, & z'u_2' + z'w'\dfrac{\partial u_2'}{\partial w'} \\ 0, & w'^2\dfrac{\partial u_2'}{\partial y'}, & w'^2\dfrac{\partial u_2'}{\partial z'}, & 2w'u_2' + w'^2\dfrac{\partial u_2'}{\partial w'} \end{vmatrix}$$

$$\equiv 4z'^2 w'^3 (z'-w') u_2'^3.$$

We also calculate

$$G' \equiv (c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4)/(c_1x' + c_2y' + c_3z' + c_4w') \equiv z'w'^3(z'-w')u_2'^3.$$

Each of these gives the four P-surfaces.

The P-plane z' meets ϕ' in l' counted four times and no variable intersection, that is, in F-lines and no P-curve. Hence $z' \sim$ an F-point, not an F-curve. Since z' is a factor of ϕ_1', ϕ_2', ϕ_3', it is the factor dropped from f' when f passes through A_4, which is the F-point which $\sim z'$.

The P-plane w' meets ϕ' in l' counted twice, l'', l''', and no variable intersection. It therefore also $\sim$ an F-point, which is A_2, since w' is a factor of ϕ_1', ϕ_3', ϕ_4'.

The P-plane $z'-w'$ meets ϕ' in l' counted twice and the variable residual

$$-c_1z'(dy'+ez')+\{c_2y'+(c_3+c_4)z'\}\{y'+(1+a+b)z'\},$$

which is a pair of P-lines through A_1'. This $\sim$ the simple F-conic ω_2'.

The quadric cone u_2' meets ϕ' in l' counted four times, l'' and ω_3', and no variable intersection. This $\sim$ the double F-point A_1.

We consider some of these correspondences in more detail. If a general surface f_μ passes through the double F-point of contact A_2, then w' is dropped from the homologue f'. If f either touches the fixed tangent plane w, or has a conic node, w'^2 is dropped; if f has a binode, one biplane being w, then w'^3 is dropped.

If f passes through A_1, each of its terms has y, z or w as a factor; after substitution, u_2' is dropped. If A_1 is i-fold on f, with $u_i(y, z, w)$ as tangent cone, then after dropping $u_2'^i$

$$f'_{4\mu-2i}\equiv\phi_1'^{\mu-i}u_i(y'z', z'w', w'^2)+u_2'.(x', y', z', w')^{4\mu-2i-2}.$$

The tangent cone u_i at A_1 answers to the variable intersection of f', u_2', consisting of $2i$ lines through A_1'. If u_i breaks up and has t as a sheet, $u_i\equiv tu_{i-2}$, then another u_2' is dropped, and the homologue is

$$f'_{4\mu-2i-2}\equiv\phi_1'^{\mu-i}w'^2u_{i-2}(y'z', z'w', w'^2)+\phi_1'^{\mu-i-1}.(y'z', z'w', w'^2)^{i+1}$$
$$+u_2'.(x', y', z', w')^{4\mu-2i-4}.$$

The variable intersection of f', u_2' is no longer a set of lines, and is no longer determined solely by the coefficients of u_i; it is a curve of degree $2i+2$ with $2i$ branches through A_1.

14. Correspondence between l and l''.

If f contains l, it is the sum of multiples of z, w, and after substitution, $w'u_2'$ is dropped; these factors $\sim$ the single points A_2, A_1 of l, and there is no P-surface which $\sim$ the rest of l.

We also have contact conditions along l. If f touches the fixed tangent plane w along l, it is the sum of multiples of z^2, w; and another w' is dropped; but this is a consequence of f touching w at the single point A_2. Similarly, if f osculates q, it is the sum of multiples of q, zw, w^2, and another u_2' is dropped; but this is due to f now having a d.p. at the vertex A_1 of the cone q. We have not yet discovered the difference of behaviour of f' according as f satisfies the conditions of incidence and contact at the points A_1, A_2 only, or all along l. We find the change to consist in f' having one more sheet through l''.

First let f_μ be general, meeting l in the μ general points $O_\alpha(\xi_\alpha, \eta_\alpha, 0, 0)$, $\alpha=1, 2, \ldots, \mu$. We shall prove the following results.

The homologue $f'_{4\mu}$ of f has μ sheets through l''; any one sheet touches a quadric f_2', associated with one of the points O. We may think of the sheet as containing the F-line l'', and as a P-line an adjacent generator of f_2', which moves, as O moves on l, so as always to be adjacent to l'', its locus reducing to l'' itself. We can associate each point O of l with a definite direction of approach to any general point O' of l'', namely along the intersection of the tangent plane to f_2' and the normal plane to l''.

If f touches l without touching w, two points O and two quadrics f_2 coincide, and two sheets of f coalesce into a cuspidal pair. But if f touches w at O, the pair become tacnodal, and at any one point of l'', they osculate separate surfaces on which l'' is simple.

Similarly, if f osculates l but not q, three sheets coalesce in a cycle; but if f osculates q, they become separable, osculating separate surfaces at any one point of l.

To prove this, we take

$$f_\mu \equiv \prod_{a=1}^{\mu} (\eta_a x - \xi_a y) + (z, w)(x, y, z, w)^{\mu-1},$$

$$f'_{4\mu} \equiv z'^{\mu}\,\Pi'\,(z'f_2' + w'^2 f_1') + w'u_2' \,.\, (z', w')(\phi_1', \phi_2', \phi_3', \phi_4')^{\mu-1},$$

where

$$f_2' \equiv \eta_a w'(x' - dy') - \xi_a y'(y' + z' + aw'),$$

$$f_1' \equiv -\eta_a(x' + ew') - \xi_a by'.$$

At points near l'', in the first instance we take

x', y', z', f_1' finite,

$w', y' + z', u_2', \phi_1', \phi_2', f_2'$ small of first order,

ϕ_3' of second, and ϕ_4' of third order.

Then the two groups in f' are small of orders $\mu, \mu + 1$; the second approximation is given by $\Pi'(f_2')$, and falls into μ quadrics, one determined by each of the points O'.

If f touches l but not w, then Π has a repeated factor, and two quadrics f_2' coincide. For the next approximation to this pair of sheets we have to assume $f_2'^2 \propto w'^3$, indicating a cuspidal sheet. But if f touches w, its second group is the sum of multiples of w, $z(\eta x - \xi y)$, z^2, and the second group in f' is the sum of multiples of

$$w'^2 u_2', \quad z'^2 w' u_2'(z'f_2' + w'^2 f_1'), \quad z'^2 w'^2 u_2'^2,$$

and becomes small of order $\mu + 2$, the same as Π'. The next approximation is of the form $(f_2', w'^2)^2$, indicating a tacnodal pair of sheets.

Similarly, if f osculates l but not q, then Π has three equal factors, and f' three sheets in contact forming one cycle. But if f osculates q, its second group is the sum of multiples of

$$q, zw, w^2, w(\eta x - \xi y), \quad z(\eta x - \xi y)^2,$$

and the second group of f' is small of order $\mu + 3$. We can assume $f_2' \propto w'^2$, and find three separate osculating surfaces at any one point of l''.

If O_μ is A_1, then $\xi_\mu = 1$, $\eta_\mu = 0$, and the last factor of Π' is $-y'u_2'$; we drop u_2', and l'' is $(\mu-1)$-fold on the homologue

$$f'_{4\mu-2} \equiv -y'z'^{\mu} \prod_{\alpha=1}^{\mu-1}{}' (z'f_2' + w'^2 f_1') + w'(z', w' \between \phi_1', \phi_2', \phi_3', \phi_4')^{\mu-1}.$$

The two groups are of orders $\mu-1$, μ; the first therefore gives $\mu-1$ tangent quadrics as before.

If also $O_{\mu-1}$ is A_2, then $\xi_{\mu-1} = 0$, $\eta_{\mu-1} = 1$, one factor of Π' is ϕ_1'/z'; we further drop w' and have

$$f'_{4\mu-3} \equiv \frac{y'z'^{\mu-1}\phi_1'}{w'} \prod_{\alpha=1}^{\mu-2}{}' (z'f_2' + w'^2 f_1') + (z', w' \between \phi_1', \phi_2', \phi_3', \phi_4')^{\mu-1},$$

on which l'' is $(\mu-2)$-fold. The groups are of orders $\mu-2$, $\mu-1$ and the first still gives the $\mu-2$ tangent quadrics.

But if f contains the whole of l, then Π vanishes identically, and

$$f'_{4\mu-3} \equiv (z', w' \between \phi_1', \phi_2', \phi_3', \phi_4')^{\mu-1}.$$

On this, l'' is $(\mu-1)$-fold, and the tangent planes are given by

$$(1, 0 \between \phi_1', \phi_2', 0, 0)^{\mu-1},$$

which reduces to

$$\{w'(x' - dy'),\quad y'(y' + z' + aw')\}^{\mu-1}.$$

There are still distinct tangent quadrics, determined by the coefficients of $z(x, y)^{\mu-1}$ in f, and therefore still associated with definite points of l, namely the $\mu-1$ points where f touches w.

15. Multiplicities of the sheets of G′ and J′.

The behaviour of the variable intersection c of ϕ, ψ, at its incidences with the elements of H, can be tabulated as on p. 348.

F-element	No. of branches of c	Contact	σ'	τ'	j'
A_1	2	Simple	3	3	u_2'
A_2	1	Simple	3	3	w'
A_4	1	None	1	2	z'
l	0	—	—	—	—
ω_2	1	None	1	1	$z' - w'$

Here σ' is the number of intersections of each branch of c with χ, and gives the multiplicity of j' in G'; and τ' is the number of conditions for each branch of c, and gives the multiplicity of j' in J'. We read off

$$G' = u_2'^3 w'^3 z'(z' - w'), \quad J' = u_2'^3 w'^3 z'^2 (z' - w'),$$

save for a numerical factor.

16. Resolution of $\mathbf{T}_{3-4}$.

This T_{3-4} can be compounded, in several different ways, of a T_{2-2} and another cubic transformation. If we apply an auxiliary transformation V to S, the family $(\phi) \sim$ another homaloidal family (ϕ'') in a third space S'', which determines a third transformation W converting (ϕ'') directly into the family of planes in S'. If we choose V to be a T_{2-2}, with its F-elements coinciding with those of (ϕ) in such a way as to make (ϕ'') a cubic family, then W is cubic and $T_{3-4} \equiv W \,.\, V$, where V is quadro-quadric and W is cubic.

If V is general, the degree of ϕ'' is $3 \times 2 = 6$. There is a reduction of 1 in the degree of ϕ'' if the quadric homaloids of V have either A_4 or l as an F-element;

a reduction of 2 if they have either A_1 or A_2 or ω_2 as an F-element, or if they touch w along l;

a reduction of 3 if they either pass through ω_2 or touch w along l, and also have as F-point either A_4, A_1 or A_2, of which A_4 is simple on (ϕ) and does not lie on ω_2 or l, and both A_1 and A_2 are double on (ϕ) and lie on both ω_2 and l;

a reduction of 3 also if they pass through A_2 and there touch the fixed biplane w, the F-conic of V consisting of l and another line neither through A_2 nor in the plane w.

Thus there are seven distinct ways of making W cubic. If, for example, we use ω_2, A_4, the equations of V are

$$\begin{aligned} x'' : y'' : z'' : w'' &= x\,(z-w) : y\,(z-w) : z\,(z-w) : \quad f_2 \quad, \\ x : y : z : w &= \quad x''w'' \quad : \quad y''w'' \quad : \quad z''w'' \quad : z''w'' - f_2'', \end{aligned}$$

where $$f_2 \equiv x\,\{y + (1+a+b)\,z\} + z\,(dy + ez).$$

After substitution, $w''f_2''$ is dropped, and ϕ becomes

$$\begin{aligned} \phi'' \equiv c_1\,\{x''z''w'' - (z''w'' - f_2'')\,(bx'' + ez'' - w'')\} + c_2 y''\,(z''w'' - f_2'') \\ + c_3 z''^2 w'' + c_4 z''\,(z''w'' - f_2''). \end{aligned}$$

The F-system of W consists of two lines of contact, $x'' = z'' = 0$, $y'' = z'' = 0$, and a conic $w'' = f_2'' = 0$. The bilinear relations hold:

$$\begin{aligned} & y''\phi_1'' + (bx'' + ez'' - w'')\,\phi_2'' \\ & \qquad + \{(1+a+b)\,x'' + dy'' + ez'' - w''\}\,\phi_3'' + w''\phi_4'' = 0, \\ & z''\phi_1'' - x''\phi_3'' + (bx'' + ez'' - w'')\,\phi_4'' = 0, \\ & z''\phi_2'' - y''\phi_4'' = 0. \end{aligned}$$

W is therefore a particular case of the bilinear T_{3-3}.

CHAPTER XVI

RESOLUTION OF SINGULARITIES OF SURFACES

I. COMPOSITION OF SPACE TRANSFORMATIONS

1. The general case (209).

Let there be a transformation $V_{n_1-n_1'}$ between S, S' and another $W_{n_2'-n_2''}$ between S', S''. Let a general point $P \mathrel{\tilde{V}} P'$ and $P' \mathrel{\tilde{W}} P''$; then $P \mathrel{\tilde{T}} P''$ in a third transformation $T \equiv W.V$ between S, S''.

The first homaloidal web (ϕ) of $T \mathrel{\tilde{V}}$ the first homaloidal web (ϕ'_W) of W; the second web (ϕ'') of $T \mathrel{\tilde{W}}$ the second web (ϕ'_V) of V. If in S' the second F-system H'_V of V and the first F-system H'_W of W are not specially related, the two degrees of T are $n = n_1 n_2'$ and $n'' = n_1' n_2''$, and

$$(\phi) \equiv H_V^{n_2'} H_W, \qquad (\phi'') \equiv H''_V H''^{n_1'}_W,$$

where
$$H_W \mathrel{\tilde{V}} H'_W \text{ and } H''_V \mathrel{\tilde{W}} H'_V.$$

Thus the total number of F-elements in the first F-system H of T is the sum of those in the first F-systems of V, W; and similarly for the second F-system H''. In the same way, the first P-system of T consists of that of V, together with the homologue by V of that of W; and similarly for the second P-system.

Now n can be lowered only by a P-surface of V dropping away from (ϕ), owing to an isolated F-point or F-curve of first species of V lying on (ϕ'_W); then either H'_V, H'_W have a common element in S', or an isolated F-point of V lies on an F-curve of W of either species. In the first case, but not in the second, a P-surface of W is dropped from (ϕ''), and n'' is lowered also. Thus we have to consider the different relations that elements in S' can have at the same time to V, W. In the first instance we consider only ordinary F-elements, and take F-elements of contact later.

A point P' of S' is of one of six kinds in relation to each of V, W:

(i) general point,
(ii) isolated F-point,
(iii) point of an F-curve of first species,
(iv) point of an F-curve of second species,
(v) P-point which $\sim$ an isolated F-point,
(vi) P-point which $\sim$ a point of an F-curve.

A curve k' is also of one of six kinds:

(1) general curve,
(2) F-curve of first species,
(3) F-curve of second species,
(4) curve on a P-surface which $\sim$ an isolated F-point,
(5) curve on a P-surface which $\sim$ an F-curve,
(6) P-curve.

2. Composition of related transformations.

(i and 1). If P' is in general position for V, it $\tilde{V}$ a definite point P; and since the homologue of P in T is that of P' in W, whatever the nature of P' for W, that of P for T is the same. Similarly if P' is in general position for W; and also for a curve or surface in S' in general position for either V or W.

(ii/ii). Let $P' \equiv O'_V$ be an isolated F-point of V; then P' $\tilde{V}$ a P-surface j_V of V, whose general point Q is associated with a definite ray l' through P'.

If also $P' \equiv O'_W$, then P' $\tilde{W}$ a P-surface j''_W, and l' is associated with a definite point Q'' of j''_W. A general surface or curve in S through Q $\tilde{V}$ another in S' touching l' at P', and this $\tilde{W}$ another in S'' through Q''. Thus Q, Q'' are ordinary homologues for T, and also j_V, j''_W, which are not P-surfaces. Compared with the general case, T has lost one F-point and one P-surface in each space, namely those that would arise from O'_V, O'_W if they were distinct.

(2/2). It is much the same if V, W have in S' a common F-curve ω' of first species. A general point O' of ω' $\tilde{V}$ a P-curve κ_V, which describes a P-surface j_V; and O' $\tilde{W}$ a P-curve κ''_W, which describes a j''_W. A general point Q of κ_V is associated with a definite plane through the tangent line to ω' at O', and this with a definite point Q'' of κ''_W. Then Q, Q''; κ_V, κ''_W; j_V, j''_W are all ordinary pairs of homologues of T. The homologues of ω' in V, W drop out of the F- and P-systems of T, and both degrees of T are lowered. This case is important in the theory of resolution of singularities (p. 369).

If i'_V, i'_W are the multiplicities of ω' on (ϕ'_V), (ϕ'_W), the P-surfaces dropped are $j_V^{i'_W}$ from (ϕ), and $j''^{i'_V}_W$ from (ϕ''); the two degrees of T are lowered accordingly, and also the multiplicities for T of any F-elements that lie on j_V, j''_W. This may cause other F-elements to be lost from T. For example, if V, W are each bilinear T_{3-3}'s, with one F-sextic in each space, then in general T is of degree 9, and $(\phi) \equiv \omega_6^3\omega_{18}$. If the two F-sextics in S' coincide, ω_{18} disappears, and we also drop $j_8 \equiv \omega_6^3$ from (ϕ), which becomes a general plane, losing both F-curves; T is a collineation.

(2/3). If ω' is of first species for V and of second for W, we have j_V $\tilde{V}$ ω' $\tilde{W}$ ω''_W, where ω''_W is also of second species for W. A general point O'' of ω''_W is associated with a definite developable f'_W of a certain pencil through ω'; any surface through O'' $\tilde{W}$ a surface touching f'_W along ω'; and this $\tilde{V}$ a surface, from which j_V is dropped, containing a definite curve κ of a certain pencil lying on j_V; and κ $\tilde{T}$ O''. Thus ω''_W is an F-curve of first species of T, which $\tilde{T}$ j_V. One element is lost from each of H, H'' and the first degree of T is lowered but not the second.

(3/3). If the coincident F-curves ω' in S' are both of second species, there is no P-surface to be dropped from either (ϕ) or (ϕ''). In general there are two different pencils (f'_V), (f'_W) of developables through ω', associated by V, W with the points of the corresponding F-curves ω_V, ω''_W of second species. Now $f'_V \;\tilde{W}$ a surface f''_V having more sheets through ω'' than the developable f''_W associated by W with a point of ω'_V. This f''_V is the surface associated by T with a general point of ω_V, and ω_V, ω''_W are F-curves of second species of T. Though H, H'' have each one F-element fewer than in the general case, the increased multiplicities of ω_V, ω''_W just compensate in the total degrees of the F-curves, and the degrees of T are unaltered.

(ii/iii). Next let $P' \equiv O'_V$ be a general point of an F-curve ω'_W of first species. We have $j_V \;\tilde{V}\; P' \;\tilde{W}\; \kappa''_W$. The different points Q'' of κ''_W are associated by W with different planes p' through the tangent line to ω'_W at P'. Each ray through P' is associated by V with a point Q of j_V, and all the rays in p' with a certain curve κ of a pencil lying on j_V. Hence $\kappa \;\tilde{T}\; Q''$, $j_V \;\tilde{T}\; \kappa''_W$, and κ''_W is an F-curve of first species of H'', replacing the F-point that would in general arise from O'_V. The numbers of F- and P-elements of T are unaltered; the first degree is lowered but not the second.

(ii/iv). If ω'_W is of second species, it $\tilde{W}$ another F-curve ω''_W of second species. Then P', regarded as a point of ω'_W is associated with a fixed developable f'', with a certain number, α say, of sheets through ω''_W. Since (ϕ'_V) contains $P'^{i'}$, it $\tilde{W}\; (\phi'')$ having a set of $i'\alpha$ sheets through ω''_W touching f''. But ϕ'_V also meets ω'_W in a certain number, β say, of variable points, equal to the degree of the curve ω_W which $\tilde{V}\; \omega'_W$, and ϕ'' has β other sets of α sheets through ω''_W, touching variable developables of the same pencil as f''. Thus ω''_W is an $\alpha(\beta + i')$-fold F-curve of T of partial contact, with α fixed tangent planes at each point Q''.

Now just as in the last case, $Q'' \;\tilde{W}\; P' \;\tilde{V}\; \kappa$; and ω'' is of first species for T, and $\tilde{T}\; j_V$; but now also $Q'' \;\tilde{W}$ every other point of ω'_W, and $\tilde{T}$ every point of ω_W; and ω''_W is at the same time an F-curve of second species of T, with ω_V as corresponding F-curve. This means that wherever a homaloidal curve c'' meets ω''_W, it touches a fixed tangent plane to ϕ'', not a variable one; for a general line l in S meets j_V but not ω_W, and $l \;\tilde{V}$ a c'_V meeting ω'_W at P' but not elsewhere.

The numbers of F- and P-elements of T are unaltered, provided we count ω''_W as two adjacent F-curves. The first degree is lowered but not the second.

(ii/v). Let $P' \equiv O'_V$ lie on a P-surface j'_W that $\tilde{W}$ an isolated F-point O''_W. Then $j'_W \;\tilde{V}$ a proper j_W from which j_V drops away; and $O''_W \;\tilde{T}$ both j_W and j_V. Now P' is associated by W with a definite ray l'' through O'', and H'' has an F-point, answering to O'_V, adjacent to O''_W in the direction l''.

(ii/vi). It is much the same if $P' \equiv O'_V$ lies on a P-curve κ'_W, generator of a P-surface j'_W; then $\kappa'_W \;\tilde{W}$ a point O'' of the F-curve ω''_W of first species that $\tilde{W}\; j'_W$. Then P' is associated by W with a definite plane p'' through the tangent line to ω''_W at O'', and H'' has an isolated point, answering to O'_V, adjacent to ω''_W at O'' in the direction of p''.

In both these cases, H'' has two adjacent elements, and one P-surface of T in S breaks up.

(2, 3/4, 5). Similarly, if an F-curve ω'_V of V, whether of first or second species, lies on a P-surface j'_W that $\tilde{W}$ an F-point O''_W or an F-curve ω''_W of first species, then (ϕ'') touches a fixed cone at O''_W or a fixed surface along ω''_W, so that these are F-elements of T of partial or total contact; H'' has an implicit F-curve of the same species as ω'_V, adjacent to the explicit element O''_W or ω''_W.

(2/6). If $j'_W \sim \omega''_W$, and ω'_V not only lies on j'_W but also coincides with one of its generating P-curves κ'_W, which $\tilde{W}$ a point O'' of ω''_W, then O'' is a hypermultiple F-point of T, lying on the F-curve ω''_W of lower multiplicity than O''.

(3/6). If now ω'_V is of second species, it does not meet c'_V, and c'' does not pass through O'', whose multiplicity on (ϕ'') is a necessary consequence of the passage of (ϕ'') through its F-curves; just as the passage of (ϕ'_V) through ω'_V is a necessary consequence of its passage through the other elements of H'_V.

In these two cases, since κ'_W meets (ϕ'_W) in F-points, which lie on ω'_V, we also have one or more of the cases (iii, iv/ii, iii, iv).

(iii/iii). Let P' be a general point of each of ω'_V, ω'_W, which have a simple intersection there; and first let each be of first species. The two homologues of P' are P-curves κ_V, κ''_W. If Q is a general point of κ_V, it is associated by V with a plane p' through the tangent line to ω'_V at P', which does not contain the tangent to ω'_W; and p' $\tilde{W}$ a ϕ''_W through κ''_W, and any general surface through Q $\tilde{T}$ a surface through κ''_W touching this ϕ''_W. Thus κ_V, κ''_W are F-curves of T of second species, which break away from the F-curves ω_W, ω''_V of T of first species, that arise from ω'_W, ω'_V, reducing their degrees; and though T has gained an F-element in each space, its degrees are unaltered.

(iii/iv). If ω'_W is of second species, it is one of a pair ω'_W, ω''_W. As before, P' $\tilde{V}$ κ_V, and this is an F-curve of T of second species, whose homologue is adjacent to ω''_W. Any plane p in S meets κ_V, ω_W say i, β times respectively, and p $\tilde{V}$ a ϕ'_V with an i-fold point P', on the i-fold F-curve ω'_V, and meeting ω'_W in β variable points; this ϕ'_V $\tilde{W}$ a ϕ'' with $i+\beta$ cycles through ω''_W, of which i touch the fixed developable f'' associated by W with P'; and ω''_W does not meet c'' at all. Thus there is a second F-curve of T of second species adjacent to ω''_W on f''.

(iv/iv). If both ω'_V, ω'_W are of second species, then in each space T has two adjacent F-curves of second species.

(iii, iv, v, vi/v, vi). The remaining combinations of the possible relations of P' to V, W are all bound to occur in the general case, since in S' every P-surface of either V or W meets every F-curve and P-surface of the other.

(iii/v). For example, if P' lies on an F-curve ω'_V of first species, and on a P-surface j'_W that $\tilde{W}$ O''_W, then it answers to some part of the intersection of the P-surfaces j_V, j_W of T in S, and to a branch of ω''_V through O''_W in S''.

(4, 5/4, 5). Similarly the curves common to the various P-surfaces in S answer to incidences of F- and P-elements of T in S, S''.

(4/6). Let the P-curve κ'_W be a general curve on a P-surface j'_V that $\tilde{V}$ O_V. Then if κ'_W $\tilde{W}$ O''_W, this is a hypermultiple point of ω''_W on j''_V. Since κ'_W meets ϕ'_V only on F-elements of V, hence O''_W is an F-point of T of total contact, adjacent to the homologues of these F-elements.

The star of planes through O'' $\tilde{W}$ a net of (ϕ'_W) through κ'_W, and these $\tilde{V}$ a net of (ϕ) touching the cone of directions at O_V associated by V with the points of κ'_W; and the homologue of any given surface through O''_W touches them also. This is not a property of the whole web (ϕ).

(5/6). If j'_V $\tilde{V}$ ω_V, then κ'_W meets ϕ'_V on certain variable P-curves κ'_V; now κ_W is adjacent to ω_V on j_W; a surface through O''_W $\tilde{T}$ a surface touching this sheet of j_W along ω_V; and O''_W is a fixed base point of the pencil (κ''_V), hypermultiple on j''_V, but not an F-point of total contact.

(6/6). Similarly, if $\kappa'_V \equiv \kappa'_W$, each of O_V, O''_W is an F-point of T of total contact, hypermultiple on a P-surface.

The only other kind of coincidence to consider is that of two P-surfaces $j'_V \equiv j'_W$. First let both $\sim$ isolated F-points O_V, O''_W. A general point of j' is associated with general directions at O_V, O''_W, and these stars correspond; but (ϕ'_V) meets j' in a set of α, say, F-elements of V, of which β, say, are of first species, one meeting c'_V in $\gamma > 0$ points; and (ϕ'') has a fixed tangent cone at O''_W, which breaks up into α sheets, one of which touches γ branches of c'', while the P-surface of T that $\sim$ O''_W breaks up into β components. Similarly O_V is an F-point of T of total contact.

If j' $\tilde{W}$ an F-curve ω''_W, then ω''_W is a curve of total contact of T, and the sheets of (ϕ'') fall into groups, answering to the separate F-elements of V on j'; but now ϕ'_W meets j' in a certain number of variable P-curves as well as F-curves, and O_V is an F-point of T of partial contact, some sheets touching fixed cones and some variable members of a certain pencil of cones. The branches of c through O_V all touch the fixed cones.

If j' $\sim$ F-curves ω_V, ω''_W in each of V, W, in general the two sets of generating P-curves are different; and ω_V, ω''_W are F-curves of T of partial contact, the branches of c, c'' meeting them all touching fixed surfaces.

We have now to consider F-elements of contact in S'. These may be singularities of any complication of the homaloids; a full treatment of composition of transformations requires the general theory of singularities, to which the present discussion is introductory. We mention a few simple cases.

Let P' be an ordinary isolated F-point of V, and an F-point of simple total contact of W, with u'_W as fixed tangent cone of (ϕ'_W). Then by W, a general direction at P' is associated with a general direction at a single P-point O''; and a direction lying on u'_W, with a curve κ''_W which generates the P-surface j''_W which $\widetilde{W}$ P'. By V, these directions are associated respectively with general points of j_V, and the points of a certain curve ω_W on j_V. Thus $j_V \widetilde{T} O''$, and $\omega_W \widetilde{T} j''_W$, and these are ordinary F- and P-elements of T; contact disappears.

Let P' be an F-point of contact of both V and W, with different fixed tangent cones u'_V, u'_W. Then $\phi'_V \widetilde{W}$ a ϕ'' touching at O'' the cone associated by W with the general rays of u'_V, and meeting j''_W in the fixed curves κ''_W associated with the common rays of u'_V, u'_W. Thus O'' is an F-point of contact of T, and the κ''_W are F-curves of second species, which may be regarded as arising from common points of implicit F-curves of V, W adjacent to P'; and similarly in S.

If the fixed tangent cones coincide, an additional j''_W is dropped from ϕ''. Then O, O''; j_V, j''_W are ordinary corresponding pairs for T.

3. A transformation with an arbitrary **F**-curve.

As an example of composition, we construct a transformation T_k, which we shall use later. Its main properties are:

(i) H contains an arbitrary curve k as an ordinary F-curve of first species;

(ii) H' consists of ordinary F-curves only, of either species, with simple intersections, and has no isolated F-points or contact conditions;

(iii) k is of higher multiplicity than any other curve of H: this ensures that ϕ has a variable tangent plane at every point of k, including its intersections with other F-curves.

A T_{Mn-n} of sufficiently high degree n satisfies (i) (p. 313). Here H consists of the first vertex O, the given curve k, the set Σl of chords of k through O and another curve ω, all the curves being simple. H' is of the same nature, consisting of O', k', ω', $\Sigma l'$, where

$O' \sim j_{O'}$, a monoid vertex O, containing k, ω, Σl;

$k' \sim j_{k'}$, the cone $O(k)$, containing k, Σl^2, but not ω;

$\omega' \sim j_{\omega'}$, the cone $O(\omega)$, containing ω, but not k or Σl;

$\Sigma l' \sim \Sigma l$, both sets being of second species.

To satisfy (ii), we must get rid of O', which is an isolated F-point and also a multiple intersection of F-curves. We compound T_M with another transformation, between S', S'' say, with O' as isolated F-point, whose second F-system consists of curves only. A convenient choice is T_{3-4} (no. 6 of Table VI), for which H'' consists of a double ω_2'', and simple ω_5'' of genus 1, and H' of O' and a simple ω_5' also of genus 1; and $O' \sim$ the plane of ω_2''.

For the compound transformation $V_k \equiv T_{3-4} . T_M$,

$$(\phi_{2n+1}) \equiv O^{2n-1} (k,\ \omega,\ \Sigma l)^2 \omega_{5n}, \quad (\phi''_{3n+1}) \equiv \omega_2''^{n+1} \omega_5''^{n} k'' \omega'' \Sigma l'',$$

where $\omega_{5n} \sim \omega_5'$ in T_M, and k'', ω'', $\Sigma l'' \sim k'$, ω', $\Sigma l'$ in T_{3-4}.

Since the F-curves of T_M pass through O' in distinct directions, general in relation to T_{3-4}, their homologues meet the P-plane of T_{3-4} in distinct points not lying on ω_2'' or ω_5''; the multiple intersection at O' is resolved, and all the intersections of F-curves of V_k in S'' are simple.

Now (ii) is satisfied, but not (iii), for k is still of the same multiplicity as ω and Σl on (ϕ_{2n+1}). If we had chosen the simpler T_{2-3} instead of T_{3-4}, these curves would have been simple, and the last F-curve also would have been of the same multiplicity as k.

We therefore compound with other transformations, and arrange to drop from the first homaloidal family the P-surfaces $j_{\omega'}, j_{k'}$, on which k is of lower multiplicity than ω, Σl respectively. Let $V_{\omega''}$ be a transformation between S'', S''', of the same kind as V_k, applied to ω'' as assigned F-curve, and otherwise general; and $V_{k'''}$ another, between S''', S'''', applied to k''' which $\sim k''$ in $V_{\omega''}$. Let

$$T_k \equiv V_{k'''} \,.\, V_{\omega''} \,.\, V_k \,.$$

The successive homologues of planes in S do not acquire isolated F-points, or multiple intersections of F-curves, and (ii) is satisfied by T_k.

By $V_{k'''}$, planes in S'''' $\sim$ homaloids (ϕ''') in S''', of degree $n_2 \geqq 2$ say, through k'''^2 and other double and simple curves. By $V_{\omega''}$, $(\phi''') \sim (\phi'')$ in S'', of degree $n_1 \geqq 2n_2 \geqq 4$, through $\omega''^{2n_2} k''^2$ and other curves of multiplicities $2n_2$, n_2, 2, 1. By V_k, $(\phi'') \sim$ the homaloids of T_k in S, from which we drop $j_{\omega'}^{2n_2} j_{k'}^2$, and which therefore contain $k^{2n_1-2} \omega^{2n_1-2n_2} \Sigma l^{2n_1-4}$ and other F-curves of multiplicities n_1, $2n_2$, n_2, 2, 1. Here k has the highest multiplicity, and (i) and (iii) are satisfied.

There are isolated F-points in S of higher multiplicity, arising from the first vertices of the T_M's which enter into the V's.

II. RESOLUTION OF SINGULARITIES OF SURFACES

4. **Statement of the problem** (66, 124, 205–210, 289, 353).

In space, both surfaces and twisted curves may have singularities to resolve; for curves it is a comparatively simple matter, very like the plane problem, and we do not discuss it further (pp. 143, 313).

For surfaces, we have to apply Cremona transformations to two problems, of which the second includes the first:

(i) to resolve a given singularity of a given surface into its elementary components;

(ii) to transform a given surface into one having only elementary singularities.

The surface f may be proper or degenerate, but has no repeated component. We regard it as a locus of points, and frame all the definitions from that point of view. There is a reciprocal theory of singularities of envelopes, which we do not touch.

The method of resolution is to apply an auxiliary transformation T to the singular element: that is, to take the element as part of the F-system of T.

The definition of an elementary singularity is to a large extent arbitrary, and different conventions have been made. The choice depends partly on the type of T, and the way of applying it; as far as possible, it is convenient to accept as elementary both the singularity that is entirely removed by T, and the accidental singularities introduced by it.

5. First definitions.

A *singular* or *multiple point* P of f is a point which is multiple on the section c of f by a general plane through P, and is therefore such that any line through P meets f in more than one point there. P is *ordinary* or *extraordinary* on f according to its nature on c (p. 128): thus P is extraordinary if and only if the tangent cone to f at P has a repeated sheet.

A *singular* or *multiple curve* k of f is a curve whose general point Q is singular; and k is ordinary or extraordinary according to the nature of Q on f and on c. The tangent cone to f at Q is a set of planes through the tangent line to k, and k is extraordinary if and only if two or more of these planes coincide for every position of Q.

The singular point P is *isolated* if it does not lie on any singular curve, and *hypermultiple* if it lies on one or more of the multiple curves Σk, and is of higher multiplicity than any one branch of Σk through it.

Since we regard f as a locus of points, the idea of a singular point is prior to that of a singular curve, which is defined as ∞ singular points. Yet we shall find some reasons, partly intrinsic and partly of convenience, for regarding a curve as elementary and not a point.

6. General effect of transformation (96, 309, 353).

The singular element, to which T is applied, is spread out into part of the second P-system of T, which is of higher dimensions unless it is an F-curve of second species; we exclude this possibility, which does not seem likely to be useful, and all F-elements are assumed of first species unless otherwise stated.

An i-fold point P of f can be taken to be either an isolated F-point of T, or a point of an F-curve. This gives two distinct ways of applying T to P, which lie at the base of two distinct methods of resolution. With the first, P, regarded as a point of S, is spread out into a P-surface j', and j'^i is dropped from f', the homologue of f. As in the case of a d.p. (p. 199), we consider that T removes an ordinary i-fold point, which is one component of P, explicit on f. But P, regarded as a point on f, is spread out into the curve of intersection k' of f', j' residual to F-curves, and the other components of the singularity of P, implicit on f, are whatever singularities f' may have at general or special points of k', other than F-points of T.

With the second way of applying T, the point P, whether regarded as a point of S or of f, is spread out into a P-curve κ', which is i-fold on f'. There is no P-surface dropped from f', and no singularity removed from f by T; no implicit singularity becomes explicit. We merely exchange an i-fold point for an i-fold curve. The advantage is that the neighbourhood of P is spread out into the neighbourhood of the whole curve κ', and components of P adjacent in different directions reappear adjacent to different points of κ'.

All this is discussed more fully below. For use in either way it is sufficient to take T to be the simplest space transformation T_{2-2}; and T_{2-3} will also serve.

If the i-fold element is a curve k, there is only one way of applying T, which must have k as an F-curve. Then, regarded as an element of space, $k \sim$ a P-surface j', and j'^i is dropped from f'; and T removes an ordinary i-fold curve, which is the explicit singularity of k on f, and the other components of the singularity of k, implicit on f, are those of f' on or along its curve of free intersection with j', other than F-points.

Here T must be of general enough type to have the arbitrary F-curve k, such as T_M or T_k (pp. 313, 358).

7. First way of applying T_{2-2} to P.

We have already seen how to remove an ordinary conical d.p. from f_μ, and to resolve an isolated binode into its components, by applying a T_{2-2} in the first way. The same treatment can be applied to any i-fold point P, other than a general point of an i-fold curve.

Let T be a T_{2-2}, with P as isolated F-point, and a general F-conic ω. The P-plane j' is dropped i times from f', on which the second F-point O' is μ-fold and the second F-conic ω', lying on j', is $(\mu-i)$-fold. These are the accidental singularities introduced by T, and have nothing to do with the composition of P. Regarded as a point of f, $P \sim$ the curve k' of intersection of f', j' other than ω'.

The points of j' are associated projectively by T with the rays of the star (P), and the points of k' with the generators of u the tangent cone to f at P. If P is ordinary, then u has no repeated sheet, and k' no repeated part, and is simple in the intersection of f', j', and therefore simple on f'. If u has no multiple edge, no point of k' is multiple on k' or on f', except the accidental singularities at the intersections of k', ω'; the original ordinary singularity of f at P has been entirely removed by T.

If u has a multiple generator, then an isolated point P' of k', not lying on ω', is multiple in the intersection of f', j', and may be multiple on f'. We then regard T as removing an ordinary i-fold point from P; the singularities, other than accidental, of f' on or along k' are the implicit components of the singularity of f, in the first neighbourhood of P.

Let P be $(0, 0, 0, 1)$; since this is i-fold on f_μ,

$$f \equiv w^{\mu-i}u_i + w^{\mu-i-1}u_{i+1} + \dots + u_\mu .$$

The tangent cone to f at P is $u \equiv u_i(x, y, z)$.

Take T_{2-2} in standard form

$$x : y : z : w = x'w' : y'w' : z'w' : q' ;$$

after dropping w'^i, the homologue of f is

$$f'_{2\mu-i} \equiv q'^{\mu-i}u'_i + q'^{\mu-i-1}w'u'_{i+1} + \dots + w'^{\mu-i}u'_\mu ,$$

and k' is the curve of degree i,

$$w' = u'_i = 0.$$

If a point P' of this, not lying on ω', is i'-fold for f', and therefore for k', it does not lie on q', and is i'-fold for u'_i; the associated generator is i'-fold on u_i; hence

$$i' \leqq i,$$

and the singular point P' of f adjacent to P lies in the direction of a singular generator of the tangent cone.

The necessary conditions that P' should be i'-fold on f' are that this generator is of multiplicities $i', \; i'-1, \; i'-2 \dots 1,$
on the cones $u_i, \quad u_{i+1}, \quad u_{i+2} \dots u_{i+i'-1}$ respectively.

8. Extraordinary singular points (129, 211).

Things may be somewhat different if P is extraordinary and u has a repeated sheet. In the simplest case of a unode,

$$f \equiv w^{\mu-2}y^2 + w^{\mu-3}u_3 + \dots + u_\mu ,$$
$$f' \equiv q'^{\mu-2}y'^2 + q'^{\mu-3}w'u_3' + \dots + w'^{\mu-2}u'_\mu ,$$

and f' has three d.p.'s, given by $u_3' = 0$, lying on the repeated line $w' = y' = 0$, which constitutes k', and associated with the lines of closest contact at P. But with a tacnode,

$$f \equiv w^{\mu-2}y^2 + w^{\mu-3}yu_2 + w^{\mu-4}u_4 + \dots + u_\mu ,$$
$$f' \equiv q'^{\mu-2}y'^2 + q'^{\mu-3}w'y'u_2' + q'^{\mu-4}w'^2u_4' + \dots + w'^{\mu-2}u'_\mu ,$$

and f' has k' as a double line which $\sim P$. Thus a tacnode has an infinitesimal double line adjacent to it.

Similarly, if P is i-fold, and u has a repeated sheet,

$$u_i \equiv u_m{}^2 u_{i-2m} ,$$

then k' has a repeated component, which is a simple curve on f' unless u_m is also a factor of u_{i+1}, when it is double on f'.

If any component is i-fold as a part of k', it is a line, $w' = y' = 0$ say; then $u_i \equiv y^i$, and k' has no other part. If y is not a factor of u_{i+1}, then k' is simple on f', which has $i+1$ d.p.'s on k', given by $u'_{i+1} = 0$. More generally, if

$$u\ , \quad u_{i+1}, \dots u_{i+\alpha-1} \qquad (\alpha < i)$$

have

$$y^i, \quad y^{i-1}, \dots y^{i-\alpha+1}$$

as factors respectively, but y is not a factor of $u_{i+\alpha}$, then k' is an α-fold line on f', on which lie $i+\alpha$ points of multiplicity $\alpha+1$, given by $u'_{i+\alpha}=0$. If $\alpha=i-1$, then k' is $(i-1)$-fold on f', with $2i-1$ points i-fold on f'. But if $\alpha=i$, the whole line is i-fold.

Thus every singular element of f adjacent to the i-fold point P is of multiplicity $\leqq i$; the only i-fold elements can be

(i) a number $\leqq 2i-1$ of isolated or hypermultiple i-fold points; or

(ii) one i-fold line, which may be of any nature on f', for if it is given arbitrarily, we can construct f from f' by reversing the transformation.

Thus we may not be able to go more than one step in resolving a singular point by this method, till we can deal with a singular curve.

If P is a simple point of an i-fold curve k_1 of f, then, by T_{2-2}, $k_1 \sim$ an i-fold curve k_1' of f', meeting j' in one point P' other than F-points. Now u is a set of i planes through the tangent line to k_1, and k' is a set of i lines in j' through P', which lies on k_1' and is i-fold for f', and $\sim$ the point of k_1 adjacent to P. The circumstances are the same in S' as in S, and a T_{2-2} applied to P' produces an i-fold point P'', which $\sim$ the point of k_1' adjacent to P', and the point of k_1 adjacent to P in the second neighbourhood. The process could be repeated indefinitely, since there are ∞ i-fold points of f adjacent to P, lying on k_1, and it is useless to try to remove them one by one.

We must therefore attack a singular curve with T_M, not T_{2-2}.

9. Singular curves (96, 289).

Since the coefficients of f can satisfy only a finite number of independent conditions, all those determining the behaviour of any one of the ∞^1 general points Q of a singular curve k are satisfied by virtue of the same finite set of conditions, and each Q is of the same nature; k can contain only a finite number of special points P of higher nature. A general plane p, meeting k in Q without touching it, cuts f in a section c on which Q is singular; and its nature is independent of the particular positions of Q and p. If c has an implicit singularity Q' adjacent to Q, then the section of f, by any general plane through any general point of k, has an implicit singularity of the same nature as Q', and the locus of these is a singular curve k' of f adjacent to k along its whole length, which acquires a separate existence when it becomes explicit by transformation.

There may be any number of such implicit curves k' in the neighbourhoods of k of any rank; then every c contains one point Q' of each k', and its singularity at Q is compounded of an explicit i-fold point and $\Sigma Q'$; its equivalent set is the same for every general Q and p. The singularity of c becomes of higher nature only if p is replaced by a plane touching k at Q, or if Q is replaced by a special point of k.

We define the *nature of the singular curve* k to be that of the plane singularity of c at Q; for this is the same for every general plane section.

This does not tell us all that there is to be known about the adjacent singular curves. For example, let k be a tacnodal line; then Q is equivalent to two adjacent d.p.'s on c, and there is adjacent to k an ordinary double curve. But this need not be a line; when it is transformed into an explicit singularity, it is a curve whose genus p may have any value, depending on the developable of tangent planes to f along k, and whose degree depends on this and also on T. If f is a homaloid and k one of its F-lines of contact, then p is one of the F-genera of the transformation determined by (f) (p. 383); but p is not revealed by the nature of k on f as just defined.

The possible singularities of f at the ∞^2 points adjacent to Q fall into sets, each of ∞^1 points of the same nature lying on one k'; whereas those adjacent to an isolated singular point P may each be of any nature independently of the others. The analysis of Q is a problem in two dimensions, that of P is in three. Thus a multiple curve, though defined as ∞^1 multiple points, is a simpler entity than a single multiple point.

On the other hand, since the postulation, equivalence and other numerical characteristics of k are finite, as regards each of these separately, k is comparable with a finite number of isolated points. Let f acquire in succession a certain number α of isolated i-fold points lying on an arbitrary curve k. For a definite value of α, depending on f and on k, there is an abrupt change, and f contains the whole of k, on which the α points are hypermultiple. As α increases further, at certain other definite values, the multiplicity of k on f increases till it reaches i, when the α points cease to be hypermultiple, and are merged as general points of the i-fold curve, and no further increase of α has any effect. The changes are analogous to the degeneration of a curve made to contain an increasing number of points lying on a curve of lower degree; and this is just what happens in our case to the curve of intersection of f with any fixed surface through k.

The singularity of Q on c can be analysed, by the methods of Chapter VII, into its equivalent set of ordinary elements, and k, considered as the aggregate of its general points apart from any special points lying on it, can be regarded as equivalent to a set of ordinary multiple curves, answering to the components of Q, and which can be made explicit on the successive homologues of f by a series of T_M's applied to k and to its homologues, as follows.

10. Application of $\mathbf{T}_\mathbf{M}$ to k.

Let T be a T_M having k as an F-curve and otherwise general (p. 313). The rest of H consists of the vertex O and, as supplementary F-curves, the chords of k through O and one other curve. We take the degree of T high enough for these to be in as general position as possible with regard to k, to f and to each other.

Regarded as a curve in space, $k \sim$ a P-cone j', whose generators are projective with the points of k; regarded as a curve of f, it $\sim$ the set $\Sigma k'$ of curves of intersection of f', j' other than F-curves. We drop j'^i from f'; and the elements of H', which is of the same nature as H, are ordinary accidental singularities of f'.

Regarded as a point of space, a general point Q of $k \sim$ a P-line κ', generator of j', whose points are projective with the pencil of planes through the tangent line to k at Q; regarded as a point of f, it $\sim$ the set $\Sigma Q'$ of i intersections of f', κ' other than F-points, associated with the i tangent planes to f at Q; and the loci of $\Sigma Q'$ are the curves $\Sigma k'$.

If k is ordinary, these i planes, and the i points Q', are distinct, $\Sigma k'$ has no repeated part, and each k' is simple on f'. If k is extraordinary, some of $\Sigma Q'$ on each κ' coincide; the locus of these is a curve multiple in the intersection of f', j', which, if other conditions hold also, may be singular on f', of multiplicity $\leqq i$. If a k' is i-fold on f', then all the $\Sigma Q'$ on each κ' coincide, and there is only one curve k', meeting each generator of j' in one distinct point other than the vertex.

A general κ' does not lie on f', being dropped as part of j'^i; and f' contains κ' only if $\kappa' \sim$ a hypermultiple point on k, which is better considered as a separate singularity, with which k is incident, than as a part of k.

Let c be the section of f by a general plane p through Q, which, without loss of generality, can be taken to pass through O; the nature of k on f is that of Q on c.

Now $p \sim$ a plane p' through κ', meeting f' in a section c' through $\Sigma Q'$, which is not general, in that p' passes through the accidental singularity O' of f'; but it is general as regards the neighbourhood of any one Q', since κ' does not lie on f'; and the nature of k' on f' is that of Q' on c'.

Between p, p', our T_M sets up a plane T_J with Q as a simple F-point, in which $c \sim c'$ and $Q \sim \kappa'$. This effects the first step in the normal resolution of the plane singularity of Q on c, resolving it into an ordinary i-fold point, removed by T_J, and the set of simpler singularities $\Sigma Q'$ of c'. The same occurs at every general point of k; we therefore regard T_M as removing an ordinary i-fold curve from k, leaving as the other components the set of curves $\Sigma k'$, each of nature simpler than k.

Since the equivalent set of the plane singularity is finite, a finite series of T_M's, having as F-curves k and all its singular homologues in later spaces, resolves the singularity, for c, and therefore for f, of all the general points Q of k, and transforms f into a surface f' on which each $Q \sim$ a set of general points of simple curves; and the only singular points and curves of f' that $\sim k$ either $\sim$ special points of k, or are accidental. The singularity of k itself as a multiple curve of f disappears entirely.

11. Special points on k (64, 65, 66).

So far we have considered general points of k only; but since Q has one degree of freedom as it moves on k, there may be a finite number of special points P, simple for Σk, that satisfy just one more condition, so that the singularity of c is of higher nature at P than at Q. For example, an ordinary double line of f_μ has in general $2\mu-4$ pinch-points at which the d.p. of c is replaced by a cusp (p. 254).

A single additional condition may raise the multiplicity from i at Q to $i+1$ at P, if all the tangents to c at Q coincide; then there is only one curve k' in the first neighbourhood of k. If the condition does not raise the explicit multiplicity, it alters the composition of Q, making it a pinch-point, and falling on the points $\Sigma Q'$ of c implicit in the first neighbourhood. It may cause two of them, Q_1', Q_2', to coincide at P' say, replacing them by their sum, as defined on p. 141; for they coalesce freely, approaching each other along the P-line κ' which $\sim Q$, which is the only fixed tangent there can be at either point. Then P' is either an intersection of k_1', k_2' of the simplest type possible, or an ordinary point of k_1' where it touches κ'.

If the fresh condition does not do this, it alters the nature of one component Q', and the same analysis applies to this: its multiplicity may be raised, or its composition altered. When k is resolved by a series of T_M's, in S or in some later space S^a, at a point P^a which $\sim P$, either the multiplicity of one component Q^a is raised by 1, or two components Q_1^a, Q_2^a are replaced by their sum; and P^a is a hypermultiple point on one k^a, or an intersection of the simplest kind of k_1^a, k_2^a, or a point of contact of k_1^a, κ^a.

In the simplest cases of these two kinds, $P^a \sim$ a point P^{a-1}, in the preceding space S^{a-1}, which is either a tacnode replacing a simple cusp on c^{a-1}, or a cusp replacing two distinct linear branches; and either k^a is a curve of simple contact of f^a, j^a, or k_1^a, k_2^a are simple on f^a and not curves of contact with j^a. In these two cases, the original point P in S is called a *normal pinch-point* of f. It follows that the intermediate homologues $P' \dots P^{a-1}$ are also normal. If P is abnormal, it leads either to a hypermultiple point, on a k^a which is either multiple on f^a or a curve of higher contact with j^a; or to an intersection of k_1^a, k_2^a, one of which is either multiple or a curve of contact with j^a.

If k is ordinary, there is only one kind of single additional condition, and at P there is a cusp of first species and $i-2$ distinct linear branches of c. This is called an *elementary pinch-point.*

Some of these special points are absorbed by any points of k where c satisfies two or more additional conditions; these include intersections of k with other branches of Σk. If P also lies on k_1 of multiplicity $i_1 \leqq i$, then the multiplicity of P on f is at least i, and if it is not higher, i_1 of the tangent planes to f coincide with the plane of the two tangents to k, k_1, so that P is a kind of pinch-point.

If P is a simple intersection of just two multiple branches k, k_1, and does not satisfy any additional condition, the singularity of c at P is the sum of those at general points Q, Q_1 of k, k_1; then P is a *normal intersection.* If k is ordinary, c has an i-fold point adjacent to which is an i_1-fold point of the same nature as Q_1. If both k and k_1 are ordinary, P is an *elementary intersection.*

For example, an intersection of multiple component curves, adjacent to an abnormal pinch-point produced by a single condition, is normal. Again, a simple intersection P, of two ordinary F-curves ω_1, ω_2 of any transformation, is normal on the homaloids (ϕ) if P is not an isolated F-point; for ϕ is not subject to any condition at P except those which arise from the nature on ϕ of the general points of ω_1, ω_2; and P is also a normal intersection on any surface whose homologue is in general position with regard to the P-surfaces which $\sim \omega_1$, ω_2.

Any singular points of f that are neither general nor normal points of its singular curves are called *abnormal.* These include all isolated and hypermultiple points, and all intersections of three or more multiple branches.

The points here defined as elementary are all extraordinary. If we wish to eliminate such from f as far as possible, we must admit ordinary hypermultiple points; the simplest ordinary point multiple on Σk is a triplanar point of f, intersection of three branches of ordinary double curves whose tangents are not coplanar. But when we transform, we may introduce pinch-points on the accidental curves, and in any convention that has been proposed, pinch-points of the simplest type are accepted as elementary. As regards intersections of multiple curves, it is a matter of taste whether we prefer to accept normal intersections, which are simple d.p.'s of Σk, and extraordinary on f; or hypermultiple points, ordinary on f, which may be multiple on Σk, whose branches touch multiple edges of the tangent cone.

12. First method of resolution (95, 96, 205–210, 353).

We have now the first standard method of dealing with a singular element of f, whether point or curve, by taking it as a complete F-element of a transformation, either T_{2-2} or T_M, which is otherwise general. This is the natural extension of the plane theory, and, as there, we define adjacence, neighbourhoods, composition, the nature of a singularity, requiring that it shall be unaltered by a regular transformation, and the sameness of a singularity on two surfaces f, f_1 requiring that it shall extend to any general member of the pencil $cf + c_1f_1$.

(i) In resolving either points or curves, implicit elements of either kind may become explicit at any stage. If the given i-fold element of f is a point P, we first apply a T_{2-2} to it. Then $P \sim$ a set of curves on f', which may consist of points all of multiplicity $< i$, or contain a finite number of i-fold points, or coincide in one i-fold line. To any isolated or hypermultiple i-fold points $P' \ldots P^a \ldots$ that arise in S' or in any later space S^a, we apply a series of

T_{2-2}'s, till all the points that $\sim P$ either are of multiplicity $< i$, or lie on i-fold curves.

This ends the first stage, which would not end if we continued to apply T_{2-2}'s to points of i-fold curves; and it has been shown that it may also become ∞, even if only a finite number of points are taken on any one i-fold curve and its homologues, for the number of such curves that arise may in that case become infinite (**208**).

(ii) When all the i-fold elements that $\sim P$ are curves Σk, we take T_M's instead of T_{2-2}'s. Now in T_M, the general points of $k \sim$ a set of curves of multiplicities $< i$, or one i-fold curve; and in addition, special points of k may $\sim$ a finite number of i-fold points. As long as any i-fold curves arise, we apply a series of T_M's to them, till either all the elements that $\sim P$ are of multiplicities $< i$, or the only i-fold elements are points. This ends the second stage.

We then start afresh and apply T_{2-2}'s to the i-fold points as in stage (i), and carry out the two stages alternately as long as there arise any i-fold elements of either kind.

It has been shown that the whole process comes to an end, and that after a finite number of finite stages, there is no i-fold element that $\sim P$. We then reduce the singular components of lower multiplicity, till the original singular point P is completely resolved into a set of simple points and curves of the transformed surface, except for the accidental singularities.

If the given i-fold element of f is a curve, we start with the second stage instead of the first. To reduce the whole surface, we treat all its extraordinary points and curves in this way; then the transformed surface has only the homologues of the original ordinary singularities and the accidental singularities introduced by the transformations. These may be of any multiplicities, but are all ordinary. By this method, no further reduction is possible; by a Riemann transformation, equivalent to two projections from higher space, f can be transformed into a surface f' having no singularities but an ordinary double curve with elementary pinch-points and triple points triplanar for f'.

13. The auxiliary transformations.

The proof of finiteness depends on the behaviour of f and its polar surfaces. To deal with these, we specialize the auxiliary transformations.

Let each F-conic of the T_{2-2} between S, S' break up into two lines intersecting in A, A' say; then the polar surface f_A of A of any order with regard to $f \sim$ the polar surface $f'_{A'}$ of A' of the same order with regard to f'. If another T_{2-2} is applied to f', we choose its first F-conic to be a pair of lines also intersecting at A', and in general directions; and so on, as long as the series of T_{2-2}'s continues. Then in S^a, the surfaces f^a, $f^a{}_{A^a}$ are the homologues of f, f_A in the compound transformation between S, S^a.

For the same reason, T_M is specialized, having the vertices O, O' as F-points of total contact. Then we can arrange that the polars of O' with regard to f' are the homologues of those of O with regard to f. If another T_M is applied to S', we choose its first vertex at O', with a different fixed tangent cone in general position. Then as long as the series of T_M's continues, the polars of O^α are the homologues of those of O (p. 318).

These specialized transformations T have F-points of contact at A', O', and introduce extraordinary accidental singularities on f'. This would spoil the effect on f as a whole. But the effects on the singular element, and on the numerical characteristics employed, do not depend on the particular type of T, or on the nature or particular positions of its other F-elements, but only on the nature for T of the assigned F-element which is singular for f. In the equivalent set, the nature of a point component depends on the explicit multiplicity of some point on some homologue of f, on the number of steps necessary to reach it, and on its grouping in the successive neighbourhoods of the earlier components. A curve component has, besides these characteristics, all those inherent to a twisted curve, degree, genus and so on.

Now we require of the nature of a singular point P that it shall be unaltered by any transformation regular at P. Thus we may include the multiplicities and groupings of components, and the genera of curves, but not their degrees. For if P'_V, P'_W or k'_V, k'_W are the explicit forms of two corresponding components, reached by two different series V, W of transformations, applied to P and its homologues in the same ways, then these are pairs of ordinary homologues in the transformation $W.V^{-1}$.

In particular, if P is either a common isolated F-point, or a general point of a common F-curve of V, W, then $W.V^{-1}$ is regular at all the points, other than accidental, that $\sim P$ in V and W; and the two homologues of f have singularities of the same nature at these points. Thus, when it is proved that the process of resolution is finite when special T's are used, we know that it remains finite when they are replaced by any set of Cremona transformations having the same assigned points or curves as ordinary F-elements, and, in particular, when we use general T_{2-2}'s and T_M's, or any other types that introduce ordinary singularities only.

14. Lines of the proof (205, 206, 210).

Now in S^α, any i-fold point P^α of f^α lies on all the polar surfaces of A^α of order $\leqq i-1$, and belongs to the curve $c^\alpha{}_{A^\alpha}$ of intersection of any two of these surfaces, among which it is convenient to include f^α itself, as the polar of order 0. But though $f^\alpha{}_{A^\alpha}, f_A$ are proper homologues, $c^\alpha{}_{A^\alpha}$ consists not only of the proper homologue of c_A, but also possibly of the lines k^α that $\sim P^{\alpha-1}$, and also certain curves $k_\beta{}^\alpha$ that $\sim P$, where $k_\beta{}^\alpha$ arises first in an earlier space S^β, as a line k^3 that $\sim P^{\beta-1}$, and on k^β and its intermediate homologues lie the i-fold points $P^\beta \ldots P^{\alpha-1}$, that are resolved in turn. This $k_\beta{}^\alpha$ is of any multiplicity $< i$; for if it were i-fold, we should apply a T_M to k^β instead of a T_{2-2} to P^β.

In the series of i-fold points $P, P' \dots P^\alpha, \dots$, to which the T_{2-2}'s of any one stage are applied, a certain number may lie on the proper homologue of c_A; but if P^α does not, neither does $P^{\alpha+1}$, nor any later point of the series. These may lie on a $k_\beta{}^\alpha$, and again this ceases to occur at a certain point, and thereafter each P^α lies only on the set of lines k^α that immediately $\sim P^{\alpha-1}$. Each of these parts of the stage demands a separate argument. In the same way, the other stages are subdivided, and also the stages are grouped in different divisions, according to the behaviour of the various parts of $c^\alpha{}_{A^\alpha}$.

For the earlier divisions, we can construct a curve in S, deduced from c_A, and of multiplicity $< i$ on f, such that its proper homologues contain all the i-fold points of the division, and thence prove that their number is finite. The other divisions require a complicated treatment of the numerical characteristics.

15. Numerical characteristics.

Each of these has a clear geometrical meaning, and is more natural than the rank of P defined below, p. 373. It is the equivalence E_P of some point P on three surfaces, one of which is a plane p, and two are chosen from $\Sigma f^\alpha{}_{A^\alpha}$. This number has two properties.

(i) It is the equivalence of P on the two plane sections. If the surfaces are f and the first polar of a point of p, these are a section c of f and one of the first polar curves of c, and E_P is the reduction in class of c due to P. In general, if $p \sim$ another plane p', the plane theory of equivalence relates E_P to the sum of its values at the points of p' that $\sim P$ (p. 132).

(ii) E_P is the number of intersections, coincident at P, of p with the curve of total intersection of the two surfaces. If p is general in relation to them, E_P is the sum, for all branches of the intersection through P, of the number of times each branch k is counted in that intersection. For a general point Q of k, this sum has only one term, which is constant as Q varies on k. If for a special point P of k we have $E_P > E_Q$, then for P there is another term in the sum, and another branch of the intersection passes through P.

But though these characteristics are simple in themselves, no combination of them decreases steadily, and an elaborate manipulation is needed for the various cases of direct and oscillating convergence. Another drawback is the use of polars of higher orders, introduced to simplify some of the intersections.

We therefore pass on to the second method of resolution, based on the second way of applying T to P, which, though less natural, is far less elaborate in detail.

III. SECOND METHOD OF RESOLUTION

16. Reduction of points to curves (66, 96).

The second way of applying a transformation to the singular point P is to take P as a general point of an F-curve. Then T replaces P by a P-curve κ' lying on f', and T^{-1} replaces κ' by P; and P, κ' are of the same multiplicity and are either both ordinary or both extraordinary. We can take any singular point of f as P, but we cannot take an arbitrary singular curve k' of f' as κ', for the P-element κ' is rational, and k' may be of any genus.

We can replace an irrational curve k' by a point P, by reversing the first way of applying T, choosing it so that a P-surface contains k'; but then P is extraordinary on f, having k' as adjacent multiple curve; this introduces needless complication, and is useless as a method of resolution.

In this second way, therefore, we regard a point and a rational curve as in some sense equivalent singularities, but an irrational curve as essentially different. We cannot replace all singular curves by points, but, as we shall show in detail, we can get rid of all singular points except general and normal points of singular curves.

Again, the accidental singularities of f', that is, the F-elements of H', cannot consist of ordinary F-points only, but always include either an F-point of contact or an F-curve (p. 309). If H' has a point of contact O', every sheet of f' through O' touches the same fixed surface, and O' is extraordinary on f'. This is a complication of f and not a reduction, so we exclude it, and choose T so that H' is free from contact conditions: then it must include a curve. But it need not contain any isolated point, and can consist of ordinary curves only.

Thus, in the accidental singularities of f', as well as in those arising from the original singularities of f, we can avoid isolated points but not curves.

This point of view is artificial, and unnecessary in the easier cases: a quadric cone is not simplified by transformation into a sextic surface with two double curves. But in more delicate cases, and in the general theory, it really simplifies to reduce the two distinct types of elementary singularities, a curve and an isolated point, to one only, which we see must be the curve and not the point.

17. Second way of applying T_{2-2} to P (96).

Now let T be a T_{2-2} with a general isolated F-point O and a general F-conic ω through P. Then $P \sim$ a P-line κ' through O' meeting ω'; every point of κ' is exactly i-fold on f', except O' and the intersection with ω', which are μ-fold.

Let λ be the tangent line to ω at P. The pencil of planes p through λ is associated projectively by T_{2-2} with the points of κ', and the pencil of lines through P in p with the pencil of planes through κ', the latter projectivity varying with p (p. 187).

A general plane p' meets κ' in one point Q', and $p' \sim$ a quadric homaloid $\phi \equiv O$, ω, touching λ at P. In relation to f, this ϕ is a general quadric through P, and its tangent plane p through λ is a general plane through P. We can set up a correspondence, regular at P, between the section c of f by p and the section c_ϕ of f by ϕ. Also the two-dimensional Cremona transformation, set up by T_{2-2} between ϕ, p', is regular at P and Q', and it converts c_ϕ into the section c' of f' by p'. These combined give a correspondence between c, c' regular at P, Q'. Thus the nature of κ' on f', defined as that of Q' on c', is the same as that of P on the general plane section c of f.

Any special point P' of κ' answers to a plane through λ containing a special ray, through P and an adjacent singular point of f. Since λ is general, no such plane contains more than one special ray, and the different singular points of f, implicit in the first neighbourhood of P, $\sim$ singularities of the same nature, implicit in the first neighbourhoods of different points of κ'.

Comparing this with the first way of applying T_{2-2} to P, we have the i-fold line κ', explicit on f', in place of an ordinary i-fold point entirely removed. When P is completely resolved, we have a set of explicit ordinary multiple curves, in place of the ordinary multiple points that would be left explicit in the first method.

18. Polar surfaces and curves.

We prove that the process of resolution is finite by defining a numerical characteristic of the singularity, which is always finite, and can be reduced in value by transformation till it reaches a definite minimum value or values, only attained when the singularity is of some definite simple type or types.

The most obvious characteristic of any singularity is the multiplicity i. This is finite, and has a definite minimum $i = 1$ for a simple point of f; but, on any convention, there are elementary singularities with all values of i, and it does not fulfil the requirements. It does not measure the complication of the element, because it takes account only of the explicit singularity, and not of those implicit in the successive neighbourhoods. For this reason, we use the polar surfaces and curves.

Every multiple point of f lies on the first polar surface f_X of any point X with regard to f. Conversely, every base point of the web (f_X), of the first polar surfaces of all the points of S, is a multiple point of f.

The total intersection of f, f_X is the *total polar curve* of X; it consists of all the singular curves Σk of f, each taken a proper number of times, at least twice, and a variable residual c_X, the *pure polar* of X, which has no repeated part if X is general. We also define an intermediate form γ_X, the *simplified polar*, consisting of c_X together with each multiple curve k taken once only.

19. Base points of the pure polars (65, 66).

A base point of the pure polars (c_X) is either an isolated multiple point, or a special point of a multiple curve, hypermultiple, multiple on Σk, or a pinch-point.

It has been shown by the methods of infinitesimal geometry that:

(i) a normal intersection does not lie on (c_X), and conversely, a simple intersection of two multiple branches which does not lie on (c_X) is normal;

(ii) a normal pinch-point is a simple base point of (c_X), and conversely, a simple intersection of Σk with (c_X) is a normal pinch-point;

(iii) an isolated or hypermultiple i-fold point is at least $2i$-fold on (c_X); it cannot be a simple base point, and if it is an ordinary d.p. of (c_X) it is an isolated conical d.p. of f.

20. Numerical characteristics in the second method (66).

Let $\alpha =$ the number of distinct cycles of branches of Σk through P;

$\beta =$ the multiplicity of P on γ_X; this is the same for all poles X not lying on the tangent cone to f at P;

$\delta =$ the reduction in class, due to the singular generator XP, of the cone $X(\gamma_X)$ projecting the simplified polar from the pole;

$\rho =$ *rank* of $P \equiv 4\delta - 5\alpha - \beta(\beta - 1)$; this is an ingenious combination which proves to be reducible by transformation; it waits for geometric interpretation.

From these definitions,

$$\alpha \leqq \beta, \qquad \delta \geqq \beta(\beta - 1),$$

whence
$$\rho \geqq \beta(3\beta - 8) \geqq 3, \text{ if } \beta \geqq 3.$$

21. Effect of T_{2-2} on the characteristics.

As before, for the part of the argument that involves transformation of polars, we use a special T_{2-2}, whose F-conics break up into pairs of lines, meeting in A, A' say, one line l_1 of the first pair passing through P, and one, l_1', of the second pair meeting κ'. Then the polar relation of f, f_A is invariant under this T_{2-2}. In the neighbourhood of P, the pure polar of A' is the proper homologue of that A, but since κ' is a new multiple line of f', the simplified polar $\gamma'_{A'}$ consists of κ' in addition to $(\gamma_A)'$ the homologue of γ_A.

Let P' be any special point of κ'.

(i) *Effect on* α. Through P' there passes κ'; any other cycles of multiple curves of $f' \sim$ some or all of the α cycles of Σk through P:

$$1 \leqq \alpha' \leqq \alpha + 1;$$

and there are $\alpha - \alpha' + 1$ cycles meeting κ' in points other than P'.

(ii) *Effect on* β. Though A is a general point, A' is not general in relation to f'; but it does not lie on the tangent cone to f' at P', which consists of i planes through κ'; for if one of these contained A', it would be the P-plane $O'l_1'$, and if f' touched this, f would touch $l_1 \equiv PA$ at P, which is not so, since l_1 is a general line through P.

Hence β' is the same as the multiplicity of $\gamma'_{A'}$, consisting of $(\gamma_A)'$ and κ'; and $\beta' - 1$, $\equiv \beta_1$ say, is the multiplicity of $(\gamma_A)'$ at P'. Let β_2, β_3 ... be the multiplicities of $(\gamma_A)'$ at the adjacent points P'', P''' ... of κ'. Then $\beta_1 + \beta_2 + \beta_3 + \dots$ intersections of $(\gamma_A)'$, κ' fall at P', and at least $\alpha - \alpha' + 1$ fall at other points of κ'; all these $\sim$ some or all of the β intersections of γ_A with l_1:

$$\beta_1 + \beta_2 + \beta_3 + \dots \leqq \beta - \alpha + \alpha' - 1.$$

(iii) *Effect on* δ. The value of δ' for a general pole $X' \leqq$ its value $\delta'_{A'}$ for the special pole A', which is the reduction in class due to $A'P'$ of the cone projecting $(\gamma_A)'$ and κ' from A'.

The component singularities of $(\gamma_A)'$, considered as a curve in space, are the same as those of γ_A, except for an explicit β-fold point at P removed by T_{2-2}, which contributes $\beta(\beta - 1)$ to δ and 0 to $\delta'_{A'}$. Of the other components, some may lie at points of κ' other than P': these also contribute to δ and not to $\delta'_{A'}$.

Others may be implicit at P', but not in the direction of κ': these contribute equally to δ, $\delta'_{A'}$.

There remain the points P', P'', P''' .... The existence of P'' shows that at P' there are β_2 tangents to $(\gamma_A)'$ touching κ' and arising from cuspidal branches at P, equivalent to β_2 simple cusps removed by T_{2-2}, which contribute an additional β_2 to δ and 0 to $\delta'_{A'}$. On the other hand, owing to the new branch κ', the multiplicity of P' on $\gamma'_{A'}$ is increased from β_1 to $\beta_1 + 1$, and the ordinary component at P' contributes $\beta_1(\beta_1 - 1)$ to δ and $(\beta_1 + 1)\beta_1$ to $\delta'_{A'}$, and similarly for P'', P''' .... Altogether:

$$\delta - \delta'_{A'} \geqq \beta(\beta - 1) + (\beta_2 + \beta_3 + \dots) - 2(\beta_1 + \beta_2 + \beta_3 + \dots),$$
$$\delta - \delta' \geqq \delta - \delta'_{A'} \geqq (\beta - 1)^2 - \beta' + \alpha - \alpha' + 1.$$

(iv) *Effect on* ρ. Hence:

$$\rho - \rho' = 4(\delta - \delta') - 5(\alpha - \alpha') - \beta(\beta - 1) + \beta'(\beta' - 1)$$
$$\geqq \beta(3\beta - 8) + (\beta - \alpha) + \alpha' + (\beta' - \tfrac{5}{2})^2 + \tfrac{7}{4}.$$

But $\beta \geqq \alpha, \quad \alpha' \geqq 1, \quad (\beta' - \tfrac{5}{2})^2 \geqq \tfrac{1}{4},$

$$\rho - \rho' \geqq 6 \qquad \text{if } \beta \geqq 3,$$

and $\rho - \rho' \geqq \alpha' - \alpha$ if $\beta = 2$.

22. Double points of the simplified polar.

Consider separately the cases $\beta \leqq 2$.

If $\beta = 0$, P is simple on f;

$$\alpha = \delta = 0, \quad \rho = 0$$

If $\beta=1$, since no isolated point of f is a simple base point of (c_X), P lies on one branch of Σk and not on (c_X);

$$\alpha=1, \quad \delta=0, \quad \rho=-5.$$

If $\beta=2$, first let P be an ordinary d.p. of (γ_X); then $\delta=2$, and either:

(i) P is an ordinary d.p. of Σk, not lying on (c_X), that is, a normal intersection;

$$\alpha=2, \quad \rho=-4,$$

and $\alpha'=2$ for each of the two special points on κ'; or

(ii) P is simple on each of Σk, (c_X), that is, a normal pinch-point;

$$\alpha=1, \quad \rho=1,$$

and $\alpha'=2$ or 1; or

(iii) P is a d.p. of (c_X) not lying on Σk, that is, an isolated d.p. of f;

$$\alpha=0, \quad \rho=6,$$

and there are no special points on κ'.

Next let the tangents to γ_X at P coincide; if $\alpha=2$, then P is at least a tacnode, $\delta \geqq 4$, $\rho \geqq 4$; if $\alpha=1$ or 0, then P is at least a cusp, $\delta \geqq 3$, $\rho \geqq 5$. In each of these cases, $\alpha'=\alpha+1$ for the only special point on κ'.

Thus for the only points that exist on f in ∞ numbers, that is, simple points and general points of Σk, we have $\rho \leqq 0$, and there are only a finite number of points of positive rank. For normal points, $\rho=1$ or -4; and for all abnormal points, $\rho \geqq 3$, and either $\beta \geqq 3$, or $\alpha'-\alpha=1$. Hence the rank of an abnormal point is always reduced by a T_{2-2} applied to it in the second way.

If P is a normal intersection, T_{2-2} separates the two branches of Σk, and we have two points P', each a normal intersection of κ' with a distinct multiple curve. If P is a normal pinch-point, there is one point P', a normal intersection of κ', k'.

Thus any singular point P can be transformed into a set of multiple curves of any nature, all of whose special points are normal. To resolve P completely, we must further resolve these curves into ordinary components. The proof that the equivalent set is finite is included in the reduction of a whole surface to elementary type.

23. Auxiliary transformations in the second method.

We therefore consider the effect of the auxiliary transformations on f as a whole, and, in particular, the accidental singularities introduced at the elements of H'. Now for T_{2-2}, these include a hypermultiple point at O', lying on κ' and $\Sigma k'$, which may be of higher rank than P; and to reduce the singularity of P may not reduce that of f.

As above, p. 369, the effect on P is the same if we replace T_{2-2} by any other transformation T, between S, S'' say, having the F-line l_1 in common with T_{2-2}. It is, however, enough if T has P as a general point of any ordinary F-curve ω of first species. Then $P \tilde{T}$ a P-curve κ'' of T, and κ', κ'' are corresponding F-curves of second species in $T \,.\, T_{2-2}^{-1}$. A singularity of f, adjacent to P on a definite ray l, reappears adjacent in definite directions to definite points P', P'' of κ', κ'', and P', P'' are of the same nature for f', f'' respectively. For each is i-fold and has only one adjacent singularity; if V', V'' are two transformations applied to κ', κ'', these implicit singularities become explicit, and are ordinary homologues in the compound transformation

$$V'' \,.\, T \,.\, T_{2-2}^{-1} \,.\, V'^{-1}.$$

We choose T so that H' consists of ordinary curves only. The simplest is the bilinear T_{3-3}, when each of H, H' is a single curve. This we now adopt as auxiliary transformation; and it has the property, proved for T_{2-2}, of reducing the rank of any abnormal point to which it is applied.

We have to show that the one accidental singular curve is ordinary, with elementary pinch-points and normal intersections with the other multiple curves of f'.

Then by applying a T_{3-3} to any abnormal point, we reduce its rank; when this becomes < 3, the point is normalized. Since f has only a finite number of abnormal points, each of finite rank, it is reduced, by a finite series of T_{3-3}'s, to a *normal surface*, that is, one having singular curves, whose general points are of any nature, but all their special points are normal. This is the first stage in the reduction of f; in the second stage, the extraordinary curves are resolved by T_M's, each supplemented by a proper set of other transformations, which bring f' back to normal form; and f is finally reduced to an *elementary surface*, having only ordinary multiple curves with elementary pinch-points and intersections.

24. Accidental singularities of $\mathbf{T}_{3-3}$.

When a T_{3-3} is applied to the i-fold point P of f, the multiple curves of f' are

(i) the homologues $\Sigma k'$ of the multiple curves Σk of f;

(ii) the line κ' which $\sim P$, and is i-fold on f';

(iii) the F-curve ω' of T, which is μ-fold on f'.

(i) Since T is regular at a pair of corresponding general points of k, k' these are of the same nature for f, f'; so are their pinch-points and intersections, and any hypermultiple or isolated points, other than P, that f may have. The only exceptions arise from F- and P-points on Σk. Now ω does not meet Σk except at P; hence the only P-points on $\Sigma k'$ are their intersections with κ', and their F-points are their intersections with ω'. These are considered under (ii) and (iii).

(ii) The point $P \sim$ a P-line κ', trisecant of ω'; the general points P' of κ' are associated with the planes p through λ the tangent line to ω at P, so that any plane p' through $P' \sim$ a ϕ_3 touching p at P. Let ϕ be mapped on a plane π by a 1, 1 transformation V regular at P and its image P_π; we have a plane Cremona transformation $W \equiv V.T^{-1}$ between p', π, regular at P', P_π. The section c' of f' by p' $\tilde{W}$ a curve c_π in π, image in V of the section c_ϕ of f by ϕ. Now ϕ is a cubic surface in general relation to f near P, and its tangent plane p is a general plane through P. We can set up a 1, 1 relation, regular at P, between c_ϕ and the section c of f by p. Thus c', c_π, c have singularities of the same nature, and the nature of κ' on f' is that of P on c.

Special positions of P' answer to special positions of p and of ϕ; now ϕ is special in its relation to T at P only if it touches the P-surface of T at P and therefore contains one of the trisecants of ω through P; then P' is an intersection of κ', ω' and is μ-fold on f'. It is also an F-point of T and of W, which removes an ordinary μ-fold point from c' and leaves P_π. These are the singularities of a plane section of f' at general points of ω', κ' respectively, and the singularity of c' is their sum: P' is a normal intersection.

Other positions of ϕ are special in relation to f, when p is properly or improperly tangent to f, touching its tangent cone u. If contact is proper, it is simple, since p passes through λ, which is a general line in relation to f; but the sheet touched may be repeated in u. Two sheets of f' of the same nature coalesce at P', and these may be simple or multiple. If they are simple, P' is a normal pinch-point; if multiple, P' is abnormal, and adjacent to it is a normal intersection of two branches of singular curves of the same multiplicity. If P is a special i-fold point on an i-fold curve, u is a set of planes, and there are no such points on κ'.

If p contains a singular generator l of u, then P' may be abnormal; if l is a tangent to Σk, then one or more branches of $\Sigma k'$ pass through P'.

In any case, if P' is abnormal, it is of lower rank than P.

(iii) A general point P' of $\omega' \sim$ a P-line κ, trisecant of ω, meeting f in μ distinct points ΣQ, associated with the tangent planes to f' along ω', which are therefore distinct also, and ω' is an ordinary μ-fold line. A special $P' \sim$ a special κ, which touches f, or meets Σk, or passes through P.

Now P' is an F-point of W, which removes a μ-fold point from c', and leaves a set of points ΣP_π of c_π, images in V of ΣQ. If κ touches f, contact is simple, two points P_π coincide at a simple point of c_π. Two tangents to c' coincide, but there is no adjacent singular point; c' has a simple cuspidal branch, and P' is an elementary pinch-point on f'.

At an intersection of κ with an i_1-fold curve k of f, i_1 of the points Q coincide; then P' is an intersection of ω', k'. Now P_π is an i_1-fold point of c_π, of the same nature as Q on f; the singularity of P' on c' is the sum of an ordinary μ-fold point and the singularity of a general point of k or of k'; and P' is a normal intersection.

If $Q \equiv P$, then P' is a normal intersection of ω', κ' considered above.

Hence ω' is an ordinary multiple curve of f', with elementary pinch-points and normal intersections.

25. Total effect of $\mathbf{T}_{3-3}$.

Thus T_{3-3} replaces the i-fold point P by the i-fold line κ', whose nature on f' is that of P on a general section of f, and on which may lie a certain number of special i-fold points, which may be abnormal, but are of lower rank than P. The singularities of f other than P are reproduced on f' unchanged in nature, and the only accidental singularity is an ordinary curve whose special points are normal.

Thus when f is normalized, any new extraordinary curve is of the same multiplicity as that of the abnormal point from which it arises.

Before starting the second stage it is convenient to remove the actual multiple points of any one multiple curve k; each is a normal intersection of k with itself, and therefore an ordinary d.p. of k; a T_{3-3} replaces it by a multiple line κ' with two distinct normal intersections with k'. As no multiple curve can acquire a fresh d.p., and the F-curves have none, we can remove all such d.p.'s. Then a normal intersection is of two different multiple curves.

26. Resolution of extraordinary curves.

There remains the second stage, reducing the normal surface to elementary type by resolving all the extraordinary curves into their ordinary components, any abnormal points being normalized as they arise, by supplementary transformations.

Let i be the highest multiplicity of any extraordinary curve on f_μ, and let k be one of these i-fold curves. Let $\Sigma k_1 \equiv \Sigma k_o$, k_e be the other multiple curves of f, where a k_o is of multiplicity $i_0 > i$, and is ordinary, and a k_e, of multiplicity $i_e \leqq i$, may be ordinary or extraordinary. Any intersection of k, k_o is hypermultiple on k. We shall transform f into another normal surface on which k is replaced by a set of ordinary curves, together with a set of extraordinary curves of multiplicities $\leqq i-1$.

After dealing thus with all the extraordinary i-fold curves, we repeat the process with $i-1$ instead of i. By a finite series of operations, i is reduced to 1; then there are no extraordinary curves, and f is transformed into an elementary surface.

Apply a transformation T to k: this removes an ordinary i-fold curve, and leaves explicit on f' the adjacent curve or curves in the first neighbourhood of k. If this is i-fold, the number of implicit i-fold curves is reduced by 1; after normalizing f', we remove another, till the explicit curves which $\sim k$ are all of multiplicities $< i$.

We could use a T_M, but it is more convenient to take a T_k of the kind described above, p. 358, whose accidental singularities are simpler.

Let Q denote a general point of k, and Q_1 of k_1, etc.; then $Q \sim$ a P-curve κ', generator of the P-surface j_k' that $\sim k$; and the points Q' of κ' are projective with the planes through the tangent line λ to k at Q. The different components of the singularity of the plane section c of f, in the first neighbourhood of Q, lie in different planes through λ, and $\sim$ different points Q_a' of κ'; the locus of each is one of the components $\Sigma k_a'$ of k, made explicit on f'.

The other multiple curves of f' are the homologues $\Sigma k_1' \equiv \Sigma k_o'$, k_e' of the other multiple curves of f, the F-curves $\Sigma \omega'$ of T, and the P-curves κ_0', of multiplicity $i_o - i$, that $\sim$ the hypermultiple points of k where it meets Σk_o.

27. The auxiliary plane transformation.

Let P, P' be any pair of corresponding points of f, f'. Let p' be a general plane through P', cutting f' in c'; then $p' \sim$ a homaloid ϕ through P, meeting f in k counted a certain number of times and a residual c_ϕ.

If P does not lie on k, it lies on c_ϕ; then ϕ is general in its relation to f near P. Let p be its tangent plane, cutting f in c, which is a general plane section of f. Between c_ϕ, c we can set up a 1, 1 relation regular at P, and P is of the same nature on both curves.

Let ϕ be mapped on a plane π by a transformation V, of the same type as T_k, with all its arbitrary elements in other general positions with regard to f and to P; let P_π, c_π be the images of P, c_ϕ. Then $W \equiv V.T^{-1}$ sets up a plane Cremona transformation between p', π, in which $P' \sim P_\pi$ and $c' \sim c_\pi$.

If P' is not an F-point of T, then W is regular at P', P_π; the nature of P' on c' is that of P_π on c_π or of P on c. Now c, c' are arbitrary plane sections of f, f'; hence P, P' are either both simple points of f, f' or both general or normal points of Σk_1, $\Sigma k_1'$, of the same nature.

Let P' be an F-point of T, lying on one F-curve ω', of multiplicity i' for f' say. Then P' $\tilde{T}$ a P-curve κ meeting f in a set of points ΣP of the curve k_j residual to k in the intersection of f with the P-surface j that $\tilde{T}$ ω'. In general, these are distinct and simple for f, and their images ΣP_π are distinct and simple for c_π. Now P' is an F-point of W, and W removes from c' an ordinary i'-fold point, leaving ΣP_π on c_π. Hence ω' is ordinary on f'.

Certain points P can be special on k_j, for κ has one degree of freedom on j. If P does not lie on k, it may $\equiv Q_1$, then k_1' passes through P'; one P_π is of the same nature as Q_1 or Q_1', and P' is a normal intersection of ω', k'. Again, P may be an ordinary d.p. of k_j, at a point of simple contact of f, j; then P_π is a simple point of c_π where it touches a P-curve of W; and P' is an elementary pinch-point of f'.

If P' is an F-point of T multiple on $\Sigma\omega'$, it is a simple intersection of two curves ω_1', ω_2' of multiplicities i_1', i_2' for f' say. Then ΣP are simple points of f, and ΣP_π of c_π; and W has two adjacent F-points, and can be resolved into two components which remove ordinary i_1'-fold and i_2'-fold points from c'; thus P' is an elementary intersection.

Next let P be a point of k general in relation to f. Then $P' \equiv Q_a'$ lying on one k_a'; one sheet of ϕ through k touches one sheet of f, and c_ϕ has one branch through P other than k, containing one adjacent point of k_α'. Now P' is not an F-point of W, but P is an F-point of V, which removes an ordinary multiple point from c_ϕ, and leaves explicit one point P_π whose nature on c_π is therefore that of Q_α' on f'. This still holds if P lies on ω; for T_k is such that P still $\sim$ a κ', and $\Sigma Q_\alpha'$ are distinct. If P' is an F-point of T, it lies on one ω'; then W removes an i'-fold point from c'. But P lies on j; its nature for f is as before, and the nature of P_π on c_π is still that of Q_α'. The singularity of c' is the sum of those of Q_α', Q_ω', and P' is a normal intersection.

28. Special points of k.

Now let P be a point of k special in relation to f. If it is a pinch-point, it is normal; then either one Q_α' is replaced by a normal pinch-point on k_α'; or one Q_α' by an isolated d.p. of f' on one simple k_α', which is normalized at once by a T_{3-3} applied to it; or else two points Q_α' are replaced by a simple intersection of two simple curves k_α', which is a simple point of f'.

There remain the normal intersections P of k with Σk_1. The effect of T_k on these is the same as that of the T_M applied to k that enters into the composition of T_k; for the other components of T_k are regular at these points and their homologues.

Let p be a general plane through P, which without loss of generality we can take to pass through the vertex O of T_M; then $p \sim$ a plane p', through the P-line κ' containing O' and all the points $\Sigma P'$ that $\sim P$, which are the intersections of κ' with $\Sigma k_\alpha'$. Also T_M sets up between p, p' a plane T_J with P as a simple F-point and κ' as corresponding P-line.

First let P be an intersection of k, k_0; then it is of multiplicity $i_0 > i$ on f. As $Q \rightarrow P$ along k, all the i tangent planes to f at $Q \rightarrow$ the plane of the tangents λ, λ_o to k, k_o, and every $Q_\alpha' \rightarrow$ one point P' of κ', its intersection with k_o', which is i_o-fold on f'.

Now κ' is $(i_o - i)$-fold on f', and if this and any k_α' are multiple, P' is certainly abnormal, lying on three multiple curves. The tangent cone to f' at P' consists of i_o planes through λ_o'; one of these contains each λ_α', and at the point adjacent to P' on the trace of the plane $\lambda_o'\lambda_\alpha'$ on p', the section c' of f' by p' has at least the singularity Q_α'.

Since P is a normal intersection and k_o is ordinary, the singularity of c is the sum of an ordinary i_o-fold point and the extraordinary i-fold singularity Q. From this, T_J removes the i_o-fold point, and transforms c into the part of c' residual to κ'; on this part therefore, the singularity of P' is that of Q, namely an i-fold point adjacent to which are $\Sigma Q_\alpha'$ in different directions. Hence $\Sigma \lambda_\alpha'$ lie in different planes through λ_o', and they are also different from κ'.

It follows that a T_k applied to k_o' scatters these intersections of k_o' with κ' and $\Sigma k_\alpha'$, replacing P' by a set of distinct normal intersections on a P-curve that $\sim P'$. We must verify that this does not introduce abnormality arising from any other special point of k_0', intersection with $\Sigma k'$. In the present case, if k were ordinary and therefore $\Sigma Q_a'$ all simple, P' would be a normal intersection of κ', k_0'.

Lastly, let P be an intersection of k, k_e, where $i_e \leqq i$. Then P is i-fold on f, and κ' does not lie on f'; in the neighbourhood of each of $\Sigma P'$, the plane p' is general in its relation to f'. Now i_e of the tangent planes to f coincide with the plane $\lambda\lambda_e$; some or all of $\Sigma k_a'$ pass through the intersection P' of κ', k_e', and P' may or may not be abnormal.

The singularity of P on c is the sum of Q, Q_e; from this, T_J removes an ordinary i-fold point. It follows that if k were ordinary, the singularity of P' on c' would be that of Q_e or Q_e', and P' would be a general point of k_e', and would not be abnormal. Since this held also of the intersection of an ordinary curve with k_0, no abnormality can arise from the transformation applied just now to the ordinary curve k_0.

If k is general, we have at P the case discussed on p. 142. It follows that if k has an adjacent i-fold curve k_a', and therefore no other adjacent curve, the removal from c by T_J of an ordinary i-fold point leaves on c' the same singularity as if the i-fold point were first removed from Q, leaving Q_a'; that is, P' is the sum of Q_a', Q_e', and is a normal intersection.

We can therefore carry on the resolution of k until there is no adjacent i-fold curve left. Then we leave k and begin the resolution of the other i-fold members of Σk_e, till no curve has an adjacent i-fold curve.

In this case there are at P either two or three adjacent i-fold points on c, but no more. We apply T_k's, first to k and then to k_e'. There may still be an i-fold point P'' that $\sim P$; if so, there is only one curve adjacent to each of k, k_e, each of multiplicity $< i$; and P'' is a hypermultiple point through which pass these two curves k_a'', k_{ea}''; and there is no i-fold point adjacent to P''. Then a T_k applied to k_a'' transforms P'' into a curve of multiplicity $i - i_a' < i$, on which there is no i-fold point left.

29. Total effect of the transformations.

Thus the total effect of one T_k on f is to replace the extraordinary i-fold curve k by a set of simpler curves, on which may lie certain abnormal points, either arising from the normal intersections of k with other multiple curves, or isolated d.p.'s arising from normal pinch-points. We have to treat the whole set of extraordinary i-fold curves together, applying to each and to its homologues as many T_k's as it has i-fold components, and possibly one more; and to supplement these by T_k's applied to the ordinary curves Σk_0 of multiplicities $> i$, and by T_{3-3}'s applied to the isolated d.p.'s.

By this finite set of transformations, $f \sim$ a normal surface whose extraordinary curves are all of multiplicities $\leqq i - 1$. A finite number of such sets reduces i to 1, and f to an elementary surface.

IV. CLASSIFICATION OF TRANSFORMATIONS

30. Ordinary transformations (209).

We can go on, and reduce not only a single surface, but a homaloidal family (ϕ) and a general transformation T to standard types. With the convention that a curve is simpler than a point, we define an *ordinary transformation* as one with no isolated F-points or elements of contact, that is, such that its F-systems H, H' consist of ordinary F-curves $\Sigma\omega^i$, $\Sigma\omega'^{i'}$, of either species, free from actual d.p.'s and multiple intersections.

Then each general homaloid ϕ, ϕ' is elementary; this alone does not make T ordinary: it would exclude isolated multiple points, and extraordinary curves, but not simple points or ordinary curves of contact. The degenerate surface $f_{12} \equiv \phi_1\phi_2$ composed of two general homaloids is also elementary; its singularities are $\Sigma\omega^{2i}$, c^2_{12}, where c_{12} is the homaloidal curve common to ϕ_1, ϕ_2. All the intersections of c_{12} with $\Sigma\omega$ are variable, and are elementary on f_{12}. The elementary character of f_{12} excludes F-curves of contact from T, which would be extraordinary on f_{12}, but not simple isolated F-points, which would be points of c_{12} general on f_{12}, or points of simple contact lying on F-curves, which would be fixed elementary intersections of c_{12}, $\Sigma\omega$. The surface $f_{123} \equiv \phi_1\phi_2\phi_3$ is elementary save for the one point of free intersection of the three general homaloids, which is hypermultiple on f_{123}, being a triple point lying on the three double curves c_{23}, c_{31}, c_{12}. This condition is sufficient to make T ordinary.

Let T be any given transformation. To reduce it to ordinary type, we first remove its isolated F-points. Any one O of these is a base point of $(c_{\alpha\beta})$, abnormal on every $f_{\alpha\beta\gamma}$. Let V be a T_{3-3} applied to O; this transforms (ϕ) into the homaloidal family (ϕ') of another transformation $T' \equiv T.V^{-1}$ such that any isolated F-point O' of (ϕ') that $\tilde{V}\,O$ is of lower rank on a general $f'_{\alpha\beta\gamma}$ than O is on f_{123}.

Now V removes one explicit F-point from H, replacing it by an F-line and possibly a set of points on it; it leaves the other elements of H unaltered in nature and adds one ordinary multiple F-sextic. The effect of V on the second F-system of T is to treble the multiplicity without altering the nature of every F-element, except those that lie on the P-surface that $\tilde{T}\,O$, which are of lower multiplicity than this; and there is a new ordinary simple F-curve. Both the new F-curves have simple intersections with the others.

We can therefore continue till all the isolated F-points disappear from H, H', and similarly remove all d.p.'s and multiple intersections from the F-curves.

Though ϕ has now no isolated F-points, it need not be normal, for it contains any such abnormal points on ω as arise from the general point by one additional condition: namely, explicit or implicit $(i+1)$-fold points on an i-fold curve of total contact, or implicit normal intersections of two components adjacent to one F-curve.

We next remove the contact conditions. Any F-curve of contact ω is extraordinary on $f_{\alpha\beta}$, and conversely, any curve extraordinary on every $f_{\alpha\beta}$ is a curve of contact of T. We reduce ω as a singularity on one of these surfaces, f_{12} say, and we must show that the same process reduces it on every $f_{\alpha\beta}$, and does not use any point or curve that has a special relation to ϕ_1, ϕ_2 which it does not have to every pair of homaloids.

A point P of higher multiplicity for ϕ_1 than for ϕ_a is an $(i+1)$-fold point on an i-fold F-curve ω of total contact of T, either explicit or implicit; and when P becomes explicit, ω still demands resolution by a transformation V applied to it. Then $P \tilde{V}$ a P-curve κ' of V, simple on ϕ_1' and not lying on (ϕ_a'); and κ' is ordinary on f_{12}' and is not used in any further reduction.

If P is special in relation to ϕ_1 only, without being hypermultiple, it is an abnormal pinch-point of ϕ_1 and of f_{12} on ω, an intersection of two adjacent curves ω_1', ω_2', of which at least one lies on ϕ_1' but not on (ϕ_a'), and is therefore simple on ϕ_1' and not a curve of contact and does not require resolution. Similarly if P is special in relation to ϕ_2 only.

The only points special in relation to the pair ϕ_1, ϕ_2, but not to either alone, are their points of simple contact at the intersections of c_{12}, ω, normal on f_{12}.

Thus the total transformation V, which reduces ω as a singularity of f_{12}, uses none of these special points, and reduces it on every $f_{\alpha\beta}$. We can thus remove all curves of contact from H, and the new F-curves introduced accidentally into H are ordinary with elementary intersections. Thus (ϕ) is reduced to ordinary type.

But V introduces isolated F-points into the second F-system of T, arising from the other vertices of the T_M's contained in V, which are not resolved in the construction of each T_k. These are abnormal on ϕ', being multiple intersections of multiple curves, but they are ordinary on f'_{123}, and their resolution by T_{3-3}'s does not introduce any fresh elements of contact or isolated points into either H or H'.

We now resolve the curves of contact of H' by a transformation V', and remove the isolated points which V' introduces into H. Then T is reduced to ordinary type, by composition with a set of transformations of three known elementary types, T_{3-3}, T_{3-4} and T_M.

31. Fundamental genera (165).

The *F-genera* of T are the genera of all its irrational F-curves, explicit or implicit. The implicit F-genera become explicit at some stage of the reduction of T to ordinary type, as the genera of curves arising from F-elements of contact of T, and not from the auxiliary transformations.

Each of these has one arbitrary F-curve, which is in general irrational. We now let this break up into rational components with simple intersections, except for the vertex of a T_M, as is always possible (p. 314). These simple intersections are normal on all the surfaces concerned, and do not affect the reduction. Then the second F-systems of T_{3-3} and T_{3-4} are entirely rational, and that of T_M has at most one irrational element, of the same genus as the curve resolved. Hence T cannot gain any F-genera during the process of reduction.

Nor can it lose any; for in each case the P-surface which may be dropped does not entirely carry away the irrational F-element, which survives on the transformed homaloidal family, whose F-genera, all explicit, are the same as those of T.

An F-genus could be lost only if a transformation is applied to several F-elements of T at once, so that we drop more than one P-surface. For example, if T is itself a T_M, we can reduce it to identity by compounding with T^{-1}, which eliminates all its F-genera.

It may happen that two irrational F-curves of T have equal genera; whether they are distinct or adjacent, they preserve their identities, and the two genera can be kept apart, just as if they were unequal; and we shall speak of them as different. In this sense, each F-genus of T occurs in at most one auxiliary transformation, and that only if, when it first becomes explicit, it belongs to a curve of contact.

The explicit irrational F-genera of the two systems of the ordinary transformation fall into pairs of the same genus; hence this is true of the explicit and implicit F-genera of T itself.

32. F-genera in composition.

We can now prove that if T is resolved into any two components, any F-genus of T is an F-genus of one or other component.

If $T \equiv V.W$, then W converts (ϕ) into the first homaloidal web (ϕ_V) of V. Let p be an F-genus of T, either explicit or implicit.

(i) Let p be explicit, then (ϕ) has an F-curve ω of genus p. This is not a P-element of W, or an F-curve of second species, since these are rational. It can have three other possible relations to W.

1. If ω is general with regard to W, then it $\widetilde{W}$ a curve of genus p on (ϕ_V), and p is explicit in V.

2. If ω is a general curve lying on a P-surface of W, it $\widetilde{W}$ the first neighbourhood of an F-element of contact of V. Then a transformation applied to this, as the first stage of reducing V, converts (ϕ_V) into a homaloidal web having an explicit F-curve in 1, 1 relation to ω, and therefore of genus p; and p is implicit in V.

3. If ω is an F-curve of first species of W, then p is explicit in W.

Thus if p is explicit in T, it is either explicit or implicit in V, or explicit in W.

(ii) Let p be implicit in T; then it is explicit in $T' \equiv T.V_1$, where V_1 is the whole or part of the reducing transformation of T, and p is not an F-genus of V_1. But $T' \equiv V.W.V_1$, and by (i), since p is explicit in T', it is either explicit or implicit in V, or explicit in $W.V_1$; and in the latter case, since it does not occur in V_1, it is either explicit or implicit in W.

Hence in any case, if p is an F-genus of T, it is an F-genus of one of V, W.

From this there follows an important, if negative, result, which shows that there is no immediate extension to space of Noether's theorem in the plane:

There is no finite set of which all Cremona space transformations are compounded. For a finite set contains only a finite number of F-genera, and a set from which all others can be compounded must between them contain all possible F-genera; and we can construct transformations, T_M's for example, having any assigned number as an F-genus.

33. Classification.

There is as yet no satisfactory classification of the group of Cremona space transformations, and no practical method of obtaining all the transformations of any proposed class, except in the simplest cases. We consider some possible criteria that might be taken as a base of division.

(i) The first degree n, with sub-division according to the second degree n'. This is simple, exhaustive and useful, but it separates T from T^{-1}, which are two aspects of the same thing.

(ii) The genus of the transformation. This is common to T and T^{-1}, and to all transformations whose general homaloids are of the same type. To determine all the transformations of genus p, we must first be able to find all the types of rational surfaces ϕ whose general plane sections are of genus p, and then determine, by Cremona's theorem, all the transformations for which each ϕ is a general homaloid.

(iii) The F-genera. These are common to T and T^{-1}; they divide the whole group more closely than the other proposals, since the number of characteristics of T taken into account is unlimited. This criterion makes a distinction between a transformation and its particular cases when an F-curve acquires an additional d.p.; this may be inconvenient, but it may be an essential distinction.

For example, two particular cases of the first singular T_{3-3} (no. 3 of Table VI), with degenerate F-sextics, are compounded of T_{2-2}'s, while the general case is not; but not all the cases with rational F-curves can be so compounded.

(iv) The degree of the F-curve of maximum multiplicity. Unlike the above, this gives only a finite number of classes (p. 167).

(v) In several connections, there occur sets of space transformations, such as the dilated punctual transformations, in 1, 1 relation to the whole set of plane transformations (pp. 226, 313, 321). From any one set we can deduce ∞ other sets, by compounding any space transformation T with all the members. Since identity is one of the dilated transformations, compounded of T_{tet} and its reverse, the new set can include T itself; thus every T belongs to one such set, and may belong to many. We may require of a classification that if a class includes any member of such a set, it shall contain them all.

(vi) We may also require that all the transformations $T_{n-n'}$ that convert a given homaloid ϕ_{n_1} into a plane p shall be classed together. Then we must accept as elementary any transformation W that converts p into another plane p', for $W.T$ converts ϕ into p'. Now W effects, between $p \equiv w$ say, and $p' \equiv w'$, a certain plane transformation T_μ, say

$$x' : y' : z' = \phi_\mu(x, y, z) : \psi_\mu : \chi_\mu,$$

and W has the form

$$x' : y' : z' : w' = \phi_\mu f_{n-\mu} + w\phi_{n-1} : \psi_\mu f_{n-\mu} + w\psi_{n-1} : \chi_\mu f_{n-\mu} + w\chi_{n-1} : w\theta_{n-1}.$$

The only condition on W is that its F-system includes a plane curve $f_{n-\mu}$ in p, and also the system of F-points of T_μ. This type is probably too little restricted to be satisfactory as an element.

The degree n of T is not necessarily equal to n_1 the degree of ϕ, unless ϕ is a proper homaloid; we may have $n = n_2 > n_1$, if ϕ is the effective part of a homaloid ϕ_{n_2} of higher degree, that breaks up into ϕ and a set of P-surfaces of T. All the T's of degree n_1 can be found when ϕ_{n_1} is given, but no method is yet known for finding all those of higher degree; even for $n = 1$, their number is infinite.

Now we must include in the same class as T the other transformations, of degree $\geqq n_2$, that convert ϕ_{n_2} into a plane. Thus ϕ_{n_1} is linked to ϕ_{n_2}, and also to other homaloids that may occur as parts of other degenerate members of (ϕ_{n_2}); and these in turn are linked to more homaloids ϕ_{n_3} of higher degree.

For example, let the first class be that which includes identity; then ϕ_{n_1} is a plane, and the class includes in the first place all T's for which one ϕ breaks up into a plane and a set of P-surfaces, such as T_{2-2}, T_{2-3}, T_{2-4}, T_{tet}, their reverses, and many others, but not the general bilinear T_{3-3}. The plane is linked to all quadrics, cubics with a nodal line, Steiner quartics, etc. Now a cubic with a nodal line is linked to a general cubic with an F-line of contact; thus the general cubic is reached in two steps. It may or may not be possible to link up all homaloids, and we do not know how many of these classes exist. If there is only one, we may classify homaloids by the number of steps needed to link them to a plane.

(vii) A classification based on composition will probably be more enlightening than any other, when it can be complete; it may incorporate the classification by F-genera. Its usefulness depends on the best choice of elementary transformations, from which to construct the others; so far, we have very little to guide us. We can form isolated classes, such as the punctual transformations, by selecting arbitrarily certain elements and compounding them in all possible ways.

In the plane, we only need T_2 as element; and it is convenient to add T_J, to shorten the process of resolving complicated higher transformations. In space, we can start with T_{2-2} as first element, but from this alone we cannot even construct the bilinear T_{3-3} and with $n = 3$ alone, the number of elements required is 34 (p. 170). To construct all space transformations, we require ∞ elements, for they must between them contain all possible F-genera.

The most natural analogue of T_J is T_M, which includes T_{2-2} and T_{tet} as its simplest cases; but from these we cannot construct the bilinear T_{3-3}; and a T_M itself may be quite complicated, and it may be possible to resolve all T_M's into certain elementary types of T_M.

CHAPTER XVII

HISTORY AND LITERATURE

1. The classical reference: Apollonius of Perga (4).

The earliest trace of a 1, 1 transformation of the points of a plane is contained in the *Plane Loci* of Apollonius of Perga, two centuries before Christ. The treatise itself is lost, but its main theorems are preserved by Pappus. The proposition stands thus in Commandinus' translation (314):

> Si duae lineae agantur, vel ab uno dato puncto, vel a duobus, et vel in rectam lineam, vel parallelae, vel datum continentes angulum, vel inter se datam proportionem habentes, vel datum comprehendentes spacium: contingat autem terminus unius locum planum positione datum, et alterius terminus locum planum positione datum continget, interdum quidem ejusdem generis, interdum vero diversum, et interdum similiter positum ad rectam lineam, interdum contrario modo.

Here *locus planus* is defined to be either a line or a circle, and the statement includes the linear and quadratic transformations: translation, rotation, similarity, inversion and combinations of these.

The idea remained undeveloped for twenty centuries.

2. The earliest modern period.

The first explicit reference to a 1, 1 transformation of the whole field occurred in 1822: the first connected treatment of the general theory in 1863. During these forty years about thirty different writers made some sort of contribution, discovering, re-discovering, enlarging, translating and spreading their ideas.

An unfortunate mistake greatly restricted the field. Magnus (222) and many others held, by false analogy with the case of a single variable, that every 1, 1 transformation must be expressible by bilinear relations, and was therefore of degree $\leqq 2$ in the plane and $\leqq 3$ in space. Magnus himself, in his second volume (223), mentions the possibility of forming space transformations of higher degree by composition, and Eckhardt, 1867–72 (120), also refers to this; but the delusion reappeared as late as 1879 (3).

As a result, attention was centred on these transformations of low degrees, which were approached from half a dozen distinct points of view. First came the two involutory types.

(i) In 1822, Poncelet published his *Traité* (330), which treats incidentally of the 1, 1 relation between pairs of points conjugate with regard to two fixed conics. This gives the general quadratic involution of the first type. In 1827–8, Bobillier (40) obtained the same transformation as a particular case of the multiple relation between a point and the group of intersections of its first polars with regard to a pencil of curves of any degree. He also obtained the corresponding space transformation, which is the general I_{3-3}. These two Frenchmen must be honoured as the first pioneers of the plane and space theories respectively. For these forms of I_2 and I_{3-3}, see also 19, 131, 133, 153, 358, 373, 378, 379.

(ii) In the same year as the *Traité*, Dandelin (92) made use of a double stereographic projection, equivalent to inversion, and for which, in his later paper (93), he uses the characteristic equations; but he regards it as a transformation between curves rather than between points. The same is true of Steiner's first reference (371), and of the MS. fragment found among his papers (45), in which inversion bears the mystical name of *Wiedergeburt und Auferstehung*. Inversion, as a transformation of points, is treated by Plücker, 1831–4 (329), Bellavitis (16), who extends it to space and generalizes it into conical inversion, Thomson (388, 389), who was led to it in his electro-static researches, Liouville (212), who elaborates Thomson's method and introduces the phrase *reciprocal radii*, Möbius (240, 241, 242), whose *Kreisverwandtschaft* became widely known, Hirst (154, 155), after whom this form of involution is often named, and many other writers of the period (19, 59, 130, 171, 172, 277, 377).

(iii) The algebraic line of approach starts from bilinear equations. Plücker, 1830 (328), in a memoir on homogeneous coordinates, gives the plane equations in their simplest form $xx' = a^2$, etc., which is the general quadratic involution of first type. Magnus gives the general non-involutory form for plane and space in the first and second volumes of his *Aufgaben und Lehrsätze* (222, 223), a shorter reference occurring already in 221, 1832. See also 156, 343, 386, 404.

(iv) Another reference in 1832 to a non-involutory transformation is the method of skew projection contained in Steiner's *Systematische Entwickelung*, 1832 (372), a space construction for the plane quadratic transformation. Battaglini (15) gives another such. De Jonquières, whose memoir (175–177), written in 1859, remained unpublished in the archives of the Paris Academy for twenty-five years, uses an idea similar to Steiner's in connection with his plane transformations of general degree. It is remarkable that the discovery of these preceded that of any isolated examples of degree > 2 in the plane or 3 in space. The first of such are Geiser's plane I_8 and space I_{7-7}, 1866–7 (132), and a plane T_5 studied by Sturm, 1869 (380).

(v) The construction of the plane T_2 by means of projective pairs of pencils was given first by Bellavitis, 1838 (17). A very special space form is Hunyadi's method (169), by reflexion of planes in the bisectors of the dihedral angles of a tetrahedron, which leads to T_{tet}. To the same order of ideas belongs an isolated paper by Geiser (133), where T_{tet} relates the foci of quadrics of revolution inscribed in a tetrahedron. A fuller extension to space was given much later by Silldorf, 1873 (362) and von Krieg, 1884 (201). See also 174, 235, 359, 378.

(vi) Jacobi, 1841–2 (**173**) and, more fully, Battaglini (**15**) construct T_2 by combining two correlations. The same idea for space occurs in the course of Cremona's memoir on cubic surfaces (p. 72 of **82**). See also **332**, **333**.

Further notes on all these papers will be found in the bibliography below. For a full historical account, see **200**; also **197**, **218**.

3. Cremona's general plane theory; Noether's theorem.

In 1863, the general theory for the plane was first stated by Cremona, in the earliest of a series of great memoirs (**80**); these laid down the main outlines once for all. Other geometers soon brought substantial additions, of which by far the most important is Noether's theorem.

The possibility of resolving every Cremona plane transformation into T_2's was first stated by Clifford, 1869 (**61**, **72**), among whose papers the resolution was found for all characteristics of degree $\leqq 8$, but no general proof. Noether also announced it in **285**; and in **286** gave a proof, essentially the same as on p. 9, that $i_1 + i_2 + i_3 > n$, from which it was then considered that the theorem followed. Independently, Rosanes (**335**) discovered the same inequality, and proved it by induction; he remarks that if the highest F-points of T_n coincide, the T_2 is specialized, having its F-points adjacent.

In 1872, Noether (**290**) noticed the first of the two cases of failure of his proof, when O_2, O_3 are adjacent to O_1 in different directions; but the possibility of cuspidal branches was still overlooked. He proves $\underset{(\gamma)}{\Sigma i_\alpha} \leqq i_1 < \underset{(\beta)}{\Sigma i_\alpha}$, and hence that a reducing trio exists, for the case which he considers.

In 1883–4, Veronese (**405**) put forward an entirely different proof, based on successive projections and sections of a surface in five dimensions.

The theorem was then generally accepted, and used extensively, especially in relation to linear systems of plane curves (**25**, **28**, **32**, **145**, **230**, etc.), until in 1901, Segre (**354**) noticed the second case of failure, when O_2, O_3 are adjacent to O_1 in the same direction, but not upon one linear branch. In this case, it may be impossible to reduce n by a single T_2, and Segre gives the example quoted on pp. 113, 145, whose degree cannot be lowered by any Cremona transformation of degree $< n$.

Castelnuovo (**58**) immediately met this with the first complete proof of the theorem; by considering adjoints, he obtains some inequalities, from which he proves the existence of a reducing T_J, and gives the algebraic resolution of T_J into T_2's (p. 145). Segre, in a footnote (**355**), adds the geometric treatment of T_J (p. 149).

Ferretti, 1902 (**125**) and Franciosi, 1916–8 (**128**) have applied these extended methods to establish fully the results on linear systems which had been based on the earlier work; see also **293**.

Three other proofs of the theorem have been published. Alexander's, 1915–6 (**2**), consists essentially in embodying Segre's treatment of T_J in Castelnuovo's reduction of (ϕ_n), thus obtaining a direct resolution into T_2's.

He does not aim directly at the reduction of n; each stage of his process consists of a group of three T_2's, and lowers either the complexity 2τ, or the number β of major F-points. When these are reduced to their lowest values, $\beta = 2$, $\tau = \frac{1}{2}$, we have $n = 2$, and the net is quadratic.

Franciosi (**128**) gave the proof whose substance is reproduced on p. 150. He bases it on a remark made to him by Bertini, that, in the second case of failure also, there holds the relation $i_1 \geqq \frac{1}{2}(n+1)$ (p. 71), proved for the first case by Noether (**290**). But this result need not be used, and the proof applies generally, not only to the exceptional cases.

Chisini's, 1921 (**67**), is the alternative proof given on p. 151. It follows Noether more closely than the others, proceeding by groups of T_2's that reduce the degree.

4. Clebsch's theorem.

The next important result was Clebsch's theorem on the symmetry of the characteristic numbers of the F-systems in the two planes. The argument based on partitions (p. 25) is the earlier, only indicated by Cremona (**80**), and set out fully by Dewulf (**107**). The first complete proof, published by Clebsch, 1871 (**70**), is expressed in terms of the rows and columns of the determinant of incidence numbers.

A different proof is due to Montesano (**270**), based on a method of tabulating the incidence numbers for a compound transformation, and then expressing that any transformation compounded with its reverse leads to identity.

5. Construction of tables.

The problem of finding the best method for constructing the tables of characteristics for the lower values of n is in a sense converse to Noether's theorem. The arithmetical method (p. 64) was probably the one used by Cremona, 1864–5 (**80**), who calculated the tables up to $n = 10$, with one omission, supplied by Cayley, 1869 (**61**). It is also followed by de Jonquières, 1886 (**181**), who gives tables for $n = 12$, 13, and by Ruffini (**336**).

Larice, 1908 (**203**), follows a quite different path, based on Veronese's five-dimensional work; but the incompleteness of the tables for $n = 14$, 15 calculated by it does not recommend the method.

Montesano, 1905 (**264**), seeks to reduce the labour of the geometrical method by restricting the examination of lower characteristics to certain special types, and fixes his attention on the complexity as well as on the degree. But in **267** he returns to the simpler course, arising directly from Noether's theorem; this memoir contains the remainder of the tables for $n = 14$, 15, calculated by Tummarello (**395**); those for $n = 16 \ldots 23$ have been calculated by Marazzo but not published hitherto; that for $n = 16$ is given below in Table I. For $n = 1 \ldots 21$ see **239·3**.

6. General space theory: Cayley, Noether, Cremona.

The extension to space came with a burst: the three outstanding geometers of England, Germany and Italy, quite independently and within a few months of each other, published the memoirs which at once raised this side of the subject to importance. In a remarkable way, they cover the same ground and study the same examples.

Cayley, 1869–70 (61), had the precedence in point of time. In his famous style, he boldly outlined the main theory from the algebraic side, with many suggestive remarks not always duly safeguarded, and leaving many points in obscurity to be cleared up by later writers; for he seems never to have seriously returned to the subject.

Noether's first contribution, 1870–1 (287), was more careful and detailed; he devoted most of his memoir to particular cases that could be treated accurately as far as he went, and his standpoint is purely geometrical. The general theory is here treated very shortly; his substantial algebraic work on postulation and equivalence (288) was published very soon after. His later work in this field is chiefly concerned with the resolution of singularities, see § 9 below.

Cremona himself wrote with the tremendous advantage of his long schooling in the plane theory. His first work on space transformations was published in various forms and places, 1871–2 (85–87). The last of these is his fullest exposition of the general algebraic theory; its promised second part was never published, and would no doubt have enlarged the pure geometrical study of examples contained in the others. Here we have already the one really illuminating result that has flowed from the first principles: Cremona's method of finding all the different transformations that convert a given homaloid into a plane (p. 167). After twelve years of silence on the subject, he contributed two studies of particular transformations to learned societies of the British Isles (88, 89).

7. Later space theory.

After this brilliant beginning, progress has been disappointingly slow; the theory as a whole has advanced little since its birth. A few important general theorems have been discovered, but they are isolated. Many particular transformations and classes of transformations have received a good deal of attention: quadratic, cubic, monoidal, involutory, punctual. Some subsidiary aspects of the subject have been dealt with systematically, such as the complexes and congruences of lines joining homologous points, when the spaces are superposed, and applications to the study of linear systems of various kinds.

Analogy with the plane theory would lead us to attach special importance to the composition and resolution of space transformations, but again results have been poor; and are confined to transformations of low degree or simple form (74, 160, 190, 191, 192), or, if general, are negative in character (165).

Autonne and Kantor have dealt at length with groups of transformations, and with periodicity (184–192).

A few text-books give more or less inadequate sections to Cremona transformations (71, 127, 338); Sturm's treatise (385) gives a considerable account, and Doehlemann's book (117) is useful as far as it goes.

8. Application to rational surfaces.

Cremona transformations have been used in dealing with many subjects, from elliptic functions (23) to triple tangents (320); but by far the most of the applications have been made for one of two purposes: the study of rational surfaces, and the resolution of singularities.

In the early days, geometers were largely occupied with the discovery and mapping of surfaces of the lower degrees. Planes and quadrics had been well-known from the dawn of mathematics; cubic surfaces became familiar ground by degrees, and were thoroughly explored by Cremona (82). Then, in the latter half of the last century, came a flood of researches on different types of quartic and quintic surfaces possessing special singularities.

To this field, the method of transformation gave a fertility and a unity impossible before. On the one hand, when a surface is known to exist, we can usually transform it into one of lower degree and simpler type. On the other hand, each new transformation, applied in different ways to quadric, cubic or other known surfaces, produces a number of new surfaces, at once proving their existence and giving an easy method of studying their properties. In fact, this very facility seems to have weakened interest in the subject, and memoirs of this type ceased to be written.

9. Application to resolution of singularities.

Of later years, more attention has been paid to the exploration of a surface in the neighbourhood of a singular point or curve, and to the resolution of a general singularity by transformation.

The first idea of resolving a singular point into simpler components goes back at least to Cramer, 1750 (79), who analyses all the types of multiple points that can occur on plane curves of degree $\leqq 6$; transformation methods had not been discovered at that time.

In 1871, Noether (289) applied Cremona transformations to plane singularities, and expanded the theory a few years later (291, 295, 296). He carries out completely the resolution of any higher singularity into a finite set of ordinary components, and the reduction of a curve to one having only ordinary multiple points, using the method of intersection with the polar. Bertini (31) presents the proof in the somewhat different form given on p. 137.

Singularities of twisted curves are dealt with by Pannelli (308), who applies T_{tet}, B. Levi (207), who applies T_{2-2}, and other writers. The problem is not essentially different from that in the plane (p. 143).

The extension of the method, to the resolution of the singularities of a surface, was suggested by Noether, 1871 (**289**), but not pursued. A serious beginning was made by Del Pezzo, 1892 (**96**); he resolves a singular curve ω by applying a series of T_M's, having ω and its homologues as F-curves of first species: an isolated singular point P is first converted into a line by a T_{2-2} having P as a general point of its F-conic.

To this Segre (see **96**) objects that there may be exceptional points lying on the homologues of ω, which in their turn give rise to multiple lines which may be no simpler than ω itself. Levi (**206**) also offers criticism.

Segre, 1896–7 (**353**), gives a valuable account of the theory of adjacent singularities, with a detailed study of d.p.'s. He is concerned with an analysis of the characteristics of a singular point P rather than with its complete resolution, or with the reduction of the surface as a whole. To prove that the equivalent set of ordinary singularities is finite, he relies on Kobb's previous analytical work (**199**), which is rejected as inadequate by Levi (**205**). Segre uses P as isolated F-point of a T_{2-2}, and considers the singularity of the first polar. Pannelli (**309**), in much the same way, uses a T_{5-5} compounded of two special cases of T_{3-3}, but without claiming to advance matters very much.

Levi, in a series of weighty memoirs, 1897–1904 (**205–211**), follows Segre's methods, and gives the first acceptable proof of the finiteness of the processes of resolution of a singular point and of reduction of a singular surface. The polar surfaces and curves of all orders play a great part in the argument, a summary of which is given on p. 369.

More recently, Chisini, 1920–1 (**66**), has returned to Del Pezzo's method of converting the singular point P into a multiple line, which is then, like any explicit singular curve, resolved by T_M's. This is completed by a laborious investigation of the singularities introduced by the transformations. All previous writers had aimed at reducing the multiplicity of P; but Chisini introduces a new characteristic for reduction, the rank (p. 373), which, though artificial, is much more searching than the multiplicity. A fuller account of this method also is given on p. 371.

10. Outstanding problems.

(i) The outstanding problem of the whole theory, in any number of dimensions, is still to determine the set of properties which are invariant for Cremona transformations, but not for the wider group of Riemann transformations: see Coble's report (**76**) and the references there given, to which may be added Klein's lectures, 1893 (**198**).

It has been doubted whether such properties exist; if they do, they are most probably to be sought in the theories of finite sets of points, linear systems of curves and surfaces, adjoints and Jacobians.

(ii) In the plane theory there is no very obvious gap: we may hope for improved methods in various directions, for example, in the construction of tables of characteristics.

(iii) But there are many plane properties which have as yet found no satisfactory analogues in space, in spite of a strong conviction that they exist. First of all is Noether's theorem. There must be an irreducible set of Cremona space transformations, of which all others can be compounded; all we know at present is that it is infinite, it contains F-curves of all genera, and it need contain no contact conditions.

(iv) Closely connected with this is the classification of space transformations, for which all present suggestions are unsatisfactory.

(v) There should be extensions of Clebsch's theorem, beyond those given by Pannelli (**310–313**) and Tummarello (**400**, **167·2**).

(vi) It may be possible to modify Cremona's or Loria's methods, of obtaining all the Cremona transformations of a given homaloid into a plane, to include those of higher degree when the given surface is the effective part of a degenerate homaloid.

(vii) There is room for much simplification in the process of resolution of singularities. One possibility to be explored is Castelnuovo's method of using P-points of the family under reduction as F-points of the auxiliary transformation.

(viii) In view of most of these problems, we want a systematic investigation, fuller than that of Chapter XVI, of the composition of space transformations when the F- and P-systems in the intermediate spaces are incident and coincident in every possible way.

(ix) Many writers expect insight into the space theory to come from work in more dimensions (**225·1**, **352**, **405**).

BIBLIOGRAPHY.

The following catalogue of papers, that deal with or make use of Cremona transformations in plane or space, is arranged in alphabetical order of authors, the works of each in order of date. References are abbreviated much as in the *Jahrbuch über der Fortschritte der Mathematik.* A date in brackets is that attached to the paper, if different from that of the volume which contains it. Any further information will be welcomed by the compiler.

The remarks do not summarize the papers, but only show their bearing on this subject.

1 **ABBIA** Catania Acc. G. Atti (5) 9 Mem. 3, 1-18. (1914) 1916.
Le trasformazioni cremoniane piane di terza classe non involutorie con curva unita.
General theorems on transformations with invariant curves; determination of the eight such types of class 3, of degrees ≤ 10. Pp. 80, 81.

2 **ALEXANDER** Amer. M. Soc. Trans. 17, 295–300. New York, (1915) 1916.
On the factorization of Cremona plane transformations.
A proof of Noether's theorem on Segre's lines (355). The author is not satisfied with Castelnuovo's proof (58). Pp. 56, 75, 151, 390.

3 **AMIGUES** Nouv. Ann. (2) 18, 548-564. Paris, 1879.
(2) 19, 433–442, 481–492. 1880.
Recherches sur deux modes de transformation des figures solides.
T_{tet} and the correlative transformation. Pp. 301, 388.

4 **APOLLONIUS** Perga, circ. 200 B.C.
Plane loci.
A lost work from which Pappus (314) quotes a clear reference to inversion. P. 388.

5 **APRILE** Catania Circ. M. Note e Mem. 1, 245–262. 1921.
Una trasformazione cremoniana dello spazio ed alcuni sistemi di quartiche gobbe di 2ª specie.
T_{4-4}, the homaloids being Steiner quartics. P. 293.

6 **AROLDI** Battaglini G. (3) 11 = 58, 175–192. Napoli, 1920.
Le trasformazioni birazionali dello spazio determinate dalla più generale superficie del quart' ordine dotata di conica doppia.
The 7 transformations, of degrees $4-7$ to $4-3$ are obtained by Loria's method (217). Pp. 171, 328.

7 **ASCHIERI** Lomb. Ist. Mem. (3) 5 = 14, 251–267. Milano, (1879) 1881.
Sopra una classe di trasformazioni razionali in spazi a tre dimensioni.
A Cremona I_{4-4} arises incidentally. P. 328.

8 —— Lomb. Ist. Rend. (2) 14, 21–26. Milano, 1881.
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9 **ASCIONE** Battaglini G. 31, 55–94. Napoli, 1893.
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T_{3-3} generated by projective pencils; many particular cases and their equations. Pp. 299, 301.

10 —— Napoli Rend. (3) $2 = 35$, 13–29. 1896.
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A cubic variety in four dimensions, with 10 d.p.'s, leads to Geiser's I_{7-7} and other space involutions of degrees 7, 11, 13, associated with linear complexes. Pp. 323, 328.

11 **AUTONNE** Paris C. R. 121, 673–676. 1895.
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Components of a P-curve which $\sim$ an F-point of higher singularity; application to the resolution of any singularity of a plane curve. Pp. 18, 129.

12 —— Paris C. R. 121, 881–883, 1129–1130. 1895.
Sur les variétés unicursales à trois dimensions.

Extension of 11 to space. P. 277.

13 —— Paris C. R. 122, 1043–1045. 1896.
Sur les substitutions régulières non linéaires.

Properties of F-systems, and a case of I_{3-3} with a double F-line (misprinted). Pp. 14, 159, 291.

14 **BAKER** London M. Soc. Proc. (2) 21, 114–133. (1922) 1923.
Remarks on Mr Wakeford's paper (*ib.* 98–133).

Equations and references for a case of the bilinear T_{3-3}. P. 300^2.

15 **BATTAGLINI** Battaglini G. 1, 321–328. Napoli, 1863.
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T_2 from two correlations; general theory, special cases, first type of involution, space constructions. Pp. 30, 40, 41, 51, 389, 390.

16 **BELLAVITIS** Ann. Sci. del Regno Lomb.-Ven. 6, 126–141. Padova, 1836.
Teoria delle figure inverse, e loro uso nella geometria elementare.

A long memoir on inversion in plane and space. Pp. 53, 208, 389.

17 —— Padova Ac. Nuovi Saggi 4, 243–288. 1838.
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T_2 from equal or projective pencils; first type of involution; adjacent F-points. Pp. 30, 36, 39, 389.

18 **BELOCH** Ann. di M. (3) 16, 27–68. Milano, 1909.
Sulle trasformazioni birazionali nello spazio.

Adjoints; maximum multiplicity of F-elements; transformations with no F-curves. Pp. 165–7, 292, 308 and Table VI.

19 **BELTRAMI** Bologna Mem. (2) 2, 361–395. (1863) 1862.
= Op. Mat. 1, 45–72. Hoepli, Milano, 1902.
Intorno alle coniche di nove punti e ad alcune quistioni che ne dipendono.

The first type of I_2; refers to inversion. Pp. 51, 53, 389^2.

20 **BELTRAMI** Battaglini G. 1, 208–217, 354–360. Napoli, 1863.
= Op. Mat. 1, 73–91. Hoepli, Milano, 1902.
Estensione allo spazio di tre dimensioni dei teoremi relativi alle coniche di nove punti.
T_{tet} occurs. P. 301.

21 **BERARDI** Battaglini G. (3) 14 = 61, 109–122. Napoli, 1923.
Le trasformazioni birazionali dello spazio determinate da sistemi omaloidici di quintiche dotate di una retta tripla e di due doppie.
States details of the 14 transformations, found by Cremona's and Montesano's methods. Pp. 170, 172, 327, 328.

22 **Van den BERG** Nieuw Arch. Wisk. 7, 78–90. Amsterdam, 1881.
Over twee met betrekking tot een driehoek symmetrische groepen van drie cirkels, en over twee dergelijke groepen van drie rechte lijnen.
$I_5\,6^2$ by symmetric circles. P. 122.

23 **BERRY** Cambridge Phil. Trans. 18, 333–347. (1899) 1900.
On quartic surfaces which admit of integrals of the first kind of total differentials.
Uses some space transformations of low degrees. Pp. 328, 393 and Table VI.

24 —— Cambridge Phil. Trans. 19, 249–296. (1902) 1904.
20, 74–112. (1904) 1908.
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Uses many space transformations of degrees 6–4 and lower. P. 328.

25 **BERTINI** Ann. di M. (2) 8, 11–23, 146. Milano, (1876) 1877.
Sopra una classe di trasformazioni univoche involutorie.
Reduction of I_J; see Noether's report (**293**). Pp. 78, 81, 96, 105, 107–110, 390.

26 —— Ann. di M. (2) 8, 244–286. Milano, 1877.
Ricerche sulle trasformazioni univoche involutorie nel piano.
Reduction of plane involutions to four independent types; I_{17}; see Noether's report (**293**). Pp. 10, 11, 21, 26, 70, 91, 96, 97, 127.

27 —— Lomb. Ist. Rend. (2) 13, 443–451. Milano, 1880.
Sulle trasformazioni univoche piane ed in particolare sulle involutorie.
P-curves of maximum degree; the four irreducible types of involutions. Pp. 22, 25, 27.

28 —— Lomb. Ist. Rend. (2) 15, 24–28. Milano, (1880) 1882.
Sui sistemi lineari.
The general member of a linear family of curves or surfaces has no variable multiple point; if it breaks up, all its variable components belong to a fixed pencil. Pp. 4, 91, 151, 160, 238, 390.

29 —— Lomb. Ist. Rend. (2) 15, 154–157. Milano, 1882.
Construzioni geometriche della trasformazione univoca di 3° ordine.
Two space constructions for T_3. P. 115.

30 —— Lomb. Ist. Rend. (2) 16, 89–101, 190–208. Milano, (1882) 1883.
Sopra alcune involuzioni piane.
General properties of involutions; constructions for I_J; determination and reduction of all of classes 1 and 2. Pp. 28, 79, 86, 90^2, 92, 93^3, 97, 105–8, 123–7.

31 **BERTINI** Lomb. Ist. Rend. (2) 21, 326–333, 413–424. Milano, 1888.
Sopra alcuni teoremi fondamentali delle curve piane algebriche.
Resolution of any plane singularity. Pp. 129, 137, 393.

32 —— Palermo Rend. 3, 5–21. (1888) 1889.
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Properties of incidences of P-curves at F-points. Pp. 6, 22^2, 390.

33 —— Lomb. Ist. Rend. (2) 22, 771–778. Milano, 1889.
Deduzione delle trasformazioni piane doppie dai tipi fondamentali delle involutorie.
All plane involutions are rational. P. 94.

34 **BERZOLARI** Ann. di M. (2) 16, 191–275. Milano, 1888.
Ricerche sulle trasformazioni piane, univoche, involutorie, e loro applicazione alla determinazione delle involuzioni di quinta classe.
Involutions constructed by pencils; determination of those of class 5. For completion see 300. Pp. 90, 93, 95^2 and Table I.

35 —— Palermo Rend. 3, 145–159. 1889.
Un nuovo teorema sulle involuzioni piane.
A property of the highest F-point. Table I.

36 —— Ann. di M. (2) 24, 165–191. Milano, 1896.
Sulle intersezioni di tre superficie algebriche.
Condition that the equivalence of a common multiple point shall be greater than the product of its multiplicities. Pp. 213, 246.

37 **BIANCHI** Battaglini G. 16, 263–266. Napoli, 1878.
Nota sulle trasformazioni univoche nel piano e nello spazio.
Composition of two T_J's; a T_M that changes one star of planes into another. Pp. 61, 306, 310 and Table IV.

38 **BIASI** Diss. 27 pp. Gallizzi, Sassari, 1905.
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If the F-points are so related that P-curves break up, the incidence relations may be modified; some theorems on composition. Pp. 18, 57, 65.

39 **BOBEK** Wien Ber. 91 Abth. II, 476–518. 1885.
Ueber gewisse eindeutige involutorische Transformationen der Ebene.
A set of $I_{3\mu+5}$'s with nine F-points constructed by pencils; but includes many merely arithmetical characteristics. P. 95.

40 **BOBILLIER** Gergonne Ann. 18, 253–269. Nismes, 1827–8.
Recherches sur les lois générales qui régissent les lignes et surfaces algébriques.
I_2 and I_{3-3}, by reciprocation with regard to a pencil of conics or net of quadrics, occur as particular cases of multiple transformations of higher degree. Pp. 51, 295, 303, 388.

41 **BONICELLI** Battaglini G. (2) 9 = 40, 184–191. Napoli, (1901) 1902.
Sopra una trasformazione birazionale dello spazio di 3° grado e una classe di superficie razionali del 6° ordine.
T_{3-6}, no. 41 of Table VI. P. 328.

42 **BOYD** Amer. J. 34, 291–324. Baltimore, 1912.
On the perspective Jonquières involutions associated with the (2, 1) *ternary correspondences.*

Properties of an I_J of even degree. Pp. 108, 110.

43 **BRETSCHNEIDER** Plauen in V. Progr. 24 pp. 1870.
Punktverwandtschaft und Linearverwandtschaft ebener Figuren.

Geometrical treatment of Magnus' T_2. P. 30.

44 **BRILL** Quart. J. 27, 356–362. London, 1895.
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In any transformation, the cross ratio of four directions through a general point is unaltered. Pp. 44, 58.

45 **BÜTZBERGER** Diss. 60 pp. Teubner, Leipzig, 1913.
Ueber bizentrische Polygone, Steinersche Kreis- und Kugelreihen und die Erfindung der Inversion.

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46 **BYDŽOVSKÝ** Prag České Ak. Rozpravy Tr. II, 18, No. 34, 1–8. 1909.
O jisté nekonečne grupě Cremonových transformací.

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47 **CALDERARA** Palermo Rend. 18, 205–217. (1903) 1904.
Le trasformazioni birazionali dello spazio inerenti ad una cubica sghemba.

T_{15-15} and I_{27-27}; rays of the associated complex meet a twisted cubic. Pp. 180, 328.

48 **CANTONE** Napoli Rend. 25, 181–190. 1886.
Teoremi sulla cubica gobba, dedotti dallo studio di una trasformazione involutoria nello spazio.

I_{3-3}; three variable d.p.'s on a cubic of contact. P. 304^2.

49 **CAPORALI** Ann. di M. (2) 7, 149–188. Milano, 1875.
= Mem. di Geometria, 1–46. Pellerano, Napoli, 1888.
Sulla superficie del quinto ordine dotata d' una curva doppia del quinto ordine.

Describes a T_{3-5}, no. 23 of Table VI. P. 328.

50 —— Napoli Rend. 18, 212–219. 1879.
= Mem. di Geometria, 116–125. Pellerano, Napoli, 1888.
Sulle trasformazioni univoche piane involutorie.

Definition of *class*; properties of Δ and Ω; all involutions of class 1 are Geiser's I_8 and degenerations. Pp. 79, 81, 86, 87, 89, 119, 123, 125.

51 —— Napoli Rend. 22, 314–320. 1883.
= Mem. di Geometria, 226–235. Pellerano, Napoli, 1888.
Relazione sul concorso pel premio accademico dell' anno 1882.

A readable account of Kantor's essay (187).

52 **CARDOSO-LAYNES** Pitagora (1) 5_2, 34–38. Palermo, 1899.
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A metrical I_2. P. 53.

53 **CARDOSO-LAYNES** Periodico di M. (2) 4 = 17, 33–41. Livorno, (1901) 1902.
Sopra una speciale trasformazione cubica del piano.
A metrical T_3. P. 115.

54 **CASSANI** Veneto Ist. Atti (7) 9 = 56, 30–34. 1897.
Sulla corrispondenza quadratica.
A space construction for T_2. P. 41.

55 **CASTELLI** Battaglini G. (3) 7 = 54, 10–26. Napoli, 1916.
Studio di una particolare trasformazione cubica dello spazio con applicazione a talune superficie del 4° e del 5° ordine.
The bilinear I_{3-3} and I_{tet}. Pp. 301, 303.

56 **CASTELNUOVO** Roma Acc. L. Rend. (5) 1_1, 47–50. 1892.
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Invariance of adjoints; reduction of a T_n with an irrational invariant curve. P. 80.

57 —— M. Ann. 44, 125–155. Leipzig, (1893) 1894.
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All plane involutions are rational. P. 94.

58 —— Torino Atti 36, 861–874 (= 539–552). 1901.
Le trasformazioni generatrici del gruppo cremoniano nel piano.
First complete proof of Noether's theorem; resolution of any T_n into T_J's, and any T_J into T_2's. Pp. 145, 150, 390.

59 **CAYLEY** Quart. J. 2, 162. London, 1858.
= Coll. Papers 3, 118–119. Cambridge, 1889–1897.
Note on the 'circular relation' of Prof. Möbius.
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60 —— London R. Soc. Phil. Trans. 159, 201–229. 1869.
= Coll. Papers 6, 329–358. Cambridge, 1889–1897.
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Gives the equivalence of a general curve. Pp. 215, 220.

61 —— London M. Soc. Proc. 3, 127–180, 196–198. (1869) 1870.
= Coll. Papers 7, 189–240. Cambridge, 1889–1897.
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The earliest memoir on the general space theory; includes Cremona's plane theory, with complete tables for $n \leqq 10$; T_{2-2}, T_{2-3}, the bilinear T_{3-3}.
Pp. 1, 6, 9, 30, 65, 144, 154, 159, 287, 294, 390–2.

62 —— M. Ann. 3, 526–529. Leipzig, 1871.
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Postulation and reduction of genus due to singular points and curves. Pp. 219, 224.

63 **ČECH** Časopis 50, 219–249. Prag, 1921.
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Uses a bilinear T_{3-3} with four adjacent F-lines, see 167. P. 298.

64 **CHISINI** Roma Acc. L. Rend. (5) 26_2, 8–12. 1917.
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The reduction in genus and class of the plane section. P. 366.

65 —— Bologna Rend. 24, 67–73. 1920.
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Exceptional points of multiple curves are base points of the pure polar curves. Pp. 366, 373.

66 —— Bologna Mem. (7) 8, 3–42. (1920) 1921.
La risoluzione della singolarità di una superficie mediante trasformazioni birazionali dello spazio.
A complete solution of the problem. Pp. 143, 287, 288, 316, 359, 366, 371, 373^2, 394.

67 —— Modena Soc. Natur. e M. Atti (5) 6, 7–13. 1921.
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A new proof of Noether's inequality and theorem; resolution of T_n into T_2's with two F-points adjacent. Pp. 10, 75, 151, 192, 391.

68 **CHIZZONI** Roma Acc. L. Mem. (3) 19, 301–343. 1883.
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Plane involutions determined by pencils or nets. Pp. 94^2, 95, 123.

69 —— Roma Acc. L. Rend. (4) 2_1, 470–476. 1886.
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I_{tet}. Pp. 301, 302.

70 **CLEBSCH** M. Ann. 4, 490–496. Leipzig, 1871.
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71 —— Ed. Lindemann. Teubner, Leipzig, 1875.
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72 **CLIFFORD** Math. Papers, 538–542. Macmillan, London, 1882.
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A fragment giving the resolution into T_2's for $n \leq 8$. Pp. 144, 390.

73 **COBLE** Amer. M. Soc. Trans. 9, 396–424. New York, 1908.
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Construction for $T_5\ 6^2$. Pp. 120, 121.

74 —— Amer. M. Soc. Trans. 17, 345–385. New York, (1914) 1916.
Point sets and allied Cremona groups (part II).
Complete 1, 1 relation between the F-points in the two planes; plane transformations with eight or fewer F-points; punctual space transformations and their incidence relations; symmetric types. Pp. 24, 70, 73, 127, 318^2, 320, 392 and Table I.

75 **COBLE** Amer. J. 41, 243–265. Baltimore, 1919.
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76 —— Amer. M. Soc. Bull. 28, 329–364. New York, 1922.
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Report on the work of previous writers, with many references. Pp. 14, 154, 394.

77 **CONNER** Amer. J. 38, 155–176. Baltimore, 1916.
Correspondences determined by the bitangents of a quartic.

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78 **COTTERILL** London M. Soc. Proc. 2, 119–125. 1868.
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$T_4\,3^2 3^1$; equations, resolution, special cases. Pp. 117, 119.

79 **CRAMER** Cramer et Philibert, Genève, 1750.
Introduction à l'analyse des lignes courbes algébriques, 400–459.

Includes an analysis of all the singular points that can occur on plane curves of degree $\leqq 8$. Pp. 128, 393.

80 **CREMONA** Bologna Mem. (2) 2, 621–630. (1863) 1862.
(2) 5, 3–35. (1864) 1865.
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Sulle trasformazioni geometriche delle figure piane.

The foundation of the general plane theory; tables for $n \leqq 10$.
Pp. 1, 18, 23, 30, 60, 64–8, 70, 78, 81, 89, 98, 100, 112, 117, 390–1, Tables II, III.

81 —— Brit. Ass. Rep. 34, 3. Bath, 1864.
= The Reader, 418. London, Oct. 1st 1864.
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82 —— Crelle J. 68, 1–133 (72). Berlin, 1868.
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Uses a T_{3-3} given by three correlations. Pp. 390, 393.

83 —— Bologna Mem. (3) 1, 365–386. 1871.
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P. 292.

84 —— Lomb. Ist. Rend. (2) 4, 140–144, 159–162. Milano, 1871.
Sulla superficie di quart' ordine, dotata di una conica doppia.

Uses a T_{2-2} with the F-point on the F-conic. P. 194.

85 —— Lomb. Ist. Rend. (2) 4, 269–279, 315–324. Milano, 1871.
Sulle trasformazioni razionali nello spazio.

His first great space memoir; a short general account, Cremona's theorem, and a large number of examples.
Pp. 154, 167, 170^3, 271, 287, 291–4, 322, 328, 392 and Table VI.

86 **CREMONA** M. Ann. 4, 213–230. Leipzig, 1871.
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General space transformation theory; many special cases of the bilinear T_{3-3}.
Pp. 154, 291–4, 327, 328, 392.

87 —— Ann. di M. (2) 5, 131–162. Milano, 1871.
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Pp. 154, 167, 170, 247, 250, 253, 257, 392.

88 —— R. Irish Ac. Trans. 28, 279–284. Dublin, 1884.
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89 —— London M. Soc. Proc. 15, 242–246. 1884.
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90 **CROCCHI** Battaglini G. 12, 378–380. Napoli, 1874.
Proprietà derivate dalle curve e superficie arguisiane.

Plane Arguesian I_3 constructed by pencils. P. 116.

91 **CZUBER** Monatshefte f. M. 5, 267–286. Wien, 1894.
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92 **DANDELIN** Bruxelles Ac. Nouv. Mém. 2, 171–202. 1822.
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93 —— Bruxelles Ac. Nouv. Mém. 4, 3–10. 1827.
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Uses inversion. Pp. 53, 389.

94 **DARBOUX** J. Univ. des Sci. et des Soc. Sav. en France et à l'Étranger.
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= Soc. Philomath. de France Bull. 5, 72–76, 77–80. Paris, 1868.
= Paris Éc. Norm. Ann. 6, 61–68. 1869.
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Space construction for T_2. P. 41.

95 **DEL PEZZO** Palermo Rend. 3, 236–240. 1889.
Sui sistemi di curve e di superficie.

Linear systems with the same singularities; criticized by Segre (**353**, and Torino Atti 32, 33); reply in Pont. Ac. Atti 27. Pp. 232, 367.

96 —— Palermo Rend. 6, 139–154. 1892.
Intorno ai punti singolari delle superficie algebriche.

Resolution of singular points by T_M's; criticized by Levi (**206**) and Segre (Torino Atti 32, 33). Pp. 313, 360, 363, 367, 371^2, 394^2.

97 **DEL PEZZO** Napoli Rend. (3) 2 = 35, 288–296. 1896.
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Homaloidal webs of cones; T_{n-n} is compounded of T_{2-2}'s.
Pp. 160^2, 194, 293, 316 and Table VI.

98 —— Napoli Rend. (3) 10 = 43, 366–372. 1904.
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Pp. 43, 46, 47.

99 **DEL RE** Battaglini G. 28, 257–282. Napoli, 1890.
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Some properties of T_2 and of the bilinear I_{3-3}. Pp. 180, 303.

100 **DE PAOLIS** Battaglini G. 13, 226–248, 282–297. Napoli, 1875.
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Monoidal transformations. P. 306.

101 —— Roma Acc. L. Mem. (3) 1, 511–544. 1877.
Le trasformazioni piane doppie.
Considers the associated involutions. P. 96 and Table I.

102 —— Roma Acc. L. Mem. (3) 2, 31–50. (1877) 1878.
La trasformazione piana doppia di secondo ordine e la sua applicazione alla geometria non euclidea.
The associated involution is I_2. P. 96.

103 —— Roma Acc. L. Mem. (3) 2, 851–878. 1878.
La trasformazione piana doppia di terzo ordine, prima genere e la sua applicazione alle curve di quarto ordine.
The associated involution is Geiser's I_8. Pp. 96, 123.

104 —— Roma Acc. L. Rend. (4) 1, 735–742, 754–758. 1885.
Alcuni particolari trasformazioni involutorie dello spazio.
The three kinds of congruence to which the associated complex can reduce.
Pp. 181, 304.

105 —— Roma Acc. L. Mem. (4) 1, 576–608. 1885.
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106 **DERUYTS** Liège Mém. (2) 14, 1–14. (1887) 1888.
Sur quelques transformations géométriques.
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107 **DEWULF** Darboux Bull. 5, 206–240. Paris, 1873.
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A presentation of Cremona's and Clebsch's work, amplified in parts.
Pp. 18, 22, 23, 391.

108 —— Paris Éc. Norm. Ann. (3) 3, 405–431. 1886.
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Continuation of **254**. P. 328.

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262 —— Napoli Rend. (3) 7 = 40, 67–106. 1901.
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263 —— Battaglini G. (2) 10 = 41, 181–189. Napoli, 1903.
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265 **MONTESANO** Napoli Rend. (3) 13 = 46, 66–88. **1907.**
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266 —— Napoli Rend. (3) 13 = 46, 256–262. **1907.**
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267 —— Napoli Atti (2) 15 No. 7, 34 pp. (1910) **1914.**
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269 —— Napoli Atti (2) 15 No. 8, 55 pp. (1911) **1914.**
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270 —— Napoli Rend. (3) 21 = 54, 30–38, 69–79, 113–119. **1915.**
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273 —— Napoli Rend. (3) 27 = 60, 116–127. **1921.**
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274 —— Napoli Rend. (3) 27 = 60, 164–175. **1921.**
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288 **NOETHER** Ann. di M. (2) 5, 163–177. Milano, 1871.
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298 **D'OCAGNE** Nouv. Ann. (3) 12, 337–352. Paris, 1893.
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301 **OSBORN** Amer. J. 46, 17–36. Baltimore, 1924.
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302 **Le PAIGE** Belg. Bull. (3) 3, 760–762. Bruxelles, 1882.
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303 —— Belg. Bull. (3) 4, 415–431. Bruxelles, 1882.
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Continuation of 302. Pp. 95^2, 111^2 and Table I.

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307 —— Battaglini G. 29, 154–162. Napoli, (1888) 1891.
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308 —— Lomb. Ist. Rend. (2) 26, 216–222. Milano, 1893.
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Resolution by T_{tet}. Pp. 143, 393.

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(5) 20_1, 404–409. 1911.
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312 **PANNELLI** Lomb. Ist. Rend. (2) 47, 1041–1052. Milano, (1912) 1914.
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The surfaces are either ruled with an $(n-1)$-fold directrix, or Steiner quartics. P. 172.

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P. 129.

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Pp. 170, 328.

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Pp. 1, 14, 27, 124, 154, 170, 327, 328, 393 and Table VI.

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Applications of **415** to higher plane curves. P. 115.

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Invariant systems and isologues in space. P. 175.

TABLE I

Plane Characteristics, $n=1$—16

Degree	Index Number		Numbers of F-points of multiplicities								Conjugate Number
		σ	8	7	6	5	4	3	2	1	
$n=1$	.	0	.	.	.	.	.	.	.	.	
$n=2$	.	3	.	.	.	.	.	.	.	3	
$n=3$	.	5	.	.	.	.	.	.	1	4	
$n=4$	1	7	.	.	.	.	.	1	.	6	
	2	6	.	.	.	.	.	.	3	3	
$n=5$	1	9	.	.	.	.	1	.	.	8	
	2	7	.	.	.	.	.	1	3	3	
	3	6	.	.	.	.	.	.	6	.	
$n=6$	1	11	.	.	.	1	.	.	.	10	1
	2	8	.	.	.	.	1	.	4	3	3
	3	8	.	.	.	.	.	3	1	4	2
	4	7	.	.	.	.	.	2	4	1	4
$n=7$	1	13	.	.	1	.	.	.	.	12	1
	2	9	.	.	.	1	.	.	5	3	3
	3	9	.	.	.	.	1	3	.	5	2
	4	8	.	.	.	.	1	2	3	2	4
	5	7	.	.	.	.	.	4	3	.	5
$n=8$	1	15	.	1	.	.	.	.	.	14	1
	2	10	.	.	1	.	.	.	6	3	5
	3	9	.	.	.	1	.	3	2	3	3
	4	8	.	.	.	1	.	2	5	.	8
	5	10	.	.	.	.	3	1	.	6	2
	6	9	.	.	.	.	3	.	3	3	6
	7	8	.	.	.	.	2	2	3	1	7
	8	8	.	.	.	.	1	5	.	2	4
	9	7	.	.	.	.	.	7	.	.	9
$n=9$	1	17	1	.	.	.	.	.	.	16	1
	2	11	.	1	.	.	.	.	7	3	5
	3	10	.	.	1	.	.	4	1	4	3
	4	9	.	.	1	.	.	3	4	1	7
	5	11	.	.	.	1	3	.	.	7	2
	6	9	.	.	.	1	2	1	3	2	6
	7	9	.	.	.	1	1	4	.	3	4
	8	8	.	.	.	1	1	3	3	.	10
	9	8	.	.	.	.	4	.	4	.	9
	10	8	.	.	.	.	3	3	1	1	8

Degree	Index Number		Numbers of F-points of multiplicities											Conjugate Number	References
		σ	11	10	9	8	7	6	5	4	3	2	1		
$n=10$	1	19	.	.	1	.	.	.	.	.	.	.	18	1	
	2	12	.	.	.	1	.	.	.	.	.	8	3	9	
	3	11	.	.	.	.	1	.	.	.	5	.	5	3	
	4	10	.	.	.	.	1	.	.	.	4	3	2	6	
	5	10	.	.	.	.	.	1	.	3	.	3	3	10	
	6	10	.	.	.	.	.	1	.	2	3	.	4	4	
	7	9	.	.	.	.	.	1	.	2	2	3	1	13	
	8	8	.	.	.	.	.	1	.	.	7	.	.	8	
	9	12	.	.	.	.	.	.	3	1	.	.	8	2	
	10	10	.	.	.	.	.	.	3	.	1	3	3	5	
	11	9	.	.	.	.	.	.	3	.	.	6	.	16	
	12	9	.	.	.	.	.	.	2	2	.	4	1	12	
	13	9	.	.	.	.	.	.	2	1	3	1	2	7	
	14	8	.	.	.	.	.	.	2	.	5	1	.	17	
	15	8	.	.	.	.	.	.	1	3	2	2	.	15	
	16	9	.	.	.	.	.	.	.	6	.	.	3	11	
	17	8	.	.	.	.	.	.	.	5	2	.	1	14	
$n=11$	1	21	.	1	.	.	.	.	.	.	.	.	20	1	
	2	13	.	.	1	.	.	.	.	.	.	9	3	9	
	3	11	.	.	.	1	.	.	.	.	5	2	3	5	267
	4	10	.	.	.	1	.	.	.	.	4	5	.	14	
	5	11	.	.	.	.	1	.	.	3	2	.	5	3	267
	6	10	.	.	.	.	1	.	.	3	1	3	2	11	
	7	9	.	.	.	.	1	.	.	2	3	3	.	17	187
	8	9	.	.	.	.	1	.	.	1	6	.	1	8	34, 187, 226
	9	13	.	.	.	.	.	1	3	.	.	.	9	2	
	10	10	.	.	.	.	.	1	2	1	.	4	2	10	226
	11	10	.	.	.	.	.	1	2	.	3	1	3	6	
	12	9	.	.	.	.	.	1	2	.	2	4	.	18	187
	13	9	.	.	.	.	.	1	1	2	2	2	1	13	187, 226, 264
	14	10	.	.	.	.	.	1	.	5	.	.	4	4	
	15	8	.	.	.	.	.	1	.	3	4	.	.	15	34, 187, 226
	16	9	.	.	.	.	.	.	4	.	.	5	.	16	226, 303
	17	9	.	.	.	.	.	.	3	1	3	.	2	7	
	18	9	.	.	.	.	.	.	2	4	.	1	2	12	
	19	8	.	.	.	.	.	.	2	3	2	1	.	19	74, 187, 226
$n=12$	1	23	1	.	.	.	.	.	.	.	.	.	22	1	
	2	14	.	1	.	.	.	.	.	.	.	10	3	15	
	3	12	.	.	1	.	.	.	.	.	6	1	4	5	267
	4	11	.	.	1	.	.	.	.	.	5	4	1	12	
	5	12	.	.	.	1	.	.	.	4	1	.	6	3	267
	6	11	.	.	.	1	.	.	.	4	.	3	3	17	270
	7	10	.	.	.	1	.	.	.	3	2	3	1	20	
	8	10	.	.	.	1	.	.	.	2	5	.	2	8	34, 226
	9	11	.	.	.	.	1	.	3	.	.	4	3	9	226, 270
	10	10	.	.	.	.	1	.	2	1	2	2	2	10	184, 226
	11	9	.	.	.	.	1	.	2	.	4	2	.	27	187
	12	11	.	.	.	.	1	.	1	4	.	.	5	4	
	13	9	.	.	.	.	1	.	1	3	1	3	.	24	187
	14	9	.	.	.	.	1	.	1	2	4	.	1	14	34, 187, 226

Degree	Index Number		Numbers of F-points of multiplicities												Conjugate Number	References
		σ	12	11	10	9	8	7	6	5	4	3	2	1		
$n=12$	15	14	.	.	.	.	.	.	3	1	.	.	.	10	2	
	16	11	.	.	.	.	.	.	3	.	1	.	4	3	16	226, 270
	17	11	.	.	.	.	.	.	3	.	.	3	1	4	6	270
	18	10	.	.	.	.	.	.	3	.	.	2	4	1	22	383
	19	10	.	.	.	.	.	.	2	2	.	.	5	1	19	226
	20	10	.	.	.	.	.	.	2	1	1	3	.	3	7	
	21	9	.	.	.	.	.	.	2	1	1	2	3	.	25	270
	22	10	.	.	.	.	.	.	2	.	4	.	1	3	18	383
	23	9	.	.	.	.	.	.	2	.	3	2	1	1	23	187, 226, 270
	24	9	.	.	.	.	.	.	1	3	.	3	1	1	13	
	25	9	.	.	.	.	.	.	1	2	3	.	2	1	21	270
	26	8	.	.	.	.	.	.	1	1	4	2	.	.	26	34, 187, 226
	27	9	.	.	.	.	.	.	.	4	2	1	.	2	11	
	28	8	.	.	.	.	.	.	.	4	1	3	.	.	29	74, 187, 267
	29	8	.	.	.	.	.	.	.	3	4	.	1	.	28	74, 187, 267
$n=13$	1	25	1	.	.	.	.	.	.	.	.	.	.	24	1	
	2	15	.	1	.	.	.	.	.	.	.	.	11	3	16	
	3	13	.	.	1	.	.	.	.	.	.	7	.	5	5	35, 267
	4	12	.	.	1	.	.	.	.	.	.	6	3	2	11	
	5	13	.	.	.	1	.	.	.	.	5	.	.	7	3	
	6	11	.	.	.	1	.	.	.	.	4	1	3	2	18	
	7	11	.	.	.	1	.	.	.	.	3	4	.	3	7	34, 226
	8	10	.	.	.	1	.	.	.	.	3	3	3	.	28	
	9	11	.	.	.	.	1	.	.	3	.	2	2	3	9	35, 101, 226
	10	10	.	.	.	.	1	.	.	3	.	1	5	.	32	
	11	12	.	.	.	.	1	.	.	2	3	.	.	6	4	
	12	10	.	.	.	.	1	.	.	2	2	1	3	1	20	270
	13	10	.	.	.	.	1	.	.	2	1	4	.	2	13	187, 226
	14	9	.	.	.	.	1	.	.	2	.	6	.	.	34	
	15	9	.	.	.	.	1	.	.	1	3	3	1	.	25	
	16	15	.	.	.	.	.	1	3	.	.	.	.	11	2	
	17	11	.	.	.	.	.	1	2	1	.	.	5	2	17	226
	18	11	.	.	.	.	.	1	2	.	1	3	.	4	6	
	19	10	.	.	.	.	.	1	2	.	1	2	3	1	21	270
	20	10	.	.	.	.	.	1	1	2	.	3	1	2	12	270
	21	10	.	.	.	.	.	1	1	1	3	.	2	2	19	270
	22	9	.	.	.	.	.	1	1	1	2	2	2	.	30	187
	23	9	.	.	.	.	.	1	1	.	4	2	.	1	23	34, 187, 226
	24	9	.	.	.	.	.	1	.	3	2	.	3	.	27	
	25	9	.	.	.	.	.	1	.	3	1	3	.	1	15	
	26	10	.	.	.	.	.	.	4	.	.	.	6	.	26	34, 226
	27	9	.	.	.	.	.	.	3	1	.	3	2	.	24	
	28	10	.	.	.	.	.	.	3	.	3	1	.	3	8	
	29	9	.	.	.	.	.	.	3	.	3	.	3	.	29	34, 187, 226
	30	9	.	.	.	.	.	.	2	2	2	1	1	1	22	187, 264
	31	8	.	.	.	.	.	.	2	.	6	.	.	.	31	34, 187, 226
	32	10	.	.	.	.	.	.	1	5	.	.	1	3	10	
	33	8	.	.	.	.	.	.	1	3	3	1	.	.	33	74, 187, 226
	34	9	.	.	.	.	.	.	.	6	1	.	.	2	14	

Degree	Index Number	Numbers of *F*-points of multiplicities													Conjugate Number	References
		13	12	11	10	9	8	7	6	5	4	3	2	1		
$n=14$	1	1	.	.	.	.	.	.	.	.	.	.	.	26	1	
	2	.	1	.	.	.	.	.	.	.	.	.	12	3	27	
	3	.	.	1	.	.	.	.	.	.	.	7	2	3	9	
	4	.	.	1	.	.	.	.	.	.	.	6	5	.	24	
	5	.	.	.	1	.	.	.	.	.	5	.	3	3	29	
	6	.	.	.	1	.	.	.	.	.	4	3	.	4	6	34
	7	.	.	.	1	.	.	.	.	.	4	2	3	1	34	
	8	.	.	.	1	.	.	.	.	.	2	7	.	.	15	
	9	.	.	.	.	1	.	.	.	3	2	.	.	7	3	
	10	.	.	.	.	1	.	.	.	3	1	1	3	2	17	
	11	.	.	.	.	1	.	.	.	3	.	4	.	3	11	35, 226
	12	.	.	.	.	1	.	.	.	3	.	3	3	.	47	
	13	.	.	.	.	1	.	.	.	2	3	.	4	.	41	
	14	.	.	.	.	1	.	.	.	2	2	3	1	1	22	
	15	.	.	.	.	1	.	.	.	.	7	.	.	2	8	
	16	.	.	.	.	.	1	.	3	.	.	.	5	3	28	
	17	.	.	.	.	.	1	.	2	1	.	3	1	3	10	
	18	.	.	.	.	.	1	.	2	1	.	2	4	.	43	
	19	.	.	.	.	.	1	.	2	.	3	.	2	3	30	
	20	.	.	.	.	.	1	.	2	.	2	2	2	1	37	
	21	.	.	.	.	.	1	.	1	2	2	.	3	1	33	
	22	.	.	.	.	.	1	.	1	2	1	3	.	2	14	
	23	.	.	.	.	.	1	.	1	1	3	2	1	.	44	187, 395
	24	.	.	.	.	.	1	.	.	5	.	.	.	6	4	
	25	.	.	.	.	.	1	.	.	4	.	3	1	.	25	34, 187, 226
	26	.	.	.	.	.	1	.	.	2	5	.	.	1	36	
	27	.	.	.	.	.	.	3	1	.	.	.	.	12	2	
	28	.	.	.	.	.	.	3	.	1	.	.	5	3	16	
	29	.	.	.	.	.	.	3	.	.	1	3	.	5	5	
	30	.	.	.	.	.	.	3	.	.	1	2	3	2	19	
	31	.	.	.	.	.	.	3	.	.	.	4	3	.	45	
	32	.	.	.	.	.	.	2	2	.	.	.	6	1	32	34, 226
	33	.	.	.	.	.	.	2	1	1	.	3	2	1	21	
	34	.	.	.	.	.	.	2	1	.	3	1	.	4	7	
	35	.	.	.	.	.	.	2	1	.	3	.	3	1	35	34, 395
	36	.	.	.	.	.	.	2	1	.	1	5	.	.	26	
	37	.	.	.	.	.	.	2	.	2	2	1	1	2	20	
	38	.	.	.	.	.	.	2	.	2	1	3	1	.	49	
	39	.	.	.	.	.	.	2	.	1	4	.	2	.	46	
	40	.	.	.	.	.	.	2	.	.	6	.	.	1	40	34, 101, 226
	41	.	.	.	.	.	.	1	3	.	.	4	.	2	13	
	42	.	.	.	.	.	.	1	2	1	2	1	2	.	42	34, 187, 226
	43	.	.	.	.	.	.	1	1	4	.	.	2	2	18	
	44	.	.	.	.	.	.	1	1	3	1	2	.	1	23	187
	45	.	.	.	.	.	.	.	4	.	3	.	.	3	31	
	46	.	.	.	.	.	.	.	4	.	2	2	.	1	39	395
	47	.	.	.	.	.	.	.	3	3	.	1	.	3	12	
	48	.	.	.	.	.	.	.	3	3	.	.	3	.	48	226
	49	.	.	.	.	.	.	.	3	2	2	.	1	1	38	
	50	.	.	.	.	.	.	.	2	3	3	.	.	.	50	74, 187, 226
	51	.	.	.	.	.	.	.	1	6	.	1	.	.	51	74, 187, 226

Degree	Index Number	Numbers of F-points of multiplicities														Conjugate Number	References
		14	13	12	11	10	9	8	7	6	5	4	3	2	1		
$n=15$	1	1	.	.	.	.	.	.	.	.	.	.	.	.	28	1	
	2	.	1	.	.	.	.	.	.	.	.	.	.	13	3	30	
	3	.	.	1	.	.	.	.	.	.	.	.	8	1	4	9	
	4	.	.	1	.	.	.	.	.	.	.	.	7	4	1	23	
	5	.	.	.	1	.	.	.	.	.	.	5	2	.	5	5	34
	6	.	.	.	1	.	.	.	.	.	.	5	1	3	2	33	
	7	.	.	.	1	.	.	.	.	.	.	4	3	3	.	51	
	8	.	.	.	1	.	.	.	.	.	.	3	6	.	1	16	
	9	.	.	.	.	1	.	.	.	.	4	1	.	.	8	3	
	10	.	.	.	.	1	.	.	.	.	4	.	1	3	3	17	
	11	.	.	.	.	1	.	.	.	.	4	.	.	6	.	60	
	12	.	.	.	.	1	.	.	.	.	3	2	.	4	1	37	
	13	.	.	.	.	1	.	.	.	.	3	1	3	1	2	20	
	14	.	.	.	.	1	.	.	.	.	3	.	5	1	.	61	
	15	.	.	.	.	1	.	.	.	.	2	3	2	2	.	45	
	16	.	.	.	.	1	.	.	.	.	1	6	.	.	3	8	
	17	.	.	.	.	.	1	.	.	3	.	.	3	1	4	10	
	18	.	.	.	.	.	1	.	.	3	.	.	2	4	1	41	
	19	.	.	.	.	.	1	.	.	2	1	2	.	3	2	32	
	20	.	.	.	.	.	1	.	.	2	1	1	3	.	3	13	
	21	.	.	.	.	.	1	.	.	2	1	1	2	3	.	55	
	22	.	.	.	.	.	1	.	.	2	.	3	2	1	1	42	
	23	.	.	.	.	.	1	.	.	1	4	.	.	.	7	4	
	24	.	.	.	.	.	1	.	.	1	3	1	.	4	.	50	
	25	.	.	.	.	.	1	.	.	1	3	.	3	1	1	25	
	26	.	.	.	.	.	1	.	.	1	1	5	.	.	2	36	395
	27	.	.	.	.	.	1	.	.	1	1	4	2	.	.	57	187
	28	.	.	.	.	.	1	.	.	.	4	1	3	.	.	28	187, 226
	29	.	.	.	.	.	1	.	.	.	3	4	.	1	.	52	
	30	.	.	.	.	.	.	1	3	.	.	.	.	.	13	2	
	31	.	.	.	.	.	.	1	2	1	.	.	.	6	2	31	226
	32	.	.	.	.	.	.	1	2	.	1	.	3	2	2	19	
	33	.	.	.	.	.	.	1	2	.	.	3	1	.	5	6	
	34	.	.	.	.	.	.	1	2	.	.	3	.	3	2	34	34
	35	.	.	.	.	.	.	1	2	.	.	2	2	3	.	53	
	36	.	.	.	.	.	.	1	2	.	.	1	5	.	1	26	
	37	.	.	.	.	.	.	1	1	2	.	.	4	.	3	12	
	38	.	.	.	.	.	.	1	1	2	.	.	3	3	.	46	
	39	.	.	.	.	.	.	1	1	1	1	2	1	2	1	39	34, 187
	40	.	.	.	.	.	.	1	1	1	.	3	3	.	.	48	
	41	.	.	.	.	.	.	1	1	.	4	.	.	2	3	18	395
	42	.	.	.	.	.	.	1	1	.	3	1	2	.	2	22	395
	43	.	.	.	.	.	.	1	1	.	3	.	4	.	.	62	395
	44	.	.	.	.	.	.	1	1	.	2	3	1	1	.	59	187, 395
	45	.	.	.	.	.	.	1	.	3	.	2	2	.	2	15	
	46	.	.	.	.	.	.	1	.	2	3	.	.	3	1	38	
	47	.	.	.	.	.	.	1	.	2	2	1	2	1	.	47	34, 187, 226
	48	.	.	.	.	.	.	1	.	1	3	3	.	.	1	40	
	49	.	.	.	.	.	.	.	4	.	.	.	.	7	.	49	101, 226
	50	.	.	.	.	.	.	.	3	1	.	.	4	1	1	24	
	51	.	.	.	.	.	.	.	3	.	1	3	.	.	4	7	395
	52	.	.	.	.	.	.	.	3	.	1	1	4	.	.	29	395
	53	.	.	.	.	.	.	.	2	2	.	3	.	1	2	35	
	54	.	.	.	.	.	.	.	2	2	.	2	2	1	.	56	
	55	.	.	.	.	.	.	.	2	1	3	.	1	1	2	21	
	56	.	.	.	.	.	.	.	2	1	2	2	.	2	.	54	
	57	.	.	.	.	.	.	.	2	.	4	1	1	.	1	27	395
	58	.	.	.	.	.	.	.	1	3	2	.	1	2	.	58	226
	59	.	.	.	.	.	.	.	1	3	1	2	1	.	1	44	
	60	.	.	.	.	.	.	.	.	6	.	.	.	1	4	11	395
	61	.	.	.	.	.	.	.	.	5	1	1	.	.	3	14	
	62	.	.	.	.	.	.	.	.	4	3	.	.	1	1	43	
	63	.	.	.	.	.	.	.	.	3	4	1	.	.	.	63	74, 187, 226

Degree	Index Number	Numbers of *F*-points of multiplicities															Conjugate Number	References
		15	14	13	12	11	10	9	8	7	6	5	4	3	2	1		
$n=16$	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	30	1	
	2	.	1	.	.	.	.	.	.	.	.	.	.	.	14	3	45	
	3	.	.	1	.	.	.	.	.	.	.	.	.	9	.	5	9	
	4	.	.	1	.	.	.	.	.	.	.	.	.	8	3	2	19	
	5	.	.	.	1	.	.	.	.	.	.	.	6	1	.	6	5	
	6	.	.	.	1	.	.	.	.	.	.	.	6	.	3	3	49	
	7	.	.	.	1	.	.	.	.	.	.	.	5	2	3	1	56	
	8	.	.	.	1	.	.	.	.	.	.	.	4	5	.	2	14	
	9	.	.	.	.	1	.	.	.	.	.	5	.	.	.	9	3	
	10	.	.	.	.	1	.	.	.	.	.	4	1	.	4	2	28	
	11	.	.	.	.	1	.	.	.	.	.	4	.	3	1	3	17	
	12	.	.	.	.	1	.	.	.	.	.	4	.	2	4	.	82	
	13	.	.	.	.	1	.	.	.	.	.	3	2	2	2	1	34	
	14	.	.	.	.	1	.	.	.	.	.	2	5	.	.	4	8	
	15	.	.	.	.	1	.	.	.	.	.	2	3	4	.	.	42	
	16	.	.	.	.	.	1	.	.	.	3	.	2	.	3	3	47	
	17	.	.	.	.	.	1	.	.	.	3	.	1	3	.	4	11	
	18	.	.	.	.	.	1	.	.	.	3	.	1	2	3	1	61	
	19	.	.	.	.	.	1	.	.	.	2	3	.	.	.	8	4	
	20	.	.	.	.	.	1	.	.	.	2	2	1	.	4	1	55	
	21	.	.	.	.	.	1	.	.	.	2	2	.	3	1	2	21	34
	22	.	.	.	.	.	1	.	.	.	2	1	2	2	2	.	71	
	23	.	.	.	.	.	1	.	.	.	2	.	5	.	.	3	52	
	24	.	.	.	.	.	1	.	.	.	2	.	4	2	.	1	65	
	25	.	.	.	.	.	1	.	.	.	1	3	1	3	.	1	25	34, 187
	26	.	.	.	.	.	1	.	.	.	1	2	4	.	1	1	58	
	27	.	.	.	.	.	.	1	.	3	.	.	.	.	6	3	27	226
	28	.	.	.	.	.	.	1	.	2	1	.	.	4	.	4	10	
	29	.	.	.	.	.	.	1	.	2	1	.	.	3	3	1	36	
	30	.	.	.	.	.	.	1	.	2	.	1	2	1	2	2	30	34
	31	.	.	.	.	.	.	1	.	2	.	1	1	3	2	.	79	
	32	.	.	.	.	.	.	1	.	2	.	.	4	.	3	.	77	
	33	.	.	.	.	.	.	1	.	2	.	.	3	3	.	1	39	
	34	.	.	.	.	.	.	1	.	1	2	.	2	2	.	3	13	
	35	.	.	.	.	.	.	1	.	1	2	.	2	1	3	.	69	
	36	.	.	.	.	.	.	1	.	1	1	3	.	.	3	2	29	
	37	.	.	.	.	.	.	1	.	1	1	2	1	2	1	1	37	34, 187
	38	.	.	.	.	.	.	1	.	1	1	.	5	1	.	.	73	
	39	.	.	.	.	.	.	1	.	1	.	3	3	.	.	2	33	
	40	.	.	.	.	.	.	1	.	1	.	3	2	2	.	.	86	187
	41	.	.	.	.	.	.	1	.	.	3	2	.	.	4	.	67	
	42	.	.	.	.	.	.	1	.	.	3	.	4	.	.	2	15	
	43	.	.	.	.	.	.	1	.	.	2	3	.	3	.	.	43	34, 187, 226
	44	.	.	.	.	.	.	1	.	.	2	2	3	.	1	.	70	

Degree	Index Number	Numbers of F-points of multiplicities								Conjugate Number	References
		8	7	6	5	4	3	2	1		
$n=16$	45	3	1	.	.	.	.	.	14	2	
	46	3	.	1	.	.	.	6	3	46	226
	47	3	.	.	1	.	3	2	3	16	
	48	3	.	.	1	.	2	5	.	74	
	49	3	.	.	.	3	1	.	6	6	
	50	3	.	.	.	3	.	3	3	50	187
	51	3	.	.	.	2	2	3	1	59	
	52	3	.	.	.	1	5	.	2	23	
	53	3	.	.	.	.	7	.	.	87	
	54	2	2	.	.	.	.	7	1	54	226
	55	2	1	1	.	.	4	1	2	20	
	56	2	1	.	1	3	.	.	5	7	
	57	2	1	.	1	2	1	3	.	68	
	58	2	1	.	1	1	4	.	1	26	
	59	2	.	2	.	3	.	1	3	51	
	60	2	.	2	.	2	2	1	1	63	
	61	2	.	1	3	.	1	1	3	18	
	62	2	.	1	3	.	.	4	.	83	
	63	2	.	1	2	2	.	2	1	60	
	64	2	.	1	1	3	2	.	.	76	
	65	2	.	.	4	1	1	.	2	24	
	66	2	.	.	3	3	.	1	.	85	187
	67	1	3	.	.	.	4	2	.	41	
	68	1	2	1	.	3	.	2	1	57	
	69	1	2	.	3	.	1	2	1	35	
	70	1	2	.	2	1	3	.	.	44	
	71	1	1	2	2	.	2	.	2	22	
	72	1	1	2	1	2	1	1	.	72	34, 187
	73	1	1	.	5	1	.	.	1	38	
	74	1	.	5	.	.	.	2	3	48	
	75	1	.	4	1	.	2	1	.	75	34, 226
	76	1	.	3	2	2	.	.	1	64	
	77	.	4	.	.	3	1	.	2	32	
	78	.	4	.	.	2	3	.	.	81	
	79	.	3	1	2	1	.	1	2	31	
	80	.	3	1	2	.	2	1	.	84	
	81	.	3	.	4	.	.	2	.	78	
	82	.	2	4	.	.	1	.	4	12	
	83	.	2	4	.	.	.	3	1	62	
	84	.	2	3	1	1	.	2	.	80	
	85	.	2	3	.	3	.	.	1	66	
	86	.	2	2	3	.	1	.	1	40	
	87	.	.	7	.	.	.	.	3	53	
	88	.	.	5	3	.	.	.	.	88	74, 187, 226

TABLE II

Characteristics of degree $\alpha\mu+\beta$, $\alpha=1, 2, 3$

n general	
$T_J \equiv C1$	$1^{n-1}(2n-2)^1$
$C2 \equiv 4$	$1^{n-2}(n-2)^2\,3^1$

$n=2\mu$		$n=2\mu+1$	
$C22 \equiv 38$	$1^{n-4}(\mu-2)^4\,3^2\,3^1$	$C28 \equiv 46$	$1^{n-4}(\mu-2)^4\,1^3\,3^2\,2^1$
$C18 \equiv 36$	$1^{n-4}(\mu-3)^4\,2^3\,3^2\,1^1$	$C26 \equiv 42$	$1^{n-4}(\mu-3)^4\,3^3\,3^2$
$C3$	$3^\mu\,1^{\mu-1}(2\mu-2)^1$	$C5$	$1^{\mu+1}\,3^\mu(2\mu-1)^1$
$R6$	$1^{\mu+1}\,3^{\mu-1}(\mu-2)^2\,3^1$	$R7_1$	$1^{\mu+1}\,2^\mu\,1^{\mu-1}(\mu-1)^2\,2^1$
$R7_0$	$3^\mu\,1^{\mu-2}(\mu-2)^2\,3^1$	$R5$	$4^\mu\,\mu^2$
$R4$	$2^\mu\,2^{\mu-1}(\mu-1)^2\,1^1$		

$n=3\mu$		$n=3\mu+1$		$n=3\mu+2$	
$C8$	$1^{n-3}(2\mu-2)^3\,1^2\,4^1$	$C12$	$1^{n-3}(2\mu-1)^3\,5^1$	$C14$	$1^{n-3}(2\mu-1)^3\,2^2\,3^1$
$C6$	$1^{n-3}(2\mu-3)^3\,4^2\,1^1$	$C10$	$1^{n-3}(2\mu-2)^3\,3^2\,2^1$	$C16$	$1^{n-3}(2\mu-2)^3\,5^2$
P	$1^{n-6}(\mu-2)^6\,3^3\,1^2\,4^1$	P	$1^{n-6}(\mu-2)^6\,1^4\,3^3\,4^1$	P	$1^{n-6}(\mu-2)^6\,1^5\,3^3\,1^2\,3^1$
P	$1^{n-6}(\mu-2)^6\,2^3\,4^2\,1^1$	P	$1^{n-6}(\mu-2)^6\,1^4\,2^3\,3^2\,1^1$	P	$1^{n-6}(\mu-2)^6\,1^5\,2^3\,4^2$
P	$1^{n-6}(\mu-3)^6\,1^5\,1^4\,3^3\,3^1$	P	$1^{n-6}(\mu-3)^6\,2^5\,3^3\,1^2\,2^1$	P	$1^{n-6}(\mu-2)^6\,2^4\,2^3\,2^2\,1^1$
P	$1^{n-6}(\mu-3)^6\,1^5\,1^4\,2^3\,3^2$	P	$1^{n-6}(\mu-3)^6\,1^5\,2^4\,2^3\,2^2$	P	$1^{n-6}(\mu-3)^6\,2^5\,1^4\,3^3\,2^1$
P	$1^{n-6}(\mu-3)^6\,3^4\,2^3\,1^2\,1^1$	P	$1^{n-6}(\mu-3)^6\,4^4\,2^3\,1^1$	P	$1^{n-6}(\mu-3)^6\,1^5\,3^4\,2^3\,1^2$
P	$1^{n-6}(\mu-4)^6\,3^5\,3^3\,1^2\,1^1$	P	$1^{n-6}(\mu-4)^6\,3^5\,1^4\,3^3\,1^1$	P	$1^{n-6}(\mu-4)^6\,4^5\,3^3\,1^2$
P	$1^{n-6}(\mu-4)^6\,1^5\,4^4\,2^3$				
P	$1^{n-6}(\mu-5)^6\,4^5\,1^4\,3^3$				
$C9$	$1^{2\mu}\,4^\mu\,1^{\mu-1}(2\mu-2)^1$	$C13$	$1^{2\mu+1}\,5^\mu(2\mu-1)^1$	$C15$	$1^{2\mu+1}\,3^{\mu+1}\,2^\mu(2\mu-1)^1$
$C7$	$1^{2\mu-1}\,1^{\mu+1}\,4^\mu(2\mu-3)^1$	$C11$	$1^{2\mu}\,2^{\mu+1}\,3^\mu(2\mu-2)^1$	$C17$	$1^{2\mu}\,5^{\mu+1}(2\mu-2)^1$
Rm	$1^{2\mu}\,4^\mu\,1^{\mu-2}(\mu-2)^2\,3^1$	Rm	$1^{2\mu+1}\,4^\mu\,1^{\mu-1}(\mu-1)^2\,2^1$	Rm	$1^{2\mu+2}\,5^\mu(\mu-1)^2\,3^1$
P	$1^{2\mu}\,3^\mu\,2^{\mu-1}(\mu-1)^2\,1^1$	P	$1^{2\mu}\,3^{\mu+1}\,2^{\mu-1}(\mu-2)^2\,3^1$	P	$1^{2\mu+1}\,3^{\mu+1}\,1^\mu\,1^{\mu-1}(\mu-1)$
P	$1^{2\mu-1}\,2^{\mu+1}\,1^\mu\,2^{\mu-1}(\mu-2)^2\,2^1$	P	$1^{2\mu}\,2^{\mu+1}\,2^\mu\,1^{\mu-1}(\mu-1)^2\,1^1$	P	$1^{2\mu+1}\,2^{\mu+1}\,3^\mu\,\mu^2$
P	$1^{2\mu-1}\,1^{\mu+1}\,3^\mu\,1^{\mu-1}(\mu-1)^2$	P	$1^{2\mu-1}\,1^{\mu+2}\,1^{\mu+1}\,3^\mu(\mu-2)^2\,2^1$	P	$1^{2\mu}\,2^{\mu+2}\,3^\mu(\mu-2)^2\,3^1$
P	$1^{2\mu-2}\,1^{\mu+2}\,4^\mu(\mu-3)^2\,3^1$	P	$1^{2\mu-1}\,3^{\mu+1}\,2^\mu(\mu-1)^2$	P	$1^{2\mu}\,1^{\mu+2}\,2^{\mu+1}\,2^\mu(\mu-1)^2\,1$
P	$1^{2\mu-2}\,2^{\mu+1}\,3^\mu(\mu-2)^2\,1^1$	P	$1^{2\mu-2}\,5^{\mu+1}(\mu-3)^2\,3^1$	P	$1^{2\mu-1}\,1^{\mu+2}\,4^{\mu+1}(\mu-2)^2\,2$

TABLE III

Characteristics of degree $4\mu+\beta$ (specimens)

	$n=4\mu$		$n=4\mu+1$		$n=4\mu+2$		$n=4\mu+3$
*C*24	$1^{n-4}(2\mu-2)^4 1^3 6^1$	*C*32	$1^{n-4}(2\mu-1)^4 7^1$	*C*40	$1^{n-4}(2\mu-2)^4 3^3 4^1$	*C*48	$1^{n-4}(2\mu-1)^4 2^3 5^1$
*C*20	$1^{n-4}(2\mu-4)^4 5^3 2^1$	*C*30	$1^{n-4}(2\mu-3)^4 4^3 3^1$	*C*34	$1^{n-4}(2\mu-4)^4 7^3$	*C*44	$1^{n-4}(2\mu-3)^4 6^3 1^1$
	$1^{n-8}(\mu-2)^8 3^4 3^2 3^1$		$1^{n-8}(\mu-2)^8 3^4 3^3 1^2$		$1^{n-8}(\mu-2)^8 1^6 3^4 2^2 3^1$		$1^{n-8}(\mu-2)^8 1^6 3^4 3^3$
*C*25	$1^{3\mu} 6^{\mu} 1^{\mu-1}(2\mu-2)^1$	*C*33	$1^{3\mu+1} 7^{\mu}(2\mu-1)^1$	*C*41	$1^{3\mu+1} 4^{\mu+1} 3^{\mu}(2\mu-2)^1$	*C*49	$1^{3\mu+2} 5^{\mu+1} 2^{\mu}(2\mu-1)^1$
*C*21	$1^{3\mu-1} 2^{\mu+1} 5^{\mu}(2\mu-4)^1$	*C*31	$1^{3\mu} 3^{\mu+1} 4^{\mu}(2\mu-3)^1$	*C*35	$1^{3\mu} 7^{\mu+1}(2\mu-4)^1$	*C*45	$1^{3\mu+1} 1^{\mu+2} 6^{\mu+1}(2\mu-3)^1$
Rm	$1^{3\mu} 6^{\mu} 1^{\mu-2}(\mu-2)^2 3^1$	*Rm*	$1^{3\mu+1} 6^{\mu} 1^{\mu-1}(\mu-1)^2 2^1$	*Rm*	$1^{3\mu+2} 7^{\mu}(\mu-1)^2 3^1$		$1^{3\mu+2} 4^{\mu+1} 3^{\mu} \mu^2$
	$1^{2\mu} 2^{2\mu-1} 1^{2\mu-2}(\mu-1)^4 2^2 1^1$		$3^{2\mu} 1^{2\mu-2}(\mu-1)^4 3^2$		$4^{2\mu} \mu^4 3^1$		$2^{2\mu+1} 2^{2\mu} \mu^4 1^2 2^1$
*C*23	$3^{2\mu} 3^{\mu} 1^{\mu-1}(2\mu-2)^1$	*C*29	$1^{2\mu+1} 2^{2\mu} 1^{\mu+1} 3^{\mu}(2\mu-2)^1$	*C*39	$3^{2\mu+1} 1^{\mu+1} 3^{\mu}(2\mu-1)^1$	*C*47	$1^{2\mu+2} 2^{2\mu+1} 3^{\mu+1} 1^{\mu}(2\mu-1)^1$
*C*19	$2^{2\mu} 1^{2\mu-1} 1^{\mu+1} 3^{\mu}(2\mu-3)^1$	*C*27	$3^{2\mu} 3^{\mu+1} 1^{\mu}(2\mu-3)^1$	*C*37	$2^{2\mu+1} 1^{2\mu} 3^{\mu+1} 1^{\mu}(2\mu-2)^1$	*C*43	$3^{2\mu+1} 1^{\mu+2} 3^{\mu+1}(2\mu-2)^1$
	$3^{2\mu} 1^{\mu+1} 3^{\mu-1}(\mu-2)^2 3^1$		$3^{2\mu} 1^{\mu+2} 3^{\mu}(\mu-1)^2$		$3^{2\mu+1} 4^{\mu} \mu^2$		$1^{2\mu+2} 2^{2\mu+1} 2^{\mu+1} 2^{\mu} \mu^2$

TABLE IV

Characteristics of degree $\alpha\mu-\beta$

$n=\alpha\mu-\beta,\ \beta=2\gamma+\epsilon,\ \epsilon=0$ *or* 1

B $\quad 1^{n-\alpha}\,(2\mu-\beta-2)^{\alpha}\,(\beta+1)^{\alpha-1}\,(2\alpha-\beta-2)^{1}$

(Includes Rm) $\quad 1^{n-\alpha}\,(2\mu-\gamma-\epsilon-2)^{\alpha}\,\epsilon^{\alpha-1}\,(\gamma+1)^{\alpha-2}\,(\alpha-\gamma-2)^{2}\,(3-\epsilon)^{1}$
$1^{n-\alpha}\,(2\mu-\gamma-\epsilon-3)^{\alpha}\,(\epsilon+2)^{\alpha-1}\,\gamma^{\alpha-2}\,(\alpha-\gamma-1)^{2}\,(1-\epsilon)^{1}$

$1^{n-2\alpha}\,(\mu-\beta-2)^{2\alpha}\quad \beta^{2\alpha-1}\quad 3^{\alpha}\,1^{\alpha-1}\,(2\alpha-\beta-2)^{1}$
$1^{n-2\alpha}\,(\mu-\beta-3)^{2\alpha}\,(\beta+1)^{2\alpha-1}\quad 1^{\alpha+1}\,3^{\alpha}\,(2\alpha-\beta-3)^{1}$

$1^{n-2\alpha}\,(\mu-\gamma-3)^{2\alpha}$	$(1-\epsilon)^{2\alpha-1}$	$(\gamma+\epsilon)^{2\alpha-2}$	$1^{\alpha+1}\,2^{\alpha}\,1^{\alpha-1}$	$(\alpha-\gamma-\epsilon-1)^{2}\,\epsilon^{1}$
$1^{n-2\alpha}\,(\mu-\gamma-\epsilon-2)^{2\alpha}$	$\epsilon^{2\alpha-1}$	$\gamma^{2\alpha-2}$	$2^{\alpha}\,2^{\alpha-1}$	$(\alpha-\gamma-1)^{2}\,(1-\epsilon)^{1}$
$1^{n-2\alpha}\,(\mu-\gamma-\epsilon-2)^{2\alpha}$	$\epsilon^{2\alpha-1}$	$\gamma^{2\alpha-2}$	$3^{\alpha}\,1^{\alpha-2}$	$(\alpha-\gamma-2)^{2}\,(3-\epsilon)^{1}$
$1^{n-2\alpha}\,(\mu-\gamma-\epsilon-3)^{2\alpha}$	$(\epsilon+2)^{2\alpha-1}$	$(\gamma-1)^{2\alpha-2}$	$3^{\alpha}\,1^{\alpha-2}$	$(\alpha-\gamma-1)^{2}\,(1-\epsilon)^{1}$
$1^{n-2\alpha}\,(\mu-\gamma-\epsilon-3)^{2\alpha}$	$\epsilon^{2\alpha-1}$	$(\gamma+1)^{2\alpha-2}$	$1^{\alpha+2}\,3^{\alpha}$	$(\alpha-\gamma-3)^{2}\,(3-\epsilon)^{1}$
$1^{n-2\alpha}\,(\mu-\gamma-\epsilon-3)^{2\alpha}$	$\epsilon^{2\alpha-1}$	$(\gamma+1)^{2\alpha-2}$	$2^{\alpha+1}\,2^{\alpha}$	$(\alpha-\gamma-2)^{2}\,(1-\epsilon)^{1}$
$1^{n-2\alpha}\,(\mu-\gamma-\epsilon-4)^{2\alpha}$	$(\epsilon+2)^{2\alpha-1}$	$\gamma^{2\alpha-2}$	$1^{\alpha+2}\,3^{\alpha}$	$(\alpha-\gamma-2)^{2}\,(1-\epsilon)^{1}$

TABLE V

Miscellaneous General Characteristics

$n=2^{\mu}$; $\quad 3^{2^{\mu-1}}\ldots 3^{2^{\alpha}}\ldots 3^{1}$

$T_{\text{semisym.}}$ $\quad n=\mu^{2}+1$; $\quad 1^{\mu(\mu-1)}\,(2\mu+1)^{\mu}$

P $\quad n=n_3\mu-n_1\alpha-(n_1-1)\,\beta-\gamma,\ \mu=n_1n_2-\rho$;
$1^{n-\mu}\,(2n_3-2-\alpha-\beta-\gamma)^{\mu}\,\gamma^{\mu-1}\,\beta^{\mu+1-n_1}\,(1+\alpha)^{\mu-n_1}\,(2n_2-2-\alpha-\rho)^{n_1}\,(\rho+1-\beta)^{n_1-1}\,(2n_1-2-\gamma-$

TABLE VI

Cubic Space Transformations

	Degrees	D.p. of contact	Binode	D.p.'s	Pt. of osculation	Pt. of contact	Ordinary pts.	*F*-curves (1) denotes a branch touching a fixed biplane	Remarks	References
1	3—2	.	.	.	.	.	.	l^2, l_1, l_2, l_3	3 generators meet double line	p. 287
2	3—3	.	.	.	.	.	.	ω_6 (genus 3)	. . .	p. 294
3		.	.	1	.	.	.	$\omega_6 \equiv O^2$ (genus 3)	. . .	85
4		1	.	.	.	.	.	$\omega_6 \equiv O^4$ (rational)	. . .	85
5		.	.	.	.	.	2	l^2, l_1, l_2	2 generators meet double line	p. 291
6	3—4	.	.	.	.	.	1	ω_5 (genus 1)	. . .	p. 170
7		.	.	1	.	.	1	$\omega_5 \equiv O_1^2$ (genus 1)	. . .	23, 97
8		.	1	.	.	.	1	$\omega_5 \equiv O_1^3$ (2) (rational)		
9		1	.	.	.	.	1	$\omega_3 \equiv O_1^2$, $l_1 \equiv O_1$, $l_2 \equiv O_1$	. . .	23
0		1	1	.	.	.	1	$\omega_2 \equiv O_1O_2$, $l \equiv O_1O_2$ (osculation)	(ϕ) touch plane along l	
1		.	.	.	.	.	4	l^2, l_1	generator meets double line	p. 291
2	3—5	.	.	.	.	.	2	ω_3 (rational), l	. . .	p. 170
3		.	.	1	.	.	2	$\omega_4 \equiv O_1$ (genus 1)	. . .	85
4		.	.	1	.	.	2	$\omega_3 \equiv O_1$ (rational), $l \equiv O_1$		
5		.	1	.	.	.	2	$\omega_4 \equiv O_1^2$ (2)		
6		.	1	.	.	.	2	$\omega_2 \equiv O_1$ (1), $l_1 \equiv O_1$ (1), $l_2 \equiv O_1$		
7		1	.	.	.	.	2	$\omega_3 \equiv O_1^2$, $l_1 \equiv O_1$		
8		1	.	.	.	.	2	$l \equiv O_1$ (contact), $l_1 \equiv O_1$, $l_2 \equiv O_1$	(ϕ) touch quadric	
9		.	.	2	.	.	2	$\omega_3 \equiv O_1O_2$ (rational), $l \equiv O_1O_2$	. . .	85
0		.	1	1	.	.	2	$\omega_2 \equiv O_1$ (1) O_2, $l_1 \equiv O_1O_2$, $l_2 \equiv O_1$ (1)		
1		1	.	1	.	.	2	$l \equiv O_1O_2$ (contact), $l_1 \equiv O_1$, $l_2 \equiv O_1$	(ϕ) touch plane	
2		1	1	.	.	.	2	$l \equiv O_1O_2$ (1) (osculation), $l_1 \equiv O_1$	(ϕ) touch plane	
3		.	.	.	.	1	.	ω_4 (rational)	. . .	49, p. 170
4		.	.	1	.	1	.	$\omega_4 \equiv O_1^2$		
5		.	1	.	.	1	.	$\omega_3 \equiv O_1^2$ (1), $l \equiv O_1$ (1)		
6		1	.	.	.	1	.	$l_1 \equiv O_1$, $l_2 \equiv O_1$, $l_3 \equiv O_1$, $l_4 \equiv O_1$		
7		.	.	.	.	.	6	l^2	. . .	p. 291
8	3—6	.	.	.	.	.	3	l (contact), l_1	. . .	p. 170
9		.	.	1	.	.	3	ω_3 (plane, genus 1)	. . .	85, 258
0		.	.	1	.	.	3	ω_2, $l \equiv O_1$		
1		.	.	1	.	.	3	$l \equiv O_1$ (contact), l_1	. . .	385
2		.	.	1	.	.	3	$l \equiv O_1$ (osculation)	(ϕ) touch quadric	
3		.	1	.	.	.	3	$\omega_2 \equiv O_1$ (1), $l \equiv O_1$		
4		1	.	.	.	.	3	$\omega_3 \equiv O_1^2$		
5		1	.	.	.	.	3	$l \equiv O_1$ (contact), $l_1 \equiv O_1$	(ϕ) touch quadric	
6		.	.	2	.	.	3	$\omega_2 \equiv O_1$, $l \equiv O_1O_2$	. . .	85
7		.	1	1	.	.	3	$\omega_2 \equiv O_1$ (1) O_2, $l \equiv O_1O_2$		
8		.	1	1	.	.	3	$l \equiv O_1O_2$, $l_1 \equiv O_1$ (1), $l_2 \equiv O_1$ (1)		
9		1	.	1	.	.	3	$l \equiv O_1O_2$ (contact), $l_1 \equiv O_1$	(ϕ) touch plane	
0		1	1	.	.	.	3	$l \equiv O_1O_2$ (1) (osculation)	(ϕ) touch plane	
1		.	.	.	.	1	1	l_1, l_2, l_3	. . .	41, p. 170
2		.	.	1	.	1	1	$\omega_3 \equiv O_1$ (rational)		
3		.	.	1	.	1	1	$l_1 \equiv O_1$, $l_2 \equiv O_1$, l_3		
4		.	1	.	.	1	1	$\omega_3 \equiv O_1^2$ (1)		
5		.	1	.	.	1	1	$\omega_2 \equiv O_1$ (1), $l \equiv O_1$ (1)		

TABLE VI (*continued*)

Cubic Space Transformations

Number	Degrees	D.p. of contact	Binode	D.p.'s	Pt. of osculation	Pt. of contact	Ordinary pts.	*F*-curves (1) denotes a branch touching a fixed biplane	Remarks	Reference
46	3—6	.	1	.	.	1	1	$l\equiv O_1$ (1) (contact), $l_1\equiv O_1$	(ϕ) touch quadric	
47		1	.	.	.	1	1	$l_1\equiv O_1$, $l_2\equiv O_1$, $l_3\equiv O_1$		
48		.	.	2	.	1	1	$l\equiv O_1O_2$, $l_1\equiv O_1$, $l_2\equiv O_2$		
49		.	1	1	.	1	1	$l\equiv O_1$ (1) O_2 (contact), $l_2\equiv O_1$	(ϕ) touch plane; O_2 on fixed plane at O_1	
50		.	.	3	.	1	1	$l_1\equiv O_2O_3$, $l_2\equiv O_3O_1$, $l_1\equiv O_1O_2$		85
51		.	.	.	1	.	.	ω_3 (rational)		p. 170
52		.	.	1	1	.	.	$\omega_3\equiv O_1^2$		
53	3—7	1	.	.	.	.	4	$l\equiv O_1$ (contact)	(ϕ) touch quadric	
54		.	.	2	.	.	4	$l\equiv O_1O_2$, l_1		85
55		.	1	1	.	.	4	$l\equiv O_1O_2$, $l_1\equiv O_1$ (1)		
56		1	.	1	.	.	4	$l\equiv O_1O_2$ (contact)	(ϕ) touch plane	
57		.	.	1	.	1	2	ω_2		85
58		.	1	.	.	1	2	$\omega_2\equiv O_1$ (1)		
59		1	.	.	.	1	2	$l_1\equiv O_1$, $l_2\equiv O_1$		
60		.	.	1	.	2	.	$l\equiv O_1$, l_1		85
61		.	1	.	.	2	.	$l_1\equiv O_1$ (1), $l_2\equiv O_1$		
62		.	1	.	.	2	.	$l\equiv O_1$ (1) (contact)	(ϕ) touch quadric	
63		.	.	2	.	2	.	$l\equiv O_1O_2$, $l_1\equiv O_1$		
64		.	1	1	.	2	.	$l\equiv O_1$ (1) O_2 (contact)	(ϕ) touch plane; O_2 on fixed plane at O_1	
65		.	.	1	1	.	1	$\omega_2\equiv O_1$		85, 258
66		.	1	.	1	.	1	$l_1\equiv O_1$ (1), $l_2\equiv O_1$ (1)		
67	3—8	.	1	1	.	.	5	$l\equiv O_1O_2$		
68		1	.	.	.	1	3	$l\equiv O_1$		
69		.	1	.	.	2	1	$l\equiv O_1$		
70		.	.	1	1	.	2	l		85
71		.	1	.	1	.	2	$l\equiv O_1$ (1)		
72	3—9	1	.	.	.	1	4			396
73		.	1	.	.	3	.			
74		.	1	.	1	.	3			
75		.	.	1	1	.	2		4-point contact at O_2	18, 85, 385

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